CHAPTER XI.

CONCERNING THE PROBABILITIES OF TESTIMONIES.

THE majority of our opinions being founded on the probability of proofs it is indeed important to submit it to calculus. Things it is true often become impossible by the difficulty of appreciating the veracity of witnesses and by the great number of circumstances which accompany the deeds they attest; but one is able in several cases to resolve the problems which have much analogy with the questions which are proposed and whose solutions may be regarded as suitable approximations to guide and to defend us againt the errors and the dangers of false reasoning to which we are exposed. An approximation of this kind, when it is well made, is always preferable to the most specious reasonings. Let us try then to give some general rules for obtaining it.

A single number has been drawn from an urn which contains a thousand of them. A witness to this drawing announces that number 79 is drawn; one asks the probability of drawing this number. Let us suppose that experience has made known that this witness deceives one time in ten, so that the probability of his testimony is $\frac{1}{10}$. Here the event observed is the witness attesting that number 79 is drawn. This event may result from the two following hypotheses, namely: that the witness utters the truth or that he deceives. Following the principle that has been expounded on the probability of causes drawn from events observed it is necessary first to determine *à priori* the probability of the event in each hypothesis. In the first, the probability that the witness will announce number 79 is the probability itself of the drawing of this number, that is to say, $\frac{1}{1000}$. It is necessary to multiply it by the probability $\frac{6}{10}$ of the veracity of the witness; one will have then $\frac{9}{10000}$ for the probability of the event observed in this hypothesis. If the witness deceives, number 79 is not drawn, and the probability of this case is $\frac{999}{1000}$. But to announce the drawing of this number the witness has to choose it among the 999 numbers not drawn; and as he is supposed to have no motive of preference for the ones rather than the others, the probability that he will choose number 79 is $\frac{1}{999}$; multiplying, then, this probability by the preceding one, we shall have $\frac{1}{1000}$ for the probability that the witness will announce number 79 in the second hypothesis. It is necessary again to multiply this probability by $\frac{1}{10}$ of the hypothesis itself, which gives $\frac{1}{10000}$ for the probability of the event relative to this hypothesis. Now if we form a fraction whose numerator is the probability relative to the first hypothesis, and whose denominator is the sum of the probabilities relative to the two hypotheses, we shall have, by the sixth principle, the probability of the first hypothesis, and this probability will be $\frac{9}{10}$; that is to say, the veracity itself of the witness. This is likewise the probability of the drawing of number 79. The probability of the falsehood of the witness and of the failure of drawing this number is $\frac{1}{10}$.

If the witness, wishing to deceive, has some interest in choosing number 79 among the numbers not drawn, —if he judges, for example, that having placed upon this number a considerable stake, the announcement of its drawing will increase his credit, the probability that he will choose this number will no longer be as at first, $\frac{1}{999}$, it will then be $\frac{1}{2}$, $\frac{1}{3}$, etc., according to the interest that he will have in announcing its drawing. Supposing it to be $\frac{1}{9}$, it will be necessary to multiply by this fraction the probability $\frac{999}{1000}$ in order to get in the hypothesis of the falsehood the probability of the event observed, which it is necessary still to multiply by $\frac{1}{10}$, which gives $\frac{111}{10000}$ for the probability of the event in the second hypothesis. Then the probability of the first hypothesis, or of the drawing of number 79, is reduced by the preceding rule to $\frac{9}{120}$. It is then very much decreased by the consideration of the interest which the witness may have in announcing the drawing of number 79. In truth this same interest increases the probability $\frac{9}{10}$ that the witness will speak the truth if number 79 is drawn. But this probability cannot exceed unity or $\frac{10}{10}$; thus the probability of the drawing of number 79 will not surpass $\frac{10}{121}$. Common sense tells us that this interest ought to inspire distrust, but calculus appreciates the influence of it.

The probability *à priori* of the number announced by the witness is unity divided by the number of the numbers in the urn; it is changed by virtue of the proof into the veracity itself of the witness; it may then be decreased by the proof. If, for example, the urn contains only two numbers, which gives $\frac{1}{2}$ for the probability *à priori* of the drawing of number 1, and if the veracity of a witness who announces it is $\frac{4}{10}$, this drawing becomes less probable. Indeed it is apparent, since the witness has then more inclination towards a falsehood than towards the truth, that his testimony ought to decrease the probability of the fact attested every time that this probability equals or surpasses $\frac{1}{2}$. But if there are three numbers in the urn the probability *à priori* of the drawing of number 1 is increased by the affirmation of a witness whose veracity surpasses $\frac{1}{3}$.

Suppose now that the urn contains 999 black balls and one white ball, and that one ball having been drawn a witness of the drawing announces that this ball is white. The probability of the event observed, determined *à priori* in the first hypothesis, will be here, as in the preceding question, equal to $\frac{9}{10000}$. But in the hypothesis where the witness deceives, the white ball is not drawn and the probability of this case is $\frac{999}{1000}$. It is necessary to multiply it by the probability $\frac{1}{10}$ of the falsehood, which gives $\frac{999}{10000}$ for the probability of the event observed relative to the second hypothesis. This probability was only $\frac{1}{10000}$ in the preceding question; this great difference results from this—that a black ball having been drawn the witness who wishes to deceive has no choice at all to make among the 999 balls not drawn in order to announce the drawing of a white ball. Now if one forms two fractions whose numerators are the probabilities relative to each hypothesis, and whose common denominator is the sum of these probabilities, one will have $\frac{9}{1008}$ for the probability of the first hypothesis and of the drawing of a white ball, and $\frac{999}{1008}$ for the probability of the second hypothesis and of the drawing of a black ball. This last probability strongly approaches certainty; it would approach it much nearer and would become $\frac{999999}{1000008}$ if the urn contained a million balls of which one was white, the drawing of a white ball becoming then much more extraordinary. We see thus how the probability of the falsehood increases in the measure that the deed becomes more extraordinary.

We have supposed up to this time that the witness was not mistaken at all; but if one admits, however, the chance of his error the extraordinary incident becomes more improbable. Then in place of the two hypotheses one will have the four following ones, namely: that of the witness not deceiving and not being mistaken at all; that of the witness not deceiving at all and being mistaken; the hypothesis of the witness deceiving and not being mistaken at all; finally, that of the witness deceiving and being mistaken. Determining *à priori* in each of these hypotheses the probability of the event observed, we find by the sixth principle the probability that the fact attested is false equal to a fraction whose numerator is the number of black balls in the urn multiplied by the sum of the probabilities that the witness does not deceive at all and is mistaken, or that he deceives and is not mistaken, and whose denominator is this numerator augmented by the sum of the probabilities that the witness does not deceive at all and is not mistaken at all, or that he deceives and is mistaken at the same time. We see by this that if the number of black balls in the urn is very great, which renders the drawing of the white ball extraordinary, the probability that the fact attested is not true approaches most nearly to certainty.

Applying this conclusion to all extraordinary deeds it results from it that the probability of the error or of the falsehood of the witness becomes as much greater as the fact attested is more extraordinary. Some authors have advanced the contrary on this basis that the view of an extraordinary fact being perfectly similar to that of an ordinary fact the same motives ought to lead us to give the witness the same credence when he affirms the one or the other of these facts. Simple common sense rejects such a strange assertion; but the calculus of probabilities, while confirming the findings of common sense, appreciates the greatest improbability of testimonies in regard to extraordinary facts.

These authors insist and suppose two witnesses equally worthy of belief, of whom the first attests that he saw an individual dead fifteen days ago whom the second witness affirms to have seen yesterday full of life. The one or the other of these facts offers no improbability. The reservation of the individual is a result of their combination; but the testimonies do not bring us at all directly to this result, although the credence which is due these testimonies ought not to be decreased by the fact that the result of their combination is extraordinary.

But if the conclusion which results from the combination of the testimonies was impossible one of them would be necessarily false; but an impossible conclusion is the limit of extraordinary conclusions, as error is the limit of improbable conclusions; the value of the testimonies which becomes zero in the case of an impossible conclusion ought then to be very much decreased in that of an extraordinary conclusion. This is indeed confirmed by the calculus of probabilities.

In order to make it plain let us consider two urns, A and B, of which the first contains a million white balls and the second a million black balls. One draws from one of these urns a ball, which he puts back into the other urn, from which one then draws a ball. Two witnesses, the one of the first drawing, the other of the second, attest that the ball which they have seen drawn is white without indicating the urn from which it has been drawn. Each testimony taken alone is not improbable; and it is easy to see that the probability of the fact attested is the veracity itself of the witness. But it follows from the combination of the testimonies that a white ball has been extracted from the urn A at the first draw, and that then placed in the urn B it has reappeared at the second draw, which is very extraordinary; for this second urn, containing then one white ball among a million black balls, the probability of drawing the white ball is $\frac{1}{1000001}$. In order to determine the diminution which results in the probability of the thing announced by the two witnesses we shall notice that the event observed is here the affirmation by each of them that the ball which he has seen extracted is white. Let us represent by $\frac{9}{10}$ the probability that he announces the truth, which can occur in the present case when the witness does not deceive and is not mistaken at all, and when he deceives and is mistaken at the same time. One may form the four following hypotheses:

1st. The first and second witness speak the truth. Then a white ball has at first been drawn from the urn A, and the probability of this event is $\frac{1}{2}$, since the ball drawn at the first draw may have been drawn either from the one or the other urn. Consequently the ball drawn, placed in the urn B, has reappeared at the second draw; the probability of this event is $\frac{1}{1000001}$, the probability of the fact announced is then $\frac{1}{2000002}$. Multiplying it by the product of the probabilities $\frac{9}{10}$ and $\frac{9}{10}$ that the witnesses speak the truth one will have $\frac{81}{200000200}$ for the probability of the event observed in this first hypothesis.

2d. The first witness speaks the truth and the second does not, whether he deceives and is not mistaken or he does not deceive and is mistaken. Then a white ball has been drawn from the urn A at the first draw, and the probability of this event is $\frac{1}{2}$. Then this ball having been placed in the urn B a black ball has been drawn from it: the probability of such drawing is $\frac{1000000}{1000001}$; one has then $\frac{1000000}{2000002}$ for the probability of the compound event. Multiplying it by the product of the two probabilities $\frac{9}{10}$ and $\frac{1}{10}$ that the first witness speaks the truth and that the second does not, one will have $\frac{9000000}{200000200}$ for the probability for the event observed in the second hypothesis.

3d. The first witness does not speak the truth and the second announces it. Then a black ball has been drawn from the urn B at the first drawing, and after having been placed in the urn A a white ball has been drawn from this urn. The probability of the first of these events is $\frac{1}{2}$ and that of the second is $\frac{1000000}{1000001}$; the probability of the compound event is then $\frac{1000000}{2000002}$. Multiplying it by the product of the probabilities $\frac{1}{10}$ and $\frac{9}{10}$ that the first witness does not speak the truth and that the second announces it, one will have $\frac{9000000}{200000200}$ for the probability of the event observed relative to this hypothesis.

4th. Finally, neither of the witnesses speaks the truth. Then a black ball has been drawn from the urn B at the first draw; then having been placed in the urn A it has reappeared at the second drawing: the probability of this compound event is $\frac{1}{2000002}$. Multiplying it by the product of the probabilities $\frac{1}{10}$ and $\frac{1}{10}$ that each witness does not speak the truth one will have $\frac{1}{200000200}$ for the probability of the event observed in this hypothesis.

Now in order to obtain the probability of the thing announced by the two witnesses, namely, that a white ball has been drawn at each draw, it is necessary to divide the probability corresponding to the first hypothesis by the sum of the probabilities relative to the four hypotheses; and then one has for this probability $\frac{81}{1800000082}$, an extremely small fraction.

If the two witnesses affirm the first, that a white ball has been drawn from one of the two urns A and B; the second that a white ball has been likewise drawn from one of the two urns A′ and B′, quite similar to the first ones, the probability of the thing announced by the two witnesses will be the product of the probabilities of their testimonies, or $\frac{81}{100}$; it will then be at least a hundred and eighty thousand times greater than the preceding one. One sees by this how much, in the first case, the reappearance at the second draw of the white ball drawn at the first draw, the extraordinary conclusion of the two testimonies decreases the value of it.

We would give no credence to the testimony of a man who should attest to us that in throwing a hundred dice into the air they had all fallen on the same face. If we had ourselves been spectators of this event we should believe our own eyes only after having carefully examined all the circumstances, and after having brought in the testimonies of other eyes in order to be quite sure that there had been neither hallucination nor deception. But after this examination we should not hesitate to admit it in spite of its extreme improbability; and no one would be tempted, in order to explain it, to recur to a denial of the laws of vision. We ought to conclude from it that the probability of the constancy of the laws of nature is for us greater than this, that the event in question has not taken place at all—a probability greater than that of the majority of historical facts which we regard as incontestable. One may judge by this the immense weight of testimonies necessary to admit a suspension of natural laws, and how improper it would be to apply to this case the ordinary rules of criticism. All those who without offering this immensity of testimonies support this when making recitals of events contrary to those laws, decrease rather than augment the belief which they wish to inspire; for then those recitals render very probable the error or the falsehood of their authors.

But that which diminishes the belief of educated men increases often that of the uneducated, always greedy for the wonderful.

There are things so extraordinary that nothing can balance their improbability. But this, by the effect of a dominant opinion, can be weakened to the point of appearing inferior to the probability of the testimonies; and when this opinion changes an absurd statement admitted unanimously in the century which has given it birth offers to the following centuries only a new proof of the extreme influence of the general opinion upon the more enlightened minds. Two great men of the century of Louis XIV.—Racine and Pascal—are striking examples of this. It is painful to see with what complaisance Racine, this admirable painter of the human heart and the most perfect poet that has ever lived, reports as miraculous the recovery of Mlle. Perrier, a niece of Pascal and a day pupil at the monastery of Port-Royal; it is painful to read the reasons by which Pascal seeks to prove that this miracle should be necessary to religion in order to justify the doctrine of the monks of this abbey, at that time persecuted by the Jesuits. The young Perrier had been afflicted for three years and a half by a lachrymal fistula; she touched her afflicted eye with a relic which was pretended to be one of the thorns of the crown of the Saviour and she had faith in instant recovery. Some days afterward the physicians and the surgeons attest the recovery, and they declare that nature and the remedies have had no part in it. This event, which took place in 1656, made a great sensation, and "all Paris rushed," says Racine, "to Port-Royal. The crowd increased from day to day, and God himself seemed to take pleasure in authorizing the devotion of the people by the number of miracles which were performed in this church." At this time miracles and sorcery did not yet appear improbable, and one did not hesitate at all to attribute to them the singularities of nature which could not be explained otherwise.

This manner of viewing extraordinary results is found in the most remarkable works of the century of Louis XIV.; even in the Essay on the Human Understanding by the philosopher Locke, who says, in speaking of the degree of assent: "Though the common experience and the ordinary course of things have justly a mighty influence on the minds of men, to make them give or refuse credit to anything proposed to their belief; yet there is one case, wherein the strangeness of the fact lessens not the assent to a fair testimony of it. For where such supernatural events are suitable to ends aimed at by him who has the power to change the course of nature, there, under such circumstances, they may be the fitter to procure belief, by how much the more they are beyond or contrary to ordinary observation." The true principles of the probability of testimonies having been thus misunderstood by philosophers to whom reason is principally indebted for its progress, I have thought it necessary to present at length the results of calculus upon this important subject.

There comes up naturally at this point the discussion of a famous argument of Pascal, that Craig, an English mathematician, has produced under a geometric form. Witnesses declare that they have it from Divinity that in conforming to a certain thing one will enjoy not one or two but an infinity of happy lives. However feeble the probability of the proofs may be, provided that it be not infinitely small, it is clear that the advantage of those who conform to the prescribed thing is infinite since it is the product of this probability and an infinite good; one ought not to hesitate then to procure for oneself this advantage.

This argument is based upon the infinite number of happy lives promised in the name of the Divinity by the witnesses; it is necessary then to prescribe them, precisely because they exaggerate their promises beyond all limits, a consequence which is repugnant to good sense. Also calculus teaches us that this exaggeration itself enfeebles the probability of their testimony to the point of rendering it infinitely small or zero. Indeed this case is similar to that of a witness who should announce the drawing of the highest number from an urn filled with a great number of numbers, one of which has been drawn and who would have a great interest in announcing the drawing of this number. One has already seen how much this interest enfeebles his testimony. In evaluating only at $\frac{1}{2}$ the probability that if the witness deceives he will choose the largest number, calculus gives the probability of his announcement as smaller than a fraction whose numerator is unity and whose denominator is unity plus the half of the product of the number of the numbers by the probability of falsehood considered *à priori* or independently of the announcement. In order to compare this case to that of the argument of Pascal it is sufficient to represent by the numbers in the urn all the possible numbers of happy lives which the number of these numbers renders infinite; and to observe that if the witnesses deceive they have the greatest interest, in order to accredit their falsehood, in promising an eternity of happiness. The expression of the probability of their testimony becomes then infinitely small. Multiplying it by the infinite number of happy lives promised, infinity would disappear from the product which expresses the advantage resultant from this promise which destroys the argument of Pascal.

Let us consider now the probability of the totality of several testimonies upon an established fact. In order to fix our ideas let us suppose that the fact be the drawing of a number from an urn which contains a hundred of them, and of which one single number has been drawn. Two witnesses of this drawing announce that number 2 has been drawn, and one asks for the resultant probability of the totality of these testimonies. One may form these two hypotheses: the witnesses speak the truth; the witnesses deceive. In the first hypothesis the number 2 is drawn and the probability of this event is $\frac{1}{100}$. It is necessary to multiply it by the product of the veracities of the witnesses, veracities which we will suppose to be $\frac{9}{10}$ and $\frac{7}{10}$: one will have then $\frac{63}{10000}$ for the probability of the event observed in this hypothesis. In the second, the number 2 is not drawn and the probability of this event is $\frac{99}{100}$. But the agreement of the witnesses requires then that in seeking to deceive they both choose the number 2 from the 99 numbers not drawn: the probability of this choice if the witnesses do not have a secret agreement is the product of the fraction $\frac{1}{99}$ by itself; it becomes necessary then to multiply these two probabilities together, and by the product of the probabilities $\frac{1}{10}$ and $\frac{3}{10}$ that the witnesses deceive; one will have thus $\frac{1}{3300000}$ for the probability of the event observed in the second hypothesis. Now one will have the probability of the fact attested or of the drawing of number 2 in dividing the probability relative to the first hypothesis by the sum of the probabilities relative to the two hypotheses; this probability will be then $\frac{2079}{2080}$, and the probability of the failure to draw this number and of the falsehood of the witnesses will be $\frac{1}{2080}$.

If the urn should contain only the numbers 1 and 2 one would find in the same manner $\frac{21}{22}$ for the probability of the drawing of number 2, and consequently $\frac{1}{22}$ for the probability of the falsehood of the witnesses, a probability at least ninety-four times larger than the preceding one. One sees by this how much the probability of the falsehood of the witnesses diminishes when the fact which they attest is less probable in itself. Indeed one conceives that then the accord of the witnesses, when they deceive, becomes more difficult, at least when they do not have a secret agreement, which we do not suppose here at all.

In the preceding case where the urn contained only two numbers the *à priori* probability of the fact attested is $\frac{1}{2}$, the resultant probability of the testimonies is the product of the veracities of the witnesses divided by this product added to that of the respective probabilities of their falsehood.

It now remains for us to consider the influence of time upon the probability of facts transmitted by a traditional chain of witnesses. It is clear that this probability ought to diminish in proportion as the chain is prolonged. If the fact has no probability itself, such as the drawing of a number from an urn which contains an infinity of them, that which it acquires by the testimonies decreases according to the continued product of the veracity of the witnesses. If the fact has a probability in itself; if, for example, this fact is the drawing of the number 2 from an urn which contains an infinity of them, and of which it is certain that one has drawn a single number; that which the traditional chain adds to this probability decreases, following a continued product of which the first factor is the ratio of the number of numbers in the urn less one to the same number, and of which each other factor is the veracity of each witness diminished by the ratio of the probability of his falsehood to the number of the numbers in the urn less one; so that the limit of the probability of the fact is that of this fact considered *à priori*, or independently of the testimonies, a probability equal to unity divided by the number of the numbers in the urn.

The action of time enfeebles then, without ceasing, the probability of historical facts just as it changes the most durable monuments. One can indeed diminish it by multiplying and conserving the testimonies and the monuments which support them. Printing offers for this purpose a great means, unfortunately unknown to the ancients. In spite of the infinite advantages which it procures the physical and moral revolutions by which the surface of this globe will always be agitated will end, in conjunction with the inevitable effect of time, by rendering doubtful after thousands of years the historical facts regarded to-day as the most certain.

Craig has tried to submit to calculus the gradual enfeebling of the proofs of the Christian religion; supposing that the world ought to end at the epoch when it will cease to be probable, he finds that this ought to take place 1454 years after the time when he writes. But his analysis is as faulty as his hypothesis upon the duration of the moon is bizarre.