THE ART OF
COMPUTER PROGRAMMING
VOLUME 4    PRE-FASCICLE 4B

A DRAFT OF SECTION 7.2.1.7:
HISTORY OF COMBINATORIAL
GENERATION

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See also http://www-cs-faculty.stanford.edu/~knuth/sgb.html for information about The Stanford GraphBase, including downloadable software for dealing with the graphs used in many of the examples in Chapter 7.


See also http://www-cs-faculty.stanford.edu/~knuth/mixware.html for downloadable software to simulate the MMIX computer.

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PREFACE

I like to work in a variety of fields
in order to spread my mistakes more thinly.
— VICTOR KLEE (1999)

This booklet contains draft material that I’m circulating to experts in the field, in hopes that they can help remove its most egregious errors before too many other people see it. I am also, however, posting it on the Internet for courageous and/or random readers who don’t mind the risk of reading a few pages that have not yet reached a very mature state. Beware: This material has not yet been proofread as thoroughly as the manuscripts of Volumes 1, 2, and 3 were at the time of their first printings. And those carefully-checked volumes, alas, were subsequently found to contain thousands of mistakes.

Given this caveat, I hope that my errors this time will not be so numerous and/or obtrusive that you will be discouraged from reading the material carefully. I did try to make it both interesting and authoritative, as far as it goes. But the field is so vast, I cannot hope to have surrounded it enough to corral it completely. Therefore I beg you to let me know about any deficiencies you discover.

To put the material in context, this is Section 7.2.1.7 of a long, long chapter on combinatorial algorithms. Chapter 7 will eventually fill three volumes (namely Volumes 4A, 4B, and 4C), assuming that I’m able to remain healthy. It will begin with a short review of graph theory, with emphasis on some highlights of significant graphs in the Stanford GraphBase, from which I will be drawing many examples. Then comes Section 7.1, which deals with the topic of bitwise manipulations. (I drafted about 60 pages about that subject in 1977, but those pages need extensive revision; meanwhile I’ve decided to work for awhile on the material that follows it, so that I can get a better feel for how much to cut.) Section 7.2 is about generating all possibilities, and it begins with Section 7.2.1: Generating Basic Combinatorial Patterns—which, in turn, begins with Section 7.2.1.1, “Generating all n-tuples,” Section 7.2.1.2, “Generating all permutations,” . . . , Section 7.2.1.6, “Generating all trees” (Readers of the present booklet should have already looked at those sections, drafts of which are available as Pre-Fascicles 2A, 2B, 3A, 3B, and 4A.) The stage is now set for the main contents of this booklet, Section 7.2.1.7: “History and further references.” Section 7.2.2 will deal with backtracking in general. And so it will continue, if all goes well; an outline of the entire Chapter 7 as currently envisaged appears on the taocp webpage that is cited on page ii.
Writing about history is extraordinarily difficult, not only because the source materials are widely scattered but also because I must operate at the limit of my ability to understand languages other than English. Furthermore, facts about real life are much more complicated than facts about mathematics. No summary can adequately convey the true feelings of an era or the true spirit of a culture. Yet the story that I'm trying to tell in this section covers many centuries of development in many different parts of the world. The story is fascinating, and many parts of it do not seem to have been told before, at least not in English. Therefore I'm keen to have professional historians of mathematics take a look at what I've been able to piece together, hoping that they will not be too shocked by blunders that have resulted from my present ignorance and/or incompetence. I hope also to get advice from people of many different cultures who know of relevant traditions that have not yet been well studied by professional historians.

The answer to exercise 6 poses two historical problems that I haven’t been able to resolve. I urgently need your help also with respect to some exercises that I made up as I was preparing this material. I certainly don’t like to receive credit for things that have already been published by others, and most of these results are quite natural “fruits” that were just waiting to be “plucked.” Therefore please tell me if you know who deserves to be credited, with respect to the ideas found in exercises 2, 8, 10, 17, 20, 26, and/or 27.

I shall happily pay a finder’s fee of $2.56 for each error in this draft when it is first reported to me, whether that error be typographical, technical, or historical. The same reward holds for items that I forgot to put in the index. And valuable suggestions for improvements to the text are worth 32¢ each. (Furthermore, if you find a better solution to an exercise, I'll actually reward you with immortal glory instead of mere money, by publishing your name in the eventual book:—)

Cross references to yet-unwritten material sometimes appear as ‘00; this impossible value is a placeholder for the actual numbers to be supplied later.

Happy reading!

Stanford, California

D. E. K.

12 October 2004
7.2.1.7. History and further references. Early work on the generation of combinatorial patterns began as civilization itself was taking shape. The story is quite fascinating, and we will see that it spans many cultures in many parts of the world, with ties to poetry, music, and religion. There is space here to discuss only some of the principal highlights; but perhaps a few glimpses into the past will stimulate the reader to dig deeper into the roots of the subject, as the world gets ever smaller and as global scholarship continues to advance.

Lists of binary $n$-tuples can be traced back thousands of years to ancient China, India, and Greece. The most notable source—because it still is a best-selling book in modern translations—is the Chinese $I$ Ching or Yijing, whose name means “the Bible of Changes.” This book, which is one of the five classics of Confucian wisdom, consists essentially of $2^6 = 64$ chapters; and each chapter is symbolized by a hexagram formed from six lines, each of which is either — ("yin") or — ("yang"). For example, hexagram 1 is pure yang, $\openbullet\openbullet\openbullet\openbullet\openbullet\openbullet$; hexagram 2 is pure yin, $\openbullet\openbullet\openbullet\openbullet\openbullet\openbullet$; and hexagram 64 intermixes yin and yang, with yang on top: $\openbullet\openbullet\openbullet\openbullet\openbullet\openbullet$

Here is the complete list:

$$
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\
25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 \\
33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 \\
41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 \\
49 & 50 & 51 & 52 & 53 & 54 & 55 & 56 \\
57 & 58 & 59 & 60 & 61 & 62 & 63 & 64 \\
\end{array}
$$

This arrangement of the 64 possibilities is called King Wen’s ordering, because the basic text of the $I$ Ching has traditionally been ascribed to King Wen (c. 1100 B.C.), the legendary progenitor of the Chou dynasty. Ancient texts are, however, notoriously difficult to date reliably, and modern historians have found no solid evidence that anyone actually compiled such a list of hexagrams before the third century B.C.

Notice that the hexagrams of (1) occur in pairs. Those with odd numbers are immediately followed by their top-to-bottom reflections, except when reflection would make no change; and the eight symmetrical diagrams are paired with their complements ($1 = 16, 27 = 40, 29 = 30, 61 = 62$). Hexagrams that are composed from two trigrams that represent the four basic elements heaven ($\openbullet\openbullet\openbullet$), earth ($\openbullet\openbullet\openbullet$), fire ($\openbullet\openbullet\openbullet$), and water ($\openbullet\openbullet\openbullet$) have also been placed judiciously. Otherwise the arrangement appears to be essentially random, as if a person untrained in mathematics kept listing different possibilities until being unable to come up with any more. A few intriguing patterns do exist between the pairs, but no more than are present by coincidence in the digits of $\pi$ (see 3.3–(1)).
Yin and yang represent complementary aspects of the elementary forces of nature, always in tension, always changing. The *I Ching* is somewhat analogous to a thesaurus in which the hexagrams serve as an index to accumulated wisdom about fundamental concepts like giving (䷊), receiving (䷋), modesty (䷌), joy (䷍), fellowship (䷎), withdrawal (䷏), peace (䷐), conflict (䷑), organization (䷒), corruption (䷓), immaturity (䷔), elegance (䷕), etc. One can choose a pair of hexagrams at random, obtaining the second from the first by, say, independently changing each yin to yang (or vice versa) with probability 1/4; this technique yields 4096 ways to ponder existential mysteries, as well as a Markov process by which change itself might perhaps give meaning to life.

A strictly logical way to arrange the hexagrams was eventually introduced about A.D. 1060 by Shao Yung. His ordering, which proceeded lexicographically from ䷊ to ䷋ to ䷸ to ䷌ to ䷍ to ䷎ to ♨ (reading each hexagram from bottom to top), was much more user-friendly than the King Wen order, because a random pattern could now be found quickly. When G. W. Leibniz learned about this sequence of hexagrams in 1702, he jumped to the erroneous conclusion that Chinese mathematicians had once been familiar with binary arithmetic. [See Frank Swetz, *Mathematics Magazine* 76 (2003), 276–291. Further details about the *I Ching* can be found, for example, in Joseph Needham’s *Science and Civilisation in China* II (Cambridge University Press, 1956), 304–345; R. J. Lynn, *The Classic of Changes* (New York: Columbia University Press, 1994).]

Another ancient Chinese philosopher, Yang Hsiung, proposed a system based on 81 ternary tetragrams instead of 64 binary hexagrams. His *Canon of Supreme Mystery*, written c. 2 B.C., has recently been translated into English by Michael Nylan (Albany, New York: 1993). Yang described a complete, hierarchical ternary tree structure in which there are 3 regions, with 3 provinces in each region, 3 departments in each province, 3 families in each department, and 9 short poems called “appraisals” for each family, hence 729 appraisals in all—making almost exactly 2 appraisals for every day in the year. His tetragrams were arranged in strict lexicographic order when read top-to-bottom: ䷊, ䷋, ䷌, ䷍, ䷎, ䷏, ䷐, ䷑, ♨. In fact, as explained on page 28 of Nylan’s book, Yang presented a simple way to compute the rank of each tetragram, as if using a radix-3 number system. Thus he would not have been surprised or impressed by Shao Yung’s systematic ordering of binary hexagrams, although Shao lived more than 1000 years later.

Indian prosody. Binary *n*-tuples were studied in a completely different context by pundits in ancient India, who investigated the poetic meters of sacred Vedic chants. Syllables in Sanskrit are either short (1) or long (5), and the study of syllable patterns is called “prosody.” Modern writers use the symbols − and — instead of 1 and 5. A typical Vedic verse consists of four lines with *n* syllables per line, for some *n* ≥ 8; prosodists therefore sought a way to classify all 2° possibilities. The classic work *Chandaḥśāstra* by Pingala, written before A.D. 400 and probably much earlier (the exact date is quite uncertain), described procedures by which one could readily find the index *k* of any given pattern of −’s and —’s, as well as to find the *k*th pattern, given *k*. In other words, Pingala explained how to *rank* any given pattern as well as to *unrank* any given index;
thus he went beyond the work of Yang Hsiung, who had considered ranking but not unranking. Piṅgala’s methods were also related to exponentiation, as we have noted earlier in connection with Algorithm 4.6.3A.

The next important step was taken by a prosodist named Kedāra in his work Vṛttaratānākara, thought to have been written in the 8th century. Kedāra gave a step-by-step procedure for listing all the \( n \)-tuples from \( \cdots \ldots \rightarrow \) to \( \cdots \ldots \rightarrow \cdots \rightarrow \cdots \rightarrow \), essentially Algorithm 7.2.1.1M in the case of radix 2. His method may well have been the first-ever explicit algorithm for combinatorial sequence generation. [See B. van Nooten, J. Indian Philos. 21 (1993), 31–50.]

Poetic meters can also be regarded as rhythms, with one beat for each \( - \) and two beats for each \( \cdots \cdots \). An \( n \)-syllable pattern can involve between \( n \) and \( 2n \) beats, but musical rhythms suitable for marching or dancing generally are based on a fixed number of beats. Therefore it was natural to consider the set of all sequences of \( - \)s and \( \cdots \) that have exactly \( m \) beats, for fixed \( m \). Such patterns are now called Morse code sequences of length \( m \), and we know from exercise 4.5.3–32 that there are exactly \( F_{m+1} \) of them. For example, the 21 sequences when \( m = 7 \) are

\[
\begin{align*}
\cdots \cdots, & \quad \cdots \cdots \cdots, \\
\cdots \cdots, & \quad \cdots \cdots \cdots, \\
\cdots \cdots, & \quad \cdots \cdots \cdots, \\
\cdots \cdots, & \quad \cdots \cdots \cdots, \\
\cdots \cdots, & \quad \cdots \cdots \cdots, \\
\cdots \cdots, & \quad \cdots \cdots \cdots, \\
\cdots \cdots, & \quad \cdots \cdots \cdots, \\
\cdots \cdots, & \quad \cdots \cdots \cdots, \\
\cdots \cdots, & \quad \cdots \cdots \cdots, \\
\cdots \cdots, & \quad \cdots \cdots \cdots.
\end{align*}
\]

In this way Indian prosodists were led to discover the Fibonacci sequence, as we have observed in Section 1.2.8.

Moreover, the anonymous author of Prākṛta Paṅgala (c. 1320) discovered elegant algorithms for ranking and unranking with respect to \( m \)-beat rhythms. To find the \( k \)th pattern, one starts by writing down \( m \) \( - \)s, then expresses the difference \( d = F_{m+1} - k \) as a sum of Fibonacci numbers \( F_{j_1} + \cdots + F_{j_r} \); here \( F_{j_i} \) is the largest Fibonacci number that is \( \leq d \) and \( F_{j_i} \) is the largest \( \leq d - F_{j_i} \), etc., continuing until the remainder is zero. Then beats \( j - 1 \) and \( j \) are to be changed from \( \cdots \cdots \) to \( \cdots \). For example, to get the 5th element of (2) we compute \( 21 - 5 = 16 = 13 + 3 = F_7 + F_4 \); the answer is \( \cdots \cdots \cdots \). A few years later, Nārāyana Pāṇḍita treated the more general problem of finding all compositions of \( m \) whose parts are \( \leq q \), where \( q \) is any given positive integer. As a consequence he discovered the \( q \)th-order Fibonacci sequence 5.4.2–(4), which was destined to be used 600 years later in polyphase sorting; he also developed the corresponding ranking and unranking algorithms. [See Parmarand Singh, Historia Mathematica 12 (1985), 229–244, and exercise 16.]

Piṅgala gave special code-names to all the three-syllable meters,

\[
\begin{align*}
\cdots \cdots & = \mathfrak{m} \text{ (m)}, & \cdots \cdots & = \mathfrak{t} \text{ (t)}, \\
\cdots \cdots & = \mathfrak{y} \text{ (y)}, & \cdots \cdots & = \mathfrak{j} \text{ (j)}, \\
\cdots \cdots & = \mathfrak{r} \text{ (r)}, & \cdots \cdots & = \mathfrak{bh} \text{ (bh)}, \\
\cdots \cdots & = \mathfrak{s} \text{ (s)}, & \cdots \cdots & = \mathfrak{n} \text{ (n)},
\end{align*}
\]
and students of Sanskrit have been expected to memorize them ever since.
Somebody long ago devised a clever way to recall these codes, by inventing
the nonsense word *yamāṭāvṛājabhānasalagām* (यमाताव्राजभानसलगाम); the point
is that the ten syllables of this word can be written

\[
\text{ya mā tā rā ja bhā na sa la gām}
\]

and each three-syllable pattern occurs just after its code name. The origin of
yamā...lagām is obscure, but Subhash Kak [Indian J. History of Science 35
(2000), 123–127] has traced it back at least to C. P. Brown’s Sanskrit Prosody
(1869), page 28; thus it qualifies as the earliest known appearance of a “de Bruijn
cycle” that encodes binary n-tuples.

**Meanwhile, in Europe.** In a similar way, classic Greek poetry was based on
groups of short and/or long syllables called “metrical feet,” analogous to bars of
music. Each basic type of foot acquired a Greek name; for example, two short
syllables ‘– ‘-’ were called a pyrrhic, and two long syllables ‘— ‘-’ were called a
spondee, because those rhythms were used respectively in a song of war (παριξη)
or a song of peace (σταυροῖς). Greek names for metric feet were soon assimilated
into Latin and eventually into modern languages, including English:

\[
\begin{align*}
\text{— arsis} & \quad \text{procelesmatic} \\
\text{— thesis} & \quad \text{fourth peon} \\
\text{— pyrrhic} & \quad \text{third peon} \\
\text{— iambus} & \quad \text{minor ionic} \\
\text{— trochee} & \quad \text{second peon} \\
\text{— spondee} & \quad \text{diëambus} \\
\text{— tribrach} & \quad \text{antispast} \\
\text{— anapest} & \quad \text{first peon} \\
\text{— amphibrach} & \quad \text{choriambus} \\
\text{— bacchius} & \quad \text{ditrochee} \\
\text{— dactyl} & \quad \text{second epitrite} \\
\text{— amphimacer} & \quad \text{major ionic} \\
\text{— palimbacchius} & \quad \text{third epitrite} \\
\text{— molossus} & \quad \text{fourth epitrite} \\
\text{— } & \quad \text{disponge}
\end{align*}
\]

(5)

Alternative names, like “choree” instead of “trochee,” or “cretic” instead of
“amphimacer,” were also in common use. Moreover, by the time Diomedes wrote
his Latin grammar (approximately A.D. 375), each of the 32 five-syllable feet
had acquired at least one name. Diomedes also pointed out the relation between
complementary patterns; he stated for example that tribrach and molossus are
“contrarius,” as are amphibrach and amphimacer. But he also regarded dactyl
as the contrary of anapest, and bacchius as the contrary of palimbacchius, al-
though the literal meaning of *palimbacchius* is actually “reverse bacchius.” Greek
prosodists had no standard order in which to list the individual possibilities, and
the form of the names makes it clear that no connection to a radix-two number system was contemplated. [See H. Keil, Grammatici Latini 1 (1857), 474–482; W. von Christ, Metrik der Griechen und Römer (1879), 78–79.]

Surviving fragments of a work by Aristozenus called Elements of Rhythm (c. 325 B.C.) show that the same terminology was applied also to music. And indeed, the same traditions lived on after the Renaissance; for example, we find on page 32 of Athanassius Kircher’s Musurgia Universalis 2 (Rome: 1650), and Kircher went on to describe all of the three-note and four-note rhythms of (5).

**Early lists of permutations.** We’ve traced the history of formulas for counting permutations in Section 5.1.2; but nontrivial lists of permutations were not published until hundreds of years after the formula $n!$ was discovered. The first such tabulation currently known was compiled by the Italian physician Shabbetai Donnolo in his commentary on the kabbalistic Sefer Yetzirah, written in A.D. 946. Table 1 shows his list for $n = 5$ as it was subsequently printed in Warsaw (1884). (The Hebrew letters in this table are typeset in a rabbinical font traditionally used for commentaries; notice that the letter מ changes its shape to ב when it appears at the left end of a word.) Donnolo went on to list 120 permutations of the six-letter word בֶּלֶתִּין all beginning with shin (ם); then he noted that 120 more could be obtained with each of the other five letters in front, making 720 in all. His lists involved groupings of six permutations, but in a haphazard fashion that led him into error (see exercise 4). Although he knew how many permutations there were supposed to be, and how many should start with a given letter, he evidently didn’t have an algorithm for generating them.

**Table 1**

<table>
<thead>
<tr>
<th>A MEDIEVAL LIST OF PERmutATIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>בֶּלֶתִּין</td>
</tr>
<tr>
<td>בֶּלֶתִּין</td>
</tr>
<tr>
<td>בֶּלֶתִּין</td>
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<td>בֶּלֶתִּין</td>
</tr>
<tr>
<td>בֶּלֶתִּין</td>
</tr>
<tr>
<td>בֶּלֶתִּין</td>
</tr>
</tbody>
</table>

A complete list of all 720 permutations of {a, b, c, d, e, f} appeared on pages 668–671 of Jeremiah Drexel’s Orbis Phaëthon (Munich: 1629; also on pages 526–531 of the Cologne edition in 1631). He offered it as proof that a man with six guests could seat them differently at lunch and dinner every day for a year——
altogether 360 days, because there were five days of fasting during Holy Week. Shortly afterwards, Marin Mersenne exhibited all 720 permutations of the six tones \{ut, re, mi, fa, sol, la\}, on pages 111–115 of his *Traitez de la Voix et des Chants* (Volume 2 of *Harmonie Universelle*, 1636); then on pages 117–128 he presented the same data in musical notation:

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & \ldots \\
\hline
1 & 2 & 3 & 4 & 5 & \\
\end{array}
\]

Drexel's table was organized lexicographically by columns; Mersenne's tables were lexicographic with respect to the order \(ut < re < mi < fa < sol < la\), beginning with "ut, re, mi, fa, sol, la" and ending with "la, sol, fa, mi, re, ut." Mersenne also prepared a "grand et immense" manuscript that listed all 40,320 permutations of eight notes on 672 folio pages, followed by ranking and unranking algorithms [Bibliothèque nationale de France, Fonds Français, no. 24256].

We saw in Section 7.2.1.2 that the important idea of plain changes, Algorithm 7.2.1.2P, was invented in England a few years later.

Methods for listing all permutations of a multiset with repeated elements were often misunderstood by early authors. For example, when Bhāskara exhibited the permutations of \{4, 5, 5, 8\} in section 271 of his *Līlāvatī* (c. 1150), he gave them in the following order:

\[
\begin{array}{cccccc}
\text{YYXX} & \text{XXYY} & \text{XYXY} & \text{YYXY} & \text{YYXX} \\
\text{YYXX} & \text{XXYY} & \text{XYXY} & \text{YYXY} & \text{YYXX} \\
\text{YYXX} & \text{XXYY} & \text{XYXY} & \text{YYXY} & \text{YYXX} \\
\end{array}
\]

Mersenne used a slightly more sensible but not completely systematic order on page 131 of his book when he listed sixty anagrams of the Latin name \(\text{I}\text{E}\text{S}\text{V}\text{S}\). When Athanasius Kircher wanted to illustrate the 30 permutations of a five-note melody on pages 10 and 11 of *Musurgia Universalis* 2 (1650), this lack of a system got him into trouble (see exercise 5):

\[
\begin{array}{cccccc}
\text{YYXX} & \text{XXYY} & \text{XYXY} & \text{YYXY} & \text{YYXX} \\
\text{YYXX} & \text{XXYY} & \text{XYXY} & \text{YYXY} & \text{YYXX} \\
\text{YYXX} & \text{XXYY} & \text{XYXY} & \text{YYXY} & \text{YYXX} \\
\end{array}
\]

But John Wallis knew better. On page 117 of his *Discourse of Combinations* (1685) he correctly listed the 60 anagrams of "messes" in lexicographic order, if we let \(m < e < s\); and on page 126 he recommended respecting alphabetic order "that we may be the more sure, not to miss any."

We will see later that the Indian mathematician Nārīyaṇa Pāṇḍita had already developed a theory of permutation generation in the 14th century, although his work remained almost totally unknown.
Seki’s list. Takakazu Seki (1642–1708) was a charismatic teacher and researcher who revolutionized the study of mathematics in 17th-century Japan. While he was studying the elimination of variables from simultaneous homogeneous equations, he was led to expressions such as $a_1b_2 - a_2b_1$ and $a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1$, which we now recognize as determinants. In 1683 he published a booklet about this discovery, introducing an ingenious scheme for listing all permutations in such a way that half of them were “alive” (even) and the other half were “dead” (odd). Starting with the case $n = 2$, when ‘12’ was alive and ‘21’ was dead, he formulated the following rules for $n > 2$:

1) Take every live permutation for $n-1$, increase all its elements by 1, and insert 1 in front. This rule produces $(n-1)!/2$ “basic permutations” of $\{1, \ldots, n\}$.

2) From each basic permutation, form $2n$ others by rotation and reflection:

$$a_1a_2\ldots a_{n-1}a_n, \quad a_2 \ldots a_{n-1}a_na_1, \quad \ldots, \quad a_na_{n-1}a_2\ldots a_{n-1};$$

$$a_na_{n-1}\ldots a_2a_1, \quad a_1a_{n-1}a_2\ldots a_{n-1}a_n.$$  

If $n$ is odd, those in the first row are alive and those in the second are dead; if $n$ is even, those in each row are alternatively alive, dead, alive, dead. For example, when $n = 3$ the only basic permutation is 123. Thus 123, 231, 312 are alive while 321, 132, 213 are dead, and we’ve successfully generated the six terms of a $3 \times 3$ determinant. The basic permutations for $n = 4$ are 1234, 1342, 1423; and from, say, 1342 we get a set of eight, namely

$$+1342 - 3421 + 4213 - 2134 + 2431 - 1243 + 3124 - 4321,$$  

alternately alive (+) and dead (−). A $4 \times 4$ determinant therefore includes the terms $a_1b_3c_4d_2 - a_3b_4c_2d_1 + \cdots - a_4b_3c_2d_1$ and sixteen others.

Seki’s rule for permutation generation is quite pretty, but unfortunately it has a serious problem: It doesn’t work when $n > 4$. His error seems to have gone unrecognized for hundreds of years. [See Y. Mikami, The Development of Mathematics in China and Japan (1913), 191–199: Takakazu Seki’s Collected Works (Osaka: 1974), 18–20, and exercises 7–8.]

Lists of combinations. The earliest exhaustive list of combinations known to have survived the ravages of time appears in the last book of Śūrūta’s well-known Sanskrit treatise on medicine, Chapter 63, written before A.D. 600 and perhaps much earlier. Noting that medicine can be sweet, sour, salty, peppery, bitter, and/or astrigent, Śūrūta’s book diligently listed the $(15, 20, 15, 6, 1, 6)$ cases that arise when those qualities occur two, three, four, five, six, and one at a time.

Bhāskara repeated this example in sections 110–114 of Lilāvatī, and observed that the same reasoning applies to six-syllable poetic meters with a given number of long syllables. But he simply mentioned the totals, $(6, 15, 20, 15, 6, 1)$, without listing the combinations themselves. In sections 274 and 275, he observed that the numbers $(n)(n-1)\ldots (n-k+1)/(k(k-1)\ldots (1))$ enumerate compositions (that is, ordered partitions) as well as combinations; again he gave no list.

To avoid proximity this is treated in a brief manner; for the science of calculation is an ocean without bounds.

— Bhāskara (c. 1150)
7.2.1.7  HISTORY AND FURTHER REFERENCES

An isolated but interesting list of combinations appeared in the remarkable algebra text  Al-Bāḥīr fī l-ḥisāb (The Shining Book of Calculation), written by al-Samaw‘al of Baghdad when he was only 19 years old (1144). In the closing part of that work he presented a list of \( \binom{10}{6} = 210 \) simultaneous linear equations in 10 unknowns:

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>०५४५२४</th>
<th>( \varphi )</th>
<th>( \omega )</th>
<th>०५४५२४</th>
<th>( \varphi )</th>
<th>( \omega )</th>
<th>( \psi )</th>
<th>( \chi )</th>
<th>( \psi )</th>
<th>( \chi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega )</td>
<td>( x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 65 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \varphi )</td>
<td>( x_1 + x_2 + x_3 + x_4 + x_5 + x_7 = 70 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \omega )</td>
<td>( x_1 + x_2 + x_3 + x_4 + x_5 + x_8 = 75 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \psi )</td>
<td>( \ldots )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \chi )</td>
<td>( (209) x_4 + x_6 + x_7 + x_8 + x_9 + x_{10} = 91 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \psi )</td>
<td>( (210) x_3 + x_6 + x_7 + x_8 + x_9 + x_{10} = 100 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Each combination of ten things taken six at a time yielded one of his equations. His purpose was evidently to demonstrate that over-determined equations can still have a unique solution—which in this case was \( (x_1, x_2, \ldots, x_{10}) = (1, 4, 9, 16, 25, 10, 15, 20, 25, 5) \). [Salah Ahmad and Rosdhi Rashed,  Al-Bāḥīr en Algèbre d’Al-Samaw‘al (Damascus: 1972), 77–82, الياف ملاكیه]

Rolling dice. Some glimmerings of elementary combinatorics arose also in medieval Europe, especially in connection with the question of listing all possible outcomes when three dice are thrown. There are, of course, \( \binom{6}{3} = 20 \) ways to choose 3 things from 6 when repetitions are allowed. Gambling was officially prohibited; yet these 20 ways became rather well known. In about A.D. 965, Bishop Wibold of Cambrai in northern France devised a game called Ludus Clericales, so that members of the clergy could enjoy rolling dice while remaining pious. His idea was to associate each possible roll with one of 56 virtues, according to the following table:

<table>
<thead>
<tr>
<th>( \square )</th>
<th>love</th>
<th>( \square )</th>
<th>perseverance</th>
<th>( \square )</th>
<th>hospitality</th>
<th>( \square )</th>
<th>mortification</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \square )</td>
<td>faith</td>
<td>( \square )</td>
<td>kindness</td>
<td>( \square )</td>
<td>economy</td>
<td>( \square )</td>
<td>innocence</td>
</tr>
<tr>
<td>( \square )</td>
<td>hope</td>
<td>( \square )</td>
<td>modesty</td>
<td>( \square )</td>
<td>patience</td>
<td>( \square )</td>
<td>contrition</td>
</tr>
<tr>
<td>( \square )</td>
<td>justice</td>
<td>( \square )</td>
<td>resignation</td>
<td>( \square )</td>
<td>zeal</td>
<td>( \square )</td>
<td>confession</td>
</tr>
<tr>
<td>( \square )</td>
<td>prudence</td>
<td>( \square )</td>
<td>gentleness</td>
<td>( \square )</td>
<td>poverty</td>
<td>( \square )</td>
<td>maturity</td>
</tr>
<tr>
<td>( \square )</td>
<td>temperance</td>
<td>( \square )</td>
<td>generosity</td>
<td>( \square )</td>
<td>softness</td>
<td>( \square )</td>
<td>solicitude</td>
</tr>
<tr>
<td>( \square )</td>
<td>courage</td>
<td>( \square )</td>
<td>wisdom</td>
<td>( \square )</td>
<td>virginity</td>
<td>( \square )</td>
<td>constancy</td>
</tr>
<tr>
<td>( \square )</td>
<td>peace</td>
<td>( \square )</td>
<td>remorse</td>
<td>( \square )</td>
<td>respect</td>
<td>( \square )</td>
<td>intelligence</td>
</tr>
<tr>
<td>( \square )</td>
<td>chastity</td>
<td>( \square )</td>
<td>joy</td>
<td>( \square )</td>
<td>piety</td>
<td>( \square )</td>
<td>sighing</td>
</tr>
<tr>
<td>( \square )</td>
<td>mercy</td>
<td>( \square )</td>
<td>sobriety</td>
<td>( \square )</td>
<td>indulgence</td>
<td>( \square )</td>
<td>weeping</td>
</tr>
<tr>
<td>( \square )</td>
<td>obedience</td>
<td>( \square )</td>
<td>satisfaction</td>
<td>( \square )</td>
<td>prayer</td>
<td>( \square )</td>
<td>cheerfulness</td>
</tr>
<tr>
<td>( \square )</td>
<td>fear</td>
<td>( \square )</td>
<td>sweetness</td>
<td>( \square )</td>
<td>affection</td>
<td>( \square )</td>
<td>compassion</td>
</tr>
<tr>
<td>( \square )</td>
<td>foresight</td>
<td>( \square )</td>
<td>cleverness</td>
<td>( \square )</td>
<td>judgment</td>
<td>( \square )</td>
<td>self-control</td>
</tr>
<tr>
<td>( \square )</td>
<td>discretion</td>
<td>( \square )</td>
<td>simplicity</td>
<td>( \square )</td>
<td>vigilance</td>
<td>( \square )</td>
<td>humility</td>
</tr>
</tbody>
</table>

Players took turns, and the first to roll each virtue acquired it. After all possibilities had arisen, the most virtuous player won. Wibold noted that love (caritas) is the best virtue of all. He gave a complicated scoring system by which two virtues could be combined if the sum of pips on all six of their dice was 21; for
example, love + humility or chastity + intelligence could be paired in this way, and such combinations ranked above any individual virtue. He also considered more complex variants of the game in which vowels appeared on the dice instead of spots, so that virtues could be claimed if their vowels were thrown.

Wibold’s table of virtues was presented in lexicographic order, as above, when it was first described by Baldéric in his Chronicon Cameracense, about 150 years later. [Patrologia Latina 134 (Paris: 1884), 1007–1016.] But another medieval manuscript presented the possible dice rolls in quite a different order:

\[
\begin{array}{cccccc}
111 & 111 & 000 & 111 & 000 & 111 \\
100 & 111 & 100 & 100 & 100 & 111 \\
001 & 001 & 101 & 010 & 110 & 101 \\
101 & 101 & 010 & 010 & 010 & 010 \\
011 & 101 & 101 & 010 & 101 & 101 \\
010 & 010 & 101 & 101 & 101 & 101 \\
110 & 110 & 110 & 110 & 110 & 110 \\
100 & 100 & 100 & 100 & 100 & 100 \\
000 & 000 & 000 & 000 & 000 & 000 \\
111 & 111 & 111 & 111 & 111 & 111
\end{array}
\]

In this case the author knew how to deal with repeated values, but had a very complicated, ad hoc way to handle the cases in which all dice were different. [See D. R. Bellhouse, International Statistical Review 68 (2000), 123–136.]

An amusing poem entitled “Chaunce of the Dyse,” attributed to John Lydgate, was written in the early 1400s for use at parties. Its opening verses invite each person to throw three dice; then the remaining verses, which are indexed in decreasing lexicographic order from 111 to 000 to … to 111, give 56 character sketches that light-heartedly describe the thrower. [The full text was published by E. P. Hammond in Englishe Studien 59 (1925), 1–16; a translation into modern English would be desirable.]

\[
\begin{align*}
\text{I pray to god that every wight may caste} \\
\text{Vpon three dye rght as is in hys herte} \\
\text{Whether he be rechelesse or stedfaste} \\
\text{So moote he lawdhen other ells smerte} \\
\text{He that is gilty his lyfe to converte} \\
\text{They that in trouthe have suffred many a throwe} \\
\text{Moot ther chaunce fal as they moote be knowe.}
\end{align*}
\]

— The Chaunce of the Dyse (c. 1410)

**Ramon Llull.** Significant ripples of combinatorial concepts also emanated from an energetic and quixotic Catalan poet, novelist, encyclopedist, educator, mystic, and missionary named Ramon Llull (c. 1232–1316). Llull’s approach to knowledge was essentially to identify basic principles and then to contemplate combining them in all possible ways.

For example, one chapter in his Ars Compendiosa Inveniendi Veritatem (c. 1274) began by enumerating sixteen attributes of God: Goodness, greatness, eternity, power, wisdom, love, virtue, truth, glory, perfection, justice, generosity, mercy, humility, sovereignty, and patience. Then Llull wrote \( \binom{16}{2} = 120 \) short essays of about 80 words each, considering God’s goodness as related to greatness,
God’s goodness as related to eternity, and so on, ending with God’s sovereignty as related to patience. In another chapter he considered seven virtues (faith, hope, charity, justice, prudence, fortitude, temperance) and seven vices (gluttony, lust, greed, sloth, pride, envy, anger), with \( \binom{14}{2} = 91 \) subchapters to deal with each pair in turn. Other chapters were systematically divided in a similar way, into \( \binom{8}{2} = 28, \binom{15}{2} = 105, \binom{4}{1} = 6 \), and \( \binom{56}{2} = 120 \) subsections. (One wonders what might have happened if he had been familiar with Wibold’s list of 56 virtues; would he have produced commentaries on all \( \binom{56}{2} = 1540 \) of their pairs?)

![Diagram](image)

**Fig. 44.** Illustrations in a manuscript presented by Ramon Lull to the doge of Venice in 1280. [From his *Ars Demonstrativa*. Biblioteca Marciana. vi 200. folio 3°.]

Lull illustrated his methodology by drawing circular diagrams like those in Figure 44. The left-hand circle in this illustration, *Deus*, names sixteen divine attributes—essentially the same sixteen listed earlier, except that love (*amor*) was now called will (*voluntas*), and the final four were now respectively simplicity, rank, mercy, and sovereignty. Each attribute is assigned a code letter, and the illustration depicts their interrelations as the complete graph \( K_{16} \) on vertices (B, C, D, E, F, G, H, I, K, L, M, N, O, P, Q, R). The right-hand figure, *virtutes et vitia*, shows the seven virtues (*b, c, d, e, f, g, h*) interleaved with the seven vices (*i, k, l, m, n, o, p*); in the original manuscript virtues appeared in blue ink while vices appeared in red. Notice that in this case his illustration depicted two independent complete graphs \( K_7 \), one of each color. (He no longer bothered to compare each individual virtue with each individual vice, since every virtue was clearly better than every vice.)

Lull used the same approach to write about medicine: Instead of juxtaposing theological concepts, his *Liber Principiorum Medicinae* (c. 1275) considered combinations of symptoms and treatments. And he also wrote books
on philosophy, logic, jurisprudence, astrology, zoology, geometry, rhetoric, and chivalry—more than 200 works in all. It must be admitted, however, that much of this material was highly repetitive; modern data compression techniques would probably reduce Llull’s output to a size much less than that of, say, Aristotle.

He eventually decided to simplify his system by working primarily with groups of nine things. See, for example, Fig. 45, where circle A now lists only the first nine of God’s attributes (B, C, D, E, F, G, H, I, K). The \( \binom{9}{3} = 36 \) associated pairs (BC, BD, . . . , IK) appear in the stair-step chart at the right of that circle. By adding two more virtues, namely patience and compassion—as well as two more vices, namely lying and inconsistency—he could treat virtues vis-à-vis virtues and vices vis-à-vis vices with the same chart. He also proposed using the same chart to carry out an interesting scheme for voting, in an election with nine candidates [see I. McLean and J. London, Studia Lulliana 32 (1992), 21–37].

The encircled triangles at the lower left of Fig. 45 illustrate another key aspect of Llull’s approach. Triangle (B, C, D) stands for (difference, concordance, contrariety); triangle (E, F, G) stands for (beginning, middle, ending); and triangle (H, I, K) stands for (greater, equal, less). These three interleaved appearances of \( K_3 \) represent three kinds of three-valued logic. Llull had experimented earlier with other such triplets, notably ‘(true, unknown, false)’. We can get an idea

\[ \text{Fig. 45. Llullian illustrations from a manuscript presented to the queen of France, c. 1325.} \]

[Badische Landesbibliothek Karlsruhe. Codex St. Peter perg. 92, folios 28º and 39º.]
of how he used the triangles by considering how he dealt with combinations of the four basic elements (earth, air, fire, water): All four elements are different; earth is concordant with fire, which concords with air, which concords with water, which concords with earth; earth is contrary to air, and fire is contrary to water; these considerations complete an analysis with respect to triangle (B, C, D). Turning to triangle (E, F, G), he noted that various processes in nature begin with one element dominating another; then a transition or middle state occurs, until a goal is reached, like air becoming warm. For triangle (H, I, K) he said that in general we have fire > air > water > earth with respect to their “spheres,” their “velocities,” and their “nobilites”; nevertheless we also have, for example, air > fire with respect to supporting life, while air and fire have equal value when they are working together.

Lhull provided the vertical table at the right of Fig. 45 as a further aid. (See exercise 11 below.) He also introduced movable concentric wheels, labeled with the letters (B, C, D, E, F, G, H, I, K) and with other names, so that many things could be contemplated simultaneously. In this way a faithful practitioner of the Lhullian art could be sure to have all the bases covered. [Lhull may have seen similar wheels that were used in nearby Jewish communities; see M. Idel, J. Warburg and Courtauld Institutes 51 (1988), 170–174 and plates 16–17.]

Several centuries later, Athanasius Kircher published an extension of Lhull’s system as part of a large tome entitled Ars Magna Scien
di sive Combinatoria (Amsterdam: 1669), with five movable wheels accompanying page 173 of that book. Kircher also extended Lhull’s repertoire of complete graphs $K_n$ by providing illustrations of complete bipartite graphs $K_{m,n}$; for example, Fig. 46 is taken from page 171 of Kircher’s book, and his page 170 contains a glorious picture of $K_{18,18}$.

![Figure 46](image)

*Fig. 46. $K_{9,9}$ as presented by Athanasius Kircher in 1669.*

*It is an investigative and inventive art. When ideas are combined in all possible ways, the new combinations start the mind thinking along novel channels and one is led to discover fresh truths and arguments.*

— MARTIN GARDNER, Logic Machines and Diagrams (1958)

The most extensive modern development of Lhull-like methods is perhaps *The Schillinger System of Musical Composition* by Joseph Schillinger (New York:
Carl Fischer, 1946), a remarkable two-volume work that presents theories of rhythm, melody, harmony, counterpoint, composition, orchestration, etc., from a combinatorial perspective. On page 56, for example, Schillinger lists the 24 permutations of \( \{a, b, c, d\} \) in the Gray-code order of plain changes (Algorithm 7.2.1.2P); then on page 57 he applies them not to pitches but rather to rhythms, to the durations of notes. On page 364 he exhibits the symmetrical cycle

\[
(2, 0, 3, 4, 2, 5, 6, 4, 0, 1, 6, 2, 3, 1, 4, 5, 3, 6, 0, 5, 1),
\]

a universal cycle of 2-combinations for the seven objects \( \{0, 1, 2, 3, 4, 5, 6\} \); in other words, (13) is an Eulerian trail in \( K_7 \): All \( \binom{7}{2} = 21 \) pairs of digits occur exactly once. Such patterns are grist for a composer’s mill. But we can be grateful that Schillinger’s better students (like George Gershwin) did not commit themselves entirely to a strictly mathematical sense of aesthetics.

Tacquet, van Schooten, and Izquierdo. Three additional books related to our story were published during the 1650s. André Tacquet wrote a popular text, *Arithmeticae Theoria et Praxis* (Louvain: 1656), that was reprinted and revised often during the next fifty years. Near the end, on pages 376 and 377, he gave a procedure for listing combinations two at a time, then three at a time, etc.

Frans van Schooten’s *Exercitationes Mathematicae* (Leiden: 1657) was more advanced. On page 373 he listed all combinations in an appealing layout

\[
\begin{array}{cccc}
\frac{a}{b} & \frac{ab}{c} \\
& & ac & be & abc & abed \\
\end{array}
\]

and he proceeded on the next few pages to extend this pattern to the letters \( e, f, g, h, i, k \), “et sic in infinitum.” On page 376 he observed that one can replace \( (a, b, c, d) \) by \( (2, 3, 5, 7) \) in (14) to get the divisors of 210 that exceed unity:

\[
\begin{array}{cccc}
\frac{2}{3} & \frac{3}{6} \\
& & 5 & 10 & 15 & 30 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
& & 7 & 14 & 21 & 42 & 35 & 70 & 105 & 210 \\
\end{array}
\]

And on the following page he extended the idea to

\[
\begin{array}{cccc}
\frac{a}{a} & \frac{aa}{b} & \frac{aab}{ab} \\
& & ac & aac & bc & abc & aabc \\
\end{array}
\]

thereby allowing two \( a \)'s. He didn’t really understand this extension, though; his next example

\[
\begin{array}{cccc}
\frac{a}{a} & \frac{aa}{a} & \frac{aaa}{ab} & \frac{aab}{bb} & \frac{aab}{abb} & \frac{aaab}{aabb} \\
& & b & ab & aab & aaab \\
\end{array}
\]

was botched, indicating the limits of his knowledge at the time. (See exercise 13.)
7.2.1.7

HISTORY AND FURTHER REFERENCES

On page 411 van Schooten observed that the weights \((a, b, c, d) = (1, 2, 4, 8)\)
could be assigned in (14), leading to

\[
\begin{array}{cccc}
1 & 2 & 3 \\
4 & 5 & 6 & 7 \\
8 & 9 & 10 & 11 & 12 & 13 & 14 & 15
\end{array}
\]

(18)
after addition. But he didn’t see the connection with radix-2 arithmetic.

Sebastián Izquierdo’s two-volume work *Pharus Scientiarum* (Lyon: 1659),
“The Lighthouse of Science,” included a nicely organized discussion of combinatorics
entitled Disputatio 29, *De Combinatione*. He gave a detailed discussion of
four key parts of Stanley’s Twelvefold Way, namely the \(n\)-tuples, \(n\)-variations,
\(n\)-multicombinations, and \(n\)-combinations of \(m\) objects that appear in the first
two rows and the first two columns of Table 7.2.1.4–1.

In Sections 81–84 of *De Combinatione* he listed all combinations of \(m\) letters
taken \(n\) at a time, for \(2 \leq n \leq 5\) and \(n \leq m \leq 9\), always in lexicographic order;
he also tabulated them for \(m = 10\) and 20 in the cases \(n = 2\) and 3. But when
he listed the \(m^n\) variations of \(m\) things taken \(n\) at a time, he chose a more
complicated ordering (see exercise 14).

Izquierdo was first to discover the formula \(\binom{m+n-1}{n}\) for combinations of \(m\) things
taken \(n\) at a time with unlimited repetition; this rule appeared in §48–51
of his work. But in §105, when he attempted to list all such combinations in the
case \(n = 3\), he didn’t know that there was a simple way to do it. In fact, his
listing of the 56 cases for \(m = 6\) was rather like the old, awkward ordering of (12).

Combinations with repetition were not well understood until James Bernoulli’s
*Ars Conjectandi*, “The Art of Guessing,” came out in 1713. In Part 2,
Chapter 5, Bernoulli simply listed the possibilities in lexicographic order, and
showed that the formula \(\binom{m+n-1}{n}\) follows by induction as an easy consequence.

[ Niccolò Tartaglia had, incidentally, come close to discovering this formula in his
*General trattato di numeri et misure* 2 (Venice: 1556), 177 and 697; so had the
Maghrebi mathematician Ibn Mun‘im in his 13th-century *Fiqh al-Hasb*.]

The null case. Before we conclude our discussion of early work on combinations,
we should not forget a small yet noble step taken by John Wallis on page 110
of his *Discourse of Combinations* (1655), where he specifically considered the
combination of \(m\) things taken \(0\) at a time: “It is manifest. That, if we would
take None, that is, if we would leave All; there can be but one case thereof, what
ever be the Number of things exposed.” Furthermore, on page 113, he knew that
\(\binom{0}{0} = 1\); “(for, here, to take all, or to leave all, is but one and the same case.)”

However, when he gave a table of \(n!\) for \(n \leq 24\), he did not go so far as to
point out that \(0! = 1\), or that there is exactly one permutation of the empty set.

The work of Nārāyana. A remarkable monograph entitled *Gaṇita Kaumudi*
(“Treatise on Calculation”), written by Nārāyana Paṇḍita in 1356, has recently
become known in detail to scholars outside of India for the first time, thanks
to an English translation by Parmanand Singh [ *Gaṇita Bhāratī* 20 (1998), 25–82; 21
Chapter 13 of his work, subtitled *Āṭika Pāṇa* ("Concatenation of Numbers"), was devoted to combinatorial generation. Indeed, although the 97 "suktas" of this chapter were rather cryptic, they presented a comprehensive theory of the subject that anticipated developments in the rest of the world by several hundred years.

For example, Nārāyaṇa dealt with permutation generation in suktas 49–55, where he gave algorithms to list all permutations of a set in decreasing colex order, together with algorithms to rank a given permutation and to unrank a given serial number. In this way he essentially discovered the factorial representation of positive integers. Then in suktas 57–60 he extended the algorithms to handle general multisets; for example, he listed the permutations of \{1, 1, 2, 4\} as

\[
1124, 1214, 2114, 1142, 1412, 4112, 1241, 2141, 1421, 4121, 2411, 4211,
\]

again in decreasing colex order.

Nārāyaṇa's suktas 88–92 dealt with systematic generation of combinations. Besides illustrating the combinations of \{1, \ldots, 8\} taken 3 at a time, namely

\[
(678, 578, 478, \ldots, 134, 124, 123),
\]

he also considered a bit-string representation of these combinations in the reverse order (increasing colex):

\[
(11100000, 11010000, 10110000, \ldots, 00010011, 00001011, 00000111).
\]

He almost, but not quite, discovered Theorem 7.2.1.3L.

Thus we can legitimately regard Nārāyaṇa Pandita as the founder of the science of combinatorial generation — even though, like many other pioneers who were significantly “ahead of their time,” his work on the subject never became well known even in his own country.

**Permutable poetry.** Let’s turn now to a curious question that attracted the attention of several prominent mathematicians in the seventeenth century, because it sheds considerable light on the state of combinatorial knowledge in Europe at that time. A Jesuit priest named Bernard Baulhuis had composed a famous one-line tribute to the Virgin Mary, in Latin hexameter:

\[
\text{Tot tibi sunt dotes, Virgo, quot sidera caelo.} \quad (19)
\]

[“Thou hast as many virtues. O Virgin, as there are stars in heaven”; see his *Epigrammatum Libri V* (Cologne: 1615), 49.] His verse inspired Erycius Puteanus, a professor at the University of Louvain, to write a book entitled *Pietatis Thaumata* (Antwerp: 1617), presenting 1022 permutations of Baulhuis’s words. For example, Puteanus wrote

\[
\begin{align*}
107 & \quad \text{Tot dotes tibi, quot caelo sunt sidera, Virgo.} \\
270 & \quad \text{Dotes tot, caelo sunt sidera quot tibi Virgo.} \\
329 & \quad \text{Dotes, caelo sunt quot sidera, Virgo tibi tot.} \\
384 & \quad \text{Sidera quot caelo, tot sunt Virgo tibi dotes.} \\
725 & \quad \text{Quot caelo sunt sidera, tot Virgo tibi dotes.} \\
949 & \quad \text{Sunt dotes Virgo, quot sidera, tot tibi caelo.} \\
1022 & \quad \text{Sunt caelo tot Virgo tibi, quot sidera, dotes.}
\end{align*}
\]
He stopped at 1022, because 1022 was the number of visible stars in Ptolemy’s well-known catalog of the heavens.

The idea of permuting words in this way was well known at the time; such wordplay was what Julius Scaliger had called “Proteus verses” in his Poetices Libri Septem (Lyon: 1561), Book 2, Chapter 30. The Latin language lends itself to permutations like (20), because Latin word endings tend to define the function of each noun, making the relative word order much less important to the meaning of a sentence than it is in English. Puteanus did state, however, that he had specifically avoided unsuitable permutations such as

Sidera tot cælo, Virgo, quot sunt tibi dotes. \hfill (21)

because they would place an upper bound on the Virgin’s virtues rather than a lower bound. [See pages 12 and 103 of his book.]

Of course there are 8! = 40,320 ways to permute the words of (19). But that wasn’t the point; most of those ways don’t “scan.” Each of Puteanus’s 1022 verses obeyed the strict rules of classical hexameter, the rules that had been followed by Greek and Latin poets since the days of Homer and Vergil, namely:

i) Each word consists of syllables that are either long (―) or short (−);

ii) The syllables of each line belong to one of 32 patterns,

\[
\begin{align*}
\{ &- &- &- &- &- \\
&- &- &- &- &- \\
&- &- &- &- &- \\
&- &- &- &- &- \\
&- &- &- &- &- \\
&- &- &- &- &- \\
&- &- &- &- &- \\
&- &- &- &- &- \\
\end{align*}
\] \hfill (22)

In other words there are six metrical feet, where each of the first four is either a dactyl or a spondee in the terminology of (5); the fifth foot should be a dactyl, and the last is either trochee or spondee.

The rules for long versus short syllables in Latin poetry are somewhat tricky in general, but the eight words of Bauluis’s verse can be characterized by the following patterns:

\[
\text{tot} = -, \quad \text{tibi} = \{\overset{\_}{\_}\}, \quad \text{sunt} = -, \quad \text{dotes} = -\_\_.,
\]

\[
\text{Virgo} = \{\overset{\_}{\_}\}, \quad \text{quot} = -, \quad \text{sidera} = -\_\_\_, \quad \text{cælo} = -\_\_. \hfill (23)
\]

Notice that poets had two choices when they used the words ‘tibi’ or ‘Virgo’. Thus, for example, (19) fits the hexameter pattern

\[
\text{Tot ti-bi sunt do- tes, Vir- gro, quot si-de-ra cæ-lo.} \hfill (24)
\]

(Dactyl, spondee, spondee, spondee, dactyl, spondee; “dum-diddy dum-dum dum-dum dum-dum dum-diddy dum-dum.” The commas represent slight pauses, called “cæsuras,” when the words are read; they don’t concern us here, although Puteanus inserted them carefully into each of his 1022 permutations.)

A natural question now arises: If we permute Bauluis’s words at random, what are the odds that they scan? In other words, how many of the permutations obey rules (i) and (ii), given the syllable patterns in (23)? G. W. Leibniz raised
this question, among others, in his *Dissertatio de Arte Combinatoria* (1666), a
work published when he was applying for a position at the University of Leipzig.
At this time Leibniz was just 19 years old, largely self-taught, and his under-
standing of combinatorics was quite limited; for example, he believed that there
are 600 permutations of \{ut, ut, re, mi, fa, sol\} and 480 of \{ut, ut, re, re, mi, fa\},
and he even stated that (22) represents 76 possibilities instead of 32. [See §5 and
§8 in his Problem 6.]

But Leibniz did realize that it would be worthwhile to develop general
methods for counting all permutations that are “useful,” in situations when
many permutations are “useless.” He considered several examples of Protesus
verses, enumerating some of the simpler ones correctly but making many errors
when the words were complicated. Although he mentioned Puteanus’s work, he
didn’t attempt to enumerate the scannable permutations of (19).

A much more successful approach was introduced a few years later by Jean
a clear exposition leading to the conclusion that exactly 2196 permutations of
Bauhaus’s verse would yield a proper hexameter. However, he soon realized that
he had forgotten to count quite a few cases—including those numbered 270,
384, and 725 in (20). So he completely rewrote this material when he published
*Nouveaux élémenst des Mathématiques* in 1689. Pages 127–133 of Prestet’s new
book were devoted to showing that the true number of scannable permutations
was 3276, almost 50% larger than his previous total.

Meanwhile John Wallis had treated the problem in his *Discourse of Combi-
nations* (London: 1685), 118–119, published as a supplement to his *Treatise of
Algebra*. After explaining why he believed the correct number to be 3096, Wallis
admitted that he may have overlooked some possibilities and/or counted some
cases more than once; “but I do not, at present, discern either the one and other.”

An anonymous reviewer of Wallis’s work remarked that the true number of
metrically correct permutations was actually 2580—but he gave no proof [*Acta
Erditorum* 5 (1686), 289]. The reviewer was almost certainly G. W. Leibniz
himself, although no clue to the reasoning behind the number 2580 has been
found among Leibniz’s voluminous unpublished notes.

Finally James Bernoulli entered the picture. In his inaugural lecture as
Dean of Philosophy at the University of Basel, 1692, he mentioned the tot-
tibi enumeration problem and stated that a careful analysis is necessary to
obtain the correct answer—which, he said, was 3312(!). His proof appeared
posthumously in the first edition of his *Ars Conjectandi* (1713), 79–81. [Those
pages were, incidentally, omitted from later editions of that famous book, and
from his collected works, because he didn’t actually intend them for publication;
a proofreader had inserted them by mistake. See *Die Werke von Jakob Bernoulli*

So who was right? Are there 2196 scannable permutations, or 3276, or 3096,
or 2580, or 3312? W. A. Whitworth and W. E. Hartley considered the question
anew in *The Mathematical Gazette* 2 (1902), 227–228, where they each presented
elegant arguments and concluded that the true total was in fact none of the
above. Their joint answer, 2880, represented the first time that any two mathematicians had independently come to the same conclusion about this problem.

But exercises 21 and 22, below, reveal the truth: Bernoulli is vindicated, and everybody else was wrong. Moreover, a study of Bernoulli’s systematic and carefully indented 3-page derivation indicates that he was successful chiefly because he adhered faithfully to a discipline that we now call the backtrack method. We shall study the backtrack method thoroughly in Section 7.2.2, where we will also see that the tot-tibi question is readily solved as a special case of the exact cover problem.

Even the wisest and most prudent people often suffer from what Logicians call insufficient enumeration of cases.

— JAMES BERNOULLI (1692)

Set partitions. The partitions of a set seem to have been studied first in Japan, where a parlor game called genji-ko (“Genji incense”) became popular among upperclass people about A.D. 1500. The host of a gathering would secretly select five packets of incense, some of which might be identical, and he would burn them one at a time. The guests would try to discern which of the scents were the same and which were different; in other words, they would try to guess which of the \( \pi_5 = 52 \) partitions of \( \{1, 2, 3, 4, 5\} \) had been chosen by their host.

Soon it became customary to represent the 52 possible outcomes by diagrams like those in Fig. 47. For example, the uppermost diagram of that illustration, when read from right to left, would indicate that the first two scents are identical and so are the last three; thus the partition is 12\345. The other two diagrams, similarly, are pictorial ways to represent the respective partitions 12\345 and 1\24\35. As an aid to memory, each of the 52 patterns was named after a chapter of Lady Murasaki’s famous 11th-century Tale of Genji, according to the following sequence [Encyclopedia Japonicae (Tokyo: Sanseido, 1910), 1299]:

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

(25)

(Once again, as we’ve seen in many other examples, the possibilities were not arranged in any particularly logical order.)
The appealing nature of these genji-ko patterns led many families to adopt them as heraldic crests. For example, the following stylized variants of \((a_5)\) were found in standard catalogs of kimono patterns early in the 20th century:

![Kimono Pattern Diagram](image)


Early in the 1700s, Takakazu Seki and his students began to investigate the number of set partitions \(\omega_n\) for arbitrary \(n\), inspired by the known result that \(\omega_5 = 52\). Yoshisuke Matsunaga found formulas for the number of set partitions when there are \(k_j\) subsets of size \(n_j\) for \(1 \leq j \leq t\), with \(k_1n_1 + \cdots + k_in_i = n\) (see the answer to exercise 1.2.5–21). He also discovered the basic recurrence relation 7.2.1.5–(14), namely

\[
\omega_{n+1} = \binom{n}{0}\omega_n + \binom{n}{1}\omega_{n-1} + \binom{n}{2}\omega_{n-2} + \cdots + \binom{n}{n}\omega_0,
\]

by which the values of \(\omega_n\) can readily be computed.

Matsunaga’s discoveries remained unpublished until Yoriyuki Arima’s book *Shūki Sanpō* came out in 1769. Problem 56 of that book asked the reader to solve the equation “\(\omega_n = 678570\)” for \(n\); and Arima’s answer, worked out in detail (with credit duly given to Matsunaga), was \(n = 11\).

Shortly afterwards, Masanobu Saka studied the number \(\binom{n}{k}\) of ways that an \(n\)-set can be partitioned into \(k\) subsets, in his work *Sanpō-Gakkai* (1782). He discovered the recurrence formula

\[
\binom{n+1}{k} = k\binom{n}{k} + \binom{n}{k-1},
\]

and tabulated the results for \(n \leq 11\). James Stirling, in his *Methodus Differentialis* (1730), had discovered the numbers \(\binom{n}{k}\) in a purely algebraic context; thus Saka was the first person to realize their combinatorial significance.


Set partitions remained virtually unknown in Europe until much later, except for three isolated incidents. First, George and/or Richard Puttenham published *The Arte of English Poesie* in 1589, and pages 70–72 of that book
contain diagrams similar to those of genjī-ko. For example, the seven diagrams

\[
\begin{align*}
\text{Diagram 1} & \quad \text{Diagram 2} & \quad \text{Diagram 3} \\
\text{Diagram 4} & \quad \text{Diagram 5} & \quad \text{Diagram 6} \\
\text{Diagram 7} & \quad \text{Diagram 8} & \quad \text{Diagram 9}
\end{align*}
\] (28)

were used to illustrate possible rhyme schemes for 5-line poems, “whereof some of them be harsher and unpleasaunter to the eare then other some be.” But this visually appealing list was incomplete (see exercise 25).

Second, an unpublished manuscript of G. W. Leibniz from the late 1600s shows that he had tried to count the number of ways to partition \(\{1, \ldots, n\}\) into three or four subsets, but with almost no success. He enumerated \(\binom{n}{3}\) by a very cumbersome method, which would not have led him to see readily that \(\binom{n}{3} = 2^n - 1\). He attempted to compute \(\binom{n}{4}\) only for \(n \leq 5\), and made several numerical slips leading to incorrect answers. [See E. Knobloch, \textit{Studia Leibniziana Suplementa} 11 (1973), 229–233; 16 (1976), 316–321.]

The third European appearance of set partitions had a completely different character. John Wallis devoted the third chapter of his \textit{Discourse of Combinations} (1685) to questions about “alque partus,” the proper divisors of numbers, and in particular he studied the set of all ways to factorize a given integer. This question is equivalent to the study of \textit{multiset} partitions; for example, the factorizations of \(p^3 q^2 r\) are essentially the same as the partitions of \(\{p, p, p, q, q, r\}\), when \(p, q, r\) are prime numbers. Wallis devised an excellent algorithm for listing all factorizations of a given integer \(n\), essentially anticipating Algorithm 7.2.1.5M (see exercise 28). But he didn’t investigate the important special cases that arise when \(n\) is the power of a prime (equivalent to integer partitions) or when \(n\) is squarefree (equivalent to set partitions). Thus, although Wallis was able to solve the more general problem, its complexities paradoxically deflected him from discovering partition numbers, Bell numbers, or Stirling subset numbers, or from devising simple algorithms that would generate integer partitions or set partitions.

**Integer partitions.** Partitions of integers arrived on the scene even more slowly. Bishop Wibold (c. 965) knew the partitions of \(n\) into exactly three parts \(\leq 6\). So did Galileo, who wrote a memo about them (c. 1627) and also studied their frequency of occurrence as rolls of three dice. [“Sopra le scoperre de i dadi,” in Galileo’s \textit{Opere}, Volume 8, 591–594; he listed partitions in decreasing lexicographic order.]

Mersenne listed the partitions of \(9\) into any number of parts, on page 130 of his \textit{Traitez de la Voix et des Chants} (1636). With each partition \(9 = a_1 + \cdots + a_k\) he also computed the multinomial coefficient \(9! / (a_1! \ldots a_k!)\); as we’ve seen earlier, he was interested in counting various melodies, and he knew for example that there are \(9! / (3!3!) = 1680\) melodies on the nine notes \(\{a, a, a, b, b, c, c, c\}\). But he failed to mention the cases \(8 + 1\) and \(3 + 2 + 1 + 1 + 1 + 1\), probably because he hadn’t listed the possibilities in any systematic way.

Leibniz considered two-part partitions in Problem 3 of his \textit{Dissertatio de Arte Combinatoria} (1666), and his unpublished notes show that he subsequently spent considerable time trying to enumerate the partitions that have three or
more summands. He called them “disceptions,” or (less frequently) “divisions”—in Latin of course—or sometimes “sections” or “dispersions” or even “partitions.” He was interested in them primarily because of their connection with the monomial symmetric functions $\sum x_1^{a_1}x_2^{a_2}\ldots$. But his many attempts led to almost total failure, except in the case of three summands, when he almost (but not quite) discovered the formula for $\binom{n}{3}$ in exercise 7.2.1.4–31. For example, he carelessly counted only 21 partitions of 8, forgetting the case $2+2+2+1+1$; and he got only 26 for $p(9)$, after missing $3+2+2+2$, $3+2+2+1+1$, $2+2+1+1+1+1$, and $2+2+1+1+1+1+1$—in spite of the fact that he was trying to list partitions systematically in decreasing lexicographic order.


Abraham de Moivre had the first real success with partitions in his paper “A Method of Raising an infinite Multinomial to any given Power, or Extracting any given Root of the same” [Philosophical Transactions 19 (1697), 619–620 and Fig. 5]. He proved that the coefficient of $z^{m+n}$ in $(az+bz^2+cz^3+\cdots)^n$ has one term for each partition of $n$; for example, the coefficient of $z^{m+6}$ is

\[
\binom{m}{0}a^{m-6}b^6 + 5\binom{m}{5}a^{m-5}b^4c + 4\binom{m}{4}a^{m-4}b^3d + 6\binom{m}{4}a^{m-4}b^2c^2
+ 3\binom{m}{3}a^{m-3}b^2d \quad \text{for } k = 1, 2, \ldots, n,
\]

and if we set $a = 1$, the term with exponents $b^i c^j d^k e^l \ldots$ corresponds to the partition with $i$ 1s, $j$ 2s, $k$ 3s, $l$ 4s, etc. Thus, for example, when $n = 6$ he essentially presented the partitions in the order

111111, 111112, 111122, 111222, 112222, 122222, 222222, 333333, 6, 111111, 111122, 111222, 122222, 222222, 333333, 6.

(30)

He explained how to list the partitions recursively, as follows (but in different language related to his own notation): For $k = 1, 2, \ldots, n$, start with $k$ and append the (previously listed) partitions of $n-k$ whose smallest part is $\geq k$.

[My solution was ordered to be published in the Transactions, not so much as a matter relating to Play, but as containing some general Speculations not unworthy to be considered by the Lovers of Truth.]

— ABRAHAM DE MOIVRE (1717)

P. R. de Montmort tabulated all partitions of numbers $\leq 9$ into $\leq 6$ parts in his Essay d’Analyse sur les Jeux de Hazard (1708), in connection with dice problems. His partitions were listed in a different order from (30); for example,

111111, 21111, 22111, 22211, 31111, 32111, 33111, 41111, 42111, 51111, 6.

(31)

He probably was unaware of de Moivre’s prior work.

So far almost none of the authors we’ve been discussing actually described the procedures by which they generated combinatorial patterns. We can only infer their methods, or lack thereof, by studying the lists that they actually published. Furthermore, in rare cases such as de Moivre’s paper where a tabulation
method was explicitly described, the author assumed that all patterns for the
first cases 1, 2, \ldots, n − 1 had been listed before it was time to tackle the case of
order n. No method for generating patterns “on the fly,” moving directly from
one pattern to its successor without looking at auxiliary tables, was actually
explained by any of the authors we have encountered, except for Kedāra and
Nārāyana. Today’s computer programmers naturally prefer methods that are
more direct and need little memory.

Roger Joseph Boscovich published the first direct algorithm for partition
generation in Giornale de’ Letterati (Rome, 1747), on pages 393–404 together
with two foldout tables facing page 404. His method, which produces for n = 6
the respective outputs

\begin{align*}
111111, & \quad 11112, \quad 1122, \quad 222, \quad 1113, \quad 123, \quad 33, \quad 114, \quad 24, \quad 15, \quad 6. \quad (32)
\end{align*}
generates partitions in precisely the reverse order from which they are visited by
Algorithm 7.2.1.4P; and his method would indeed have been featured in Section
7.2.1.4, except for the fact that the reverse order turns out to be slightly easier
and faster than the order that he had chosen.

Boscovich published sequels in Giornale de’ Letterati (Rome, 1748), 12–27
and 84–99, extending his algorithm in two ways. First, he considered generating
only partitions whose parts belong to a given set S, so that symbolic multinomials
with sparse coefficients could be raised to the mth power. (He said that the gcd
of all elements of S should be 1; in fact, however, his method could fail if 1 \not\in S.)
Second, he introduced an algorithm for generating partitions of n into m parts,
given m and n. Again he was unlucky: A slightly better way to do that task,
Algorithm 7.2.1.4H, was found subsequently, diminishing his chances for fame.

**Hindenburg’s hype.** The inventor of Algorithm 7.2.1.4H was Carl Friedrich
Hindenburg, who also rediscovered Nārāyana’s Algorithm 7.2.1.2L, a winning
technique for generating multiset permutations. Unfortunately, these small suc-
cesses led him to believe that he had made revolutionary advances in mathemat-
ic—although he did condescend to remark that other people such as de Moivre,
Euler, and Lambert had come close to making similar discoveries.

Hindenburg was a prototypical overachiever, extremely energetic if not in-
spired. He founded or cofounded Germany’s first professional journals of math-
ematics (published 1786–1789 and 1794–1800), and contributed long articles to
each. He served several times as academic dean at the University of Leipzig,
where he was also the Rector in 1792. If he had been a better mathematician,
German mathematics might well have flourished more in Leipzig than in Berlin
or Göttingen.

But his first mathematical work, Beschreibung einer ganz neuen Art, nach
einem bekannten Gesetze fortgehende Zahlen durch Abzählen oder Abmessen
bequem und sicher zu finden (Leipzig: 1776), amply foreshadowed what was to
come: His “ganz neue” (completely new) idea in that booklet was simply to give
combinatorial significance to the digits of numbers written in decimal notation.
Incredibly, he concluded his monograph with large foldout sheets that contained
a table of the numbers 0000 through 9999 — followed by two other tables that listed the even numbers and odd numbers separately (1).

Hindenburg published letters from people who praised his work, and invited them to contribute to his journals. In 1796 he edited *Sammlung combinatorisch-analytischer Abhandlungen*, whose subtitle stated (in German) that de Moivre’s multinomial theorem was “the most important proposition in all of mathematical analysis.” About a dozen people joined forces to form what became known as Hindenburg’s Combinatorial School, and they published thousands of pages filled with esoteric symbolism that must have impressed many nonmathematicians.

The work of this School was not completely trivial from the standpoint of computer science. For example, H. A. Rothe, who was Hindenburg’s best student, noticed that there is a simple way to go from a Morse code sequence to its lexicographic successor or predecessor. Another student, J. C. Burkhardt, observed that Morse code sequences of length \( n \) could also be generated easily by first considering those with no dashes, then one dash, then two, etc. Their motivation was not to tabulate poetic meters of \( n \) beats, as it had been in India, but rather to list the terms of the continuant polynomials \( K(x_1, x_2, \ldots, x_n) \), Eq. 4.5.3–(4). [See Archiv für reine und angewandte Mathematik 1 (1794), 154–194.] Furthermore, on page 53 of Hindenburg’s 1796 *Sammlung* cited above, G. S. Klügel introduced a way to list all permutations that has subsequently become known as Ord-Smith’s algorithm; see Eqs. (23)–(26) in Section 7.2.1.2.

Hindenburg believed that his methods deserved equal time with algebra, geometry, and calculus in the standard curriculum. But he and his disciples were combinatorialists who only made combinatorial lists. Burying themselves in formulas and formalisms, they rarely discovered any new mathematics of real interest. Eugen Netto has admirably summarized their work in M. Cantor’s *Geschichte der Mathematik* 4 (1908), 201–219. “For a while they controlled the German market; however, most of what they dug up soon sank into a not-entirely-deserved oblivion.”

The sad outcome was that combinatorial studies in general got a bad name. Gösta Mittag-Leffler, who assembled a magnificent library of mathematical literature about 100 years after Hindenburg’s death, decided to place all such work on a special shelf marked “Dekadenter.” And this category still persists in the library of Sweden’s Institut Mittag-Leffler today, even as that institute attracts world-class combinatorial mathematicians whose research is anything but decadent.

Looking on the bright side, we may note that at least one good book did emerge from all of this activity. Andreas von Ettingshausen’s *Die kombinatorische Analysis* (Vienna: 1826) is noteworthy as the first text to discuss combinatorial generation methods in a perspicuous way. He discussed the general principles of lexicographic generation in §8, and applied them to construct good ways to list all permutations (§11), combinations (§30), and partitions (§41–§44).

**Where were the trees?** We’ve now seen that lists of tuples, permutations, combinations, and partitions were compiled rather early in human history, by
interested and interesting researchers. Thus we’ve accounted for the evolution of the topics studied in Sections 7.2.1.1 through 7.2.1.5, and our story will be complete if we can trace the origins of tree generation. Section 7.2.1.6.

But the historical record of that topic before the advent of computers is virtually a blank page, with the exception of a few 19th-century papers by Arthur Cayley. Cayley’s major work on trees, originally published in 1875 and reprinted on pages 427–460 of his Collected Mathematical Papers. Volume 4, was climaxed by a large foldout illustration that exhibited all the free trees with 9 or fewer unlabeled vertices. Earlier in that paper he had also illustrated the nine oriented trees with 5 vertices. The methods he used to produce those lists were quite complicated, completely different from Algorithm 7.2.1.60 and exercise 7.2.1.6–90. All free trees with up to 10 vertices were listed many years later by F. Harary and G. Prins [Acta Math. 101 (1958), 158–162], who also went up to \( n = 12 \) in the cases of free trees with no nodes of degree 2 or with no symmetries.

The trees most dearly beloved by computer scientists—binary trees or the equivalent ordered forests or nested parentheses—are however strangely absent from the literature. We saw in Section 2.3.4.5 that many mathematicians of the 1700s and 1800s had learned how to count binary trees, and we also know that the Catalan numbers \( C_n \) enumerate dozens of different kinds of combinatorial objects. Yet nobody seems to have published an actual list of the \( C_4 = 14 \) objects of order 4 in any of these guises, much less the \( C_5 = 42 \) objects of order 5, before 1950. (Except indirectly: The 42 genji-ko diagrams in (25) that have no intersecting lines turn out to be equivalent to the 5-node binary trees and forests. But this fact was not learned until the 20th century.)

There are a few isolated instances where authors of yore did prepare lists of \( C_3 = 5 \) Catalan-related objects. Cayley, again, was first; he illustrated the binary trees with 3 internal nodes and 4 leaves as follows in Philosophical Magazine 18 (1859), 374–378:

(33)

(That same paper also illustrated another species of tree, equivalent to so-called weak orderings.) Then, in 1901, E. Netto listed the five ways to insert parentheses into the expression ‘\( a + b + c + d \)’:

\[
(a+b)+(c+d), \ [(a+b)+c]+d, \ [a+(b+c)]+d, \ a+[(b+c)+d], \ a+[b+(c+d)].
\]

(34)

[Lehrbuch der Combinatorik, §122.] And the five permutations of \{+1, +1, +1, –1, –1, –1\} whose partial sums are nonnegative were listed in the following way by Paul Erdős and Irving Kaplansky [Scripta Math. 12 (1946), 73–75]:

\[
1+1+1-1-1-1, \quad 1+1-1+1-1-1, \quad 1+1-1+1+1-1, \\
1-1+1+1-1-1, \quad 1-1+1-1+1-1. \quad (35)
\]

Even though only five objects are involved, we can see that the orderings in (33) and (34) were basically catch-as-catch-can; only (35), which matches Algorithm 7.2.1.6P, was systematic and lexicographic.
We should also note briefly the work of Walther von Dyck, since many recent papers use the term “Dyck words” to refer to strings of nested parentheses. Dyck was an educator known for co-founding the Deutsches Museum in Munich, among other things. He wrote two pioneering papers about the theory of free groups [Math. Annalen 20 (1882), 1–44; 22 (1883), 70–108]. Yet the so-called Dyck words have at best a tenuous connection to his actual research: He studied the words on \( \{ x_1, x_1^{-1}, \ldots, x_k, x_k^{-1} \} \) that reduce to the empty string after repeatedly erasing adjacent letter-pairs of the forms \( x_i x_i^{-1} \) or \( x_i^{-1} x_i \); the connection with parentheses and trees arises only when we limit erasures to the first case, \( x_i x_i^{-1} \).

Thus we may conclude that, although an explosion of interest in binary trees and their cousins occurred after 1950, such trees represent the only aspect of our story whose historical roots are rather shallow.

**After 1950.** Of course the arrival of electronic computers changed everything. The first computer-oriented publication about combinatorial generation methods was a note by C. B. Tompkins, “Machine attacks on problems whose variables are permutations” [Proc. Symp. Applied Math. 6 (1956), 202–205]. Thousands more were destined to follow.


The main publications relevant to particular algorithms that we’ve studied have already been cited in previous sections, so there is no need to repeat them here. But textbooks and monographs that first put pieces of the subject together in a coherent framework were also of great importance. Three books, in particular, were especially noteworthy with respect to establishing general principles:

- **Combinatorial Algorithms: Theory and Practice** by Edward M. Reingold, Jurg Nievergelt, and Narsingh Deo (Prentice-Hall, 1977), especially the material in Chapter 5.

We noted above that algorithms to generate Catalan-counted objects were not invented until computer programmers developed an appetite for them. The first such algorithms to be published were not cited in Section 7.2.1.6 because they have been superseded by better techniques; but it is appropriate to list them here. First, H. I. Scoincs gave two recursive algorithms for ordered tree generation, in the same paper we have cited with respect to the generation of oriented trees [Machine Intelligence 3 (1968), 43–60]. His algorithms dealt with binary trees represented as bit strings that were essentially equivalent to Polish prefix notation or to nested parentheses. Then Mark Wells, in Section 5.5.4 of his book cited above, generated binary trees by representing them as noncrossing set partitions. And Gary Knott [CACM 20 (1977), 113–115] gave recursive ranking and unranking algorithms for binary trees, representing them via the inorder-to-preorder permutations $q_1 \ldots q_n$ of Table 7.2.1.6–3.

Algorithms to generate all spanning trees of a given graph have been published by numerous authors ever since the 1950s, motivated originally by the study of electrical networks. Among the earliest such papers were works of N. Nakagawa, IRE Trans. CT-5 (1958), 122–127; W. Mayeda, IRE Trans. CT-6 (1959), 136–137, 394; H. Watanabe, IRE Trans. CT-7 (1960), 296–302; S. Hakimi, J. Franklin Institute 272 (1961), 347–359.

A recent introduction to the entire subject can be found in Chapters 2 and 3 of Combinatorial Algorithms: Generation, Enumeration, and Search by Donald L. Kreher and Douglas R. Stinson (CRC Press, 1999).

Frank Ruskey is preparing a book entitled Combinatorial Generation that will contain a thorough treatment and a comprehensive bibliography. He has made working drafts of several chapters available on the Internet.

**EXERCISES**

Many of the exercises below ask a modern reader to find and/or to correct errors in the literature of bygone days. The point is not to gloat over how smart we are in the 21st century; the point is rather to understand that even the pioneers of a subject can stumble. One good way to learn that a set of ideas is not really as simple as it might seem to today’s computer scientists and mathematicians is to observe that some of the world’s leading thinkers had to struggle with the concepts when they were new.

1. [15] Does the notion of “computing” arise in the I Ching?

2. (Correlated) DNA molecules are strings of “nucleotides” on the 4-letter alphabet {T, C, A, G}, and most protein molecules are strings of “amino acids” on the 20-letter alphabet {A, C, D, E, F, G, H, I, K, L, M, N, P, Q, R, S, T, V, W, Y}. Three consecutive nucleotides $xyz$ form a “codon” and a strand $x_1 y_1 z_1 x_2 y_2 z_2 \ldots$ of DNA specifies the protein $f(x_1, y_1, z_1) f(x_2, y_2, z_2) \ldots$ where $f(x, y, z)$ is the element in row $z$ and column $y$ of matrix $x$ in the array

\[
\begin{array}{cccc}
F & S & Y & C \\
F & S & Y & C \\
L & S & W & - \\
L & S & W & - \\
\end{array} \\
\begin{array}{cccc}
L & P & H & R \\
L & P & Q & R \\
L & P & Q & R \\
L & P & Q & R \\
\end{array} \\
\begin{array}{cccc}
I & T & N & S \\
I & T & N & S \\
I & T & K & R \\
I & T & K & R \\
\end{array} \\
\begin{array}{cccc}
V & A & D & G \\
V & A & D & G \\
V & A & E & G \\
V & A & E & G \\
\end{array} \\
\]

(Here $(T, C, A, G) = (1, 2, 3, 4)$; for example, $f(CAT)$ is the element in row 1 and column 3 of matrix 2, namely H.) Encoding proceeds until a codon leads to the stopper ‘-‘.
a) Show that there is a simple way to map each codon into a hexagram of the I Ching
with the property that the 21 possible outcomes \{A, C, D, ... , W, Y, -\} correspond
to 21 consecutive hexagrams of the King Wen ordering (1).
b) Is that a sensational discovery?

3. [20] What is the milliönth meter that has 30 beats, in colex ordering analogous
to (2)? What is the rank of $\sim \cdots \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim$?

4. [19] Analyze the imperfections of Donnolo's list of permutations in Table 1.

5. [16] What's wrong with Kircher's list of five-note permutations in (7)?

Traitez de la Voix et des Chants (1636). His value for 64! was \(\approx 2.2 \times 10^{89}\); but it should
have been \(\approx 1.3 \times 10^{89}\). Find a copy of his book and try to figure out where he erred.

7. [20] What permutations of \{1, 2, 3, 4, 5\} are "alive" and "dead" according to Seki's
rules (8) and (9)?

- 8. [M27] Make a patch to (g) so that Seki's procedure will be correct.

9. [15] From (11), deduce the Arabic way to write the Arabic numerals (0, 1, ..., 9).

- 10. [HM27] In Ludus ClericaUs, what is the expected number of times the three dice
are rolled before all possible virtues are acquired?

11. [27] Decipher Lhull's vertical table at the right of Fig. 45. What 20 combinatorial
objects does it represent? Hint: Don't be misled by typographic errors.

12. [M20] Relate Schilling's universal cycle (13) to the universal cycle of Poinset in
exercise 7.2.1–106.

13. [27] What should van Schooten have written, instead of (17)? Give also the
Corresponding tableau for combinations of the multiset \{a, a, a, b, c\}.

- 14. [20] Complete the following sequence, from §95 of Izquierdo's De Combinaciones:

    ABC ABD ABE ACD ACE ACB ADE ADB ADC AEB ... 

15. [15] If all \(n\)-combinations of \{1, ..., \(m\)\} with repetition are listed in lexicographic
order, how many of them begin with the number \(j\)?

16. [20] (Nârâyana Paññita. 1356.) Design an algorithm to generate all compositions
of \(n\) into parts \(\leq q\), namely all ordered partitions \(n = a_1 + \cdots + a_r\), where \(1 \leq a_j \leq q\)
for \(1 \leq j \leq t\) and \(t\) is arbitrary. Illustrate your method when \(n = 7\) and \(q = 3\).

17. [HM27] Analyze the algorithm of exercise 15.

18. [10] Trick question: Leibniz published his Dissertatio de Arte Combinatoria in
1666. Why was that a particularly auspicious year, permutationwise?

19. [17] In which of Putenes's verses (20) is 'tibi' treated as \(\sim \sim\) instead of \(\sim \sim\)?

20. [M25] To commemorate the visit of three illustrious noblemen to Dresden in 1617,
a poet published 1617 permutations of the hexameter verse

    Dant tria jam DresDé, cæ sol dat, luminæ lucem.

"Three give now to Dresden, as the sun gives, lights to light." [Gregor Kleppis, Poetaus
Poeticius (Leipzig: 1617)]. How many permutations of those words would actually scan
properly? Hint: The verse has dactyli in the first and fifth feet, spondees elsewhere.
21. [HM30] Let \( f(p, q; r; s, t) \) be the number of ways to make \((o^p, o^q, o^r)\) by concatenating the strings \(\{s \cdot o, t \cdot oo\}\), when \( p + q + r = s + 2t \). For example, \( f(2, 3, 2, 3, 2) = 5 \) because the five ways are 

\[
(oo \cdot oo \cdot oo), \quad (oo \cdot oo \cdot oo), \quad (oo \cdot oo \cdot oo), \quad (oo \cdot oo \cdot oo).
\]

a) Show that \( f(p, q; r; s, t) = [u^p v^q w^r z^t] / [(1 - zw - u^2)(1 - zv - v^2)(1 - zw - t^2)] \).
b) Use the function \( f \) to enumerate the scannable permutations of \((19)\), subject to the additional condition that the fifth foot doesn’t begin in the middle of a word.
c) Now enumerate the remaining cases.

▷ 22. [M40] Look up the original discussions of the tot-tibi problem that were published by Prested, Wallis, Whitworth, and Hartley. What errors did they make?

23. [20] What order of the 52 genji-ko diagrams corresponds to Algorithm 7.2.1.5H?

▷ 24. [23] Early in the 1800s, Toshiaki Honda gave a recursive rule for generating all partitions of \(\{1, \ldots, n\}\). His algorithm produced them in the following order when \(n = 4\):

\[
\begin{array}{cccccccc}
\hline
\text{a} & \text{b} & \text{c} & \text{d} & \text{e} & \text{f} & \text{g} & \text{h} \\
\hline
\end{array}
\]

Can you guess the corresponding order for \(n = 5\)? \textit{Hint:} See (26).

25. [15] The 16th-century author of \textit{The Arte of English Poesie} was interested only in rhyme schemes that are “complete” in the sense of exercise 7.2.1.5–35; in other words, every line should rhyme with at least one other. Furthermore, the scheme should be “indecomposable” in the sense of exercise 7.2.1.2–100: A partition like \(\{123\}45\) decomposes into a 2-line poem followed by a 3-line poem. And the scheme shouldn’t consist trivially of lines that all rhyme with each other. Under these conditions, is (28) a complete list of 5-line rhyme schemes?

▷ 26. [HM25] How many \(n\)-line rhyme schemes satisfy the constraints of exercise 24?

27. [HM31] The set partition \(14\{25\}36\) can be represented by a genji-ko diagram such as \(\frown\), but every such diagram for this partition must have at least three places where lines cross, and crossings are sometimes considered undesirable. How many partitions of \(\{1, \ldots, n\}\) have a genji-ko diagram in which the lines cross at most once?

▷ 28. [25] Let \(a\), \(b\), and \(c\) be prime numbers. John Wallis listed all possible factorizations of \(a^b b^c\) as follows: \(\text{cbbac}, \text{cbca}, \text{abaa}, \text{bbab}, \text{cbaa}, \text{cbba}, \text{bbaa}, \text{bos}, \text{aabb}, \text{baab}, \text{abab}, \text{abcc}, \text{bacc}, \text{bca}, \text{aca}, \text{cab}, \text{abc}, \text{acbb}, \text{cbac}, \text{bcab}, \text{acbc}, \text{bcac}, \text{acbc}, \text{acbc}, \text{acbc}\). What algorithm did he use to generate them in this order?

▷ 29. [24] In what order would Wallis have generated all factorizations of the number \(\text{abcde} = 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17\)? Give your answer as a sequence of genji-ko diagrams.

30. [M20] What is the coefficient of \(a_1^2 a_2^3 \cdots z^{m+n}\) in \((a_0 z + a_1 z^2 + a_2 z^3 + \cdots)^m\)? (See (29).)

31. [20] Compare de Moivre’s and de Montmort’s orders for partitions, (30) and (31), with Algorithm 7.2.1.4P.

32. [21] (R. J. Boscovich, 1748.) List all partitions of 20 for which all parts are 1, 7, or 10. Also design an algorithm that lists all such partitions of any given integer \(n > 0\).
SECTION 7.2.1.7

1. Perhaps under hexagram 21, “crunching” (䷁); however, the ancient commentators related this hexagram more to law enforcement than to the interaction of electrons.

2. (a) For the first nucleotide in the codon, let (T, C, A, G) be respectively represented by (รว รว รว รว ); represent the second nucleotide similarly, by (รว รว รว รว ); represent the third by (รว รว รว รว ); and superimpose those three representations. Thus, for example, hexagram number 34 is รว รว รว รว รว ; it represents the codon TTC, which maps to the amino acid F. Under this correspondence, hexagrams 34 through 54 inclusive map into the respective values (F, G, L, Q, W, D, S, – P, Y, K, A, I, T, N, H, M, R, V, E, C). Moreover, the three hexagrams that map to ‘-‘ are numbers 1, 9, and 41, namely รว รว รว and รว, which mean “creation”, “taming,” and “removal of excess” in the I Ching— all quite appropriate for the notion of completing a protein.

(b) Consider the 2402880402175789790003993681964551328451668718750185553920000000 ≈ 2.4 × 10^{63} ways to permute the elements of the 4 × 4 × 4 genetic code array. Exactly

of them contain at least one run of 21 distinct consecutive elements. [Using the principle of inclusion and exclusion one can show that any multiset \(\{x_1, \ldots, x_r\}\) with \(r\) distinct elements and \(n_r = 0\) has exactly

\[
(n+1){n \choose n_1, \ldots, n_r} r! - \sum_{k=1}^{r} (n+1-k)!(r-k)!a_k \sum_{0 \leq d_1, \ldots, d_r \leq 1, d_1 + \cdots + d_r = k} \left( \begin{array}{c} n-k \\ n_1-d_1, \ldots, n_r-d_r \end{array} \right)
\]

such permutations where \(n = n_1 + \cdots + n_r\) and \(a_k\) is the number of indecomposable permutations with \(k\) elements (exercise 7.2.1.2–100).] Thus only about one out of every million permutations has the stated property.

But there are 4!^{1/6}\left(2,2,2\right) = 1244160 ways to represent codons as in part (a), and most of them correspond to different permutations of the amino acids (except for interchanging the representations of T and C in third position).

Empirically, in fact, about 31% of all permutations of the 64 hexagrams turn out to have suitable codon mappings. Thus the construction in part (a) gives no reason to believe that the authors of the I Ching anticipated the genetic code in any way.

3. Since \(F_{31} = 10^6 = F_{28} + F_{22} + F_{20} + F_{18} + F_{16} + F_{14} + F_{9}\), the millionth is

Going the other way is easier: \(F_{31} - (F_3 + F_8 + F_{18} + F_{16} + F_{14} + F_{13} + F_{30}) = 314159\).

4. One of the two appearances of 𫸩징 on line 4 should be ䷁; this glitch may simply be a typographical error. Similarly, one ䷁ on line 8 should be ䷁. But the six cases with rightmost letters ䷁ appear twice in lines 3 and 4, while the cases with rightmost ䷁ are missing. Donnolo himself must be responsible for this flaw.

5. The last one should have been ䷁, not ䷁.

6. The nth value \(m_n\) in Mersenne’s list agrees with \(n!\) only for \(1 \leq n \leq 13\) and 15 \(\leq n \leq 38\). Mersenne knew that 14! = 87178291200 ≠ \(m_{14}\) = 8778291200, because he inserted the missing ‘1’ in his personal copy of the book (now owned by the Bibliothèque Nationale; a facsimile was published in 1975). But the other errors in his table were not merely typographical, because they propagated into subsequent entries, except in the case of \(m_{50}\): \(m_{39} = 39! + 10^{66} - 10^{10}; m_{40} = 40m_{39}; m_{41} = 41m_{40} - 4 \cdot 10^{25} - 14 \cdot 10^{11};\)
\[ m_n = n m_{n-1} \] for \( n = 42, 43, 44, 46, 47, 48, 49, 55, 60, \text{ and } 62; \] \[ m_{50} = 50 m_{49} + 10^{56}; \]
m_{51} = 51 \cdot 50 \cdot m_{49}. \] When he computed \( m_{45} = 9 \cdot 45 \cdot m_{44} - 10^{40} + 10^{29}, \) he apparently decided to take a shortcut, because it’s easy to multiply by 5 or by 9; but he multiplied twice by 9. Most of his errors indicate an unreliable multiplication technique, which may have depended on an abacus: \( m_{52} = 52 m_{51} + 5 \cdot 10^{56} - 2 \cdot 10^{67} + 10^{34}; \] \( m_{53} = 53 m_{52} - 4 \cdot 10^{25}; \] \( m_{54} = 54 m_{53} + 10^{16}; \] \( m_{55} = 57 m_{54} + 10^{33} + 10^{24}; \] \( m_{56} = 58 m_{55} + 10^{67} - 10^{35} + 10^{12} + 11 \cdot 10^{26}; \] \( m_{57} = 59 m_{56} + 10^{60} + 10^{49} - 10^{28}; \] \( m_{58} = 61 m_{57} + 5 \cdot 10^{81}; \] \( m_{59} = 63 m_{62} + 10^{22} - 10^{74}; m_{64} = 64 m_{63} + 3 \cdot 10^{71} + 10^{67} + 2 \cdot 10^{48} - 2 \cdot 10^{33} - 10^{23}. \]

The remaining case, \( m_{56} \approx 10.912 m_{55} \) is baffling; it is \( \equiv 56 m_{55} \) (modulo \( 10^{17} \)), but its other digits seem to satisfy neither rhyme nor reason. Can they be easily explained?

**Notes:** Athanasius Kircher must have copied from Merseenne when he tabulated \( n^! \) for \( 1 \leq n \leq 50 \) on page 157 of his Ars Magna Scientiae (1669), because he repeated all of Merseenne’s mistakes. Kircher did, however, list the values \( 10 m_{14}, m_{45}/10, \text{ and } 10 m_{49} \) instead of \( m_{14}, m_{45}, \text{ and } m_{49}; \) perhaps he was trying to make the sequence grow more steadily. It is not clear who first calculated the correct value of \( 39^!; \) exercise 1.2.5–4 tells the story of 1000!.

**7.** The basic permutations are \( 12345, 13254, 14523, 15432, 12453, 14235, 15324, 13542, 12534, 15243, 13425, 14352. \] But then we find that all 60 of the even permutations are both alive and dead, because \( g \) differs by an even permutation from \( \iota g \). (Moreover, if we somehow repair the case \( n = 5 \), half of the live permutations for \( n = 6 \) will turn out to be odd.)

**8.** For example, we can replace \( g \) by
\[ a_6 a_3 \cdots a_{n-1} a_2 a_1; \] \[ a_1 a_2 \cdots a_{n-1} a_n; \]
thus flipping the ends and cyclically shifting the other elements in the permutations of \( \iota g \). This modification works because all permutations have the correct parity, and because the live and dead ones both have \( a_1 \) in every possible position. (We essentially have a dual Sims table for the alternating group, as in Eq. 7.2.1.2–32; but our elements are named \( a, n - 1, \ldots, 1 \) instead of \( 0, 1, \ldots, n - 1 \).)

A simpler way to generate permutations with the proper signs was published by É. Bézout [Mémoires Acad Royale des Sciences (Paris. 1764). 292]: Each permutation \( a_1 \cdots a_{n-1} \) of \( \{1, \ldots, n - 1\} \) yields \( n \) others, \( a_1 \cdots a_{n-2} a_n a_{n-1} + \cdots \)

**9.** \( (\times, \wedge, \vee, \cdot, \circ, \wedge, \vee, \Lambda, \Lambda); \) or perhaps we should say \( (\times, \wedge, \vee, \cdot, \circ, \wedge, \vee, \Lambda, \Lambda); \) Notes: A different system was used for the index numbers of the equations; for example, \( \iota j \) stood for 200. Moreover, it should be noted that \( (11) \) is actually a transcription of al-Samaw’al’s work into modern Arabic; Ahmad and Rashid based their work on a 14th-century copy that used similar but older forms of the digits: \( (\times, \wedge, \vee, \cdot, \circ, \times, \wedge, \vee, \Lambda, \Lambda). \) Al-Samaw’al himself may well have used numerals of an even earlier vintage.

**10.** If the 56 cases were equally likely, the answer would be \( 56 H_{56} \approx 258.2. \) as in the coupon collector’s problem (exercise 3.3.2–8). But \( (6,30,20) \) cases occur with the respective probabilities \( (1/216, 1/72, 1/36); \) so the correct answer turns out to be
\[ \int_0^\infty \left( 1 - (1 - e^{-t/216})^6 \right) \left( 1 - (1 - e^{-t/72})^{30} \right) (1 - e^{-t/36})^{20} \, dt \approx 546.6. \]


**11.** It tabulates the \( \binom{4}{2} = 20 \) combinations of \( (b, c, d, B, C, D) \) taken three at a time; furthermore, they appear in lexicographic order if we regard \( b < c < d < B < C < D. \)
The letter $t$ (Carol) means "shift from lowercase to uppercase." [See A. Bonner, Selected Works of Ramon Lull (Princeton: 1985), 596–597.] There are two typos: 'd' should be 'b' at the beginning of line 6; 'c' should be 'd' at the end of line 18. Line 1 would have been more consistent with the others if Lull had presented it as

\[ \begin{array}{ccc} b & c & d \end{array} \]

but in that line, of course, no case shift was needed.

12. Multiply Poinot’s cycle by $5 \pmod{7}$.

13. It’s best to have just $n$ lines when there are $n$ different letters:

\[
\begin{array}{c}
\text{a} \\
\text{aa} \\
\text{aaa}
\end{array}
\begin{array}{c}
\text{b} \\
\text{ab} \\
\text{aabb } \text{bb} \text{.abb} \text{.aabbb}
\end{array}
\]

Then, assigning the weights $(a, b) = (1, 4)$ gives the numbers 1 through 11 as in (18). (The first line of (16) should also be omitted.) Similarly, for \{a, a, a, b, b, c\} we would implicitly give $c$ the weight 12 and add the additional line

\[
\begin{array}{c}
\text{c} \\
\text{ac} \\
\text{aac} \\
\text{aacc acbac} \\
\text{abc. aabc. aabbc. bbc. abbc. aabbc. aaabbc.}
\end{array}
\]

[J. Bernoulli almost did it right in Ars Conjectandi, Part 2, Chapter 6.]

14. ABC ABD ABE ACD ACE ACF ADE ADB ADC AEC AFD AEF AGE AGF AGB AHE AIL AEM AEN AEP AGP AKQ

BDA BDC BEA BEC BDE BAD BAC CAD CDA DCE EDA EBD EDC DBA DCE EBC BCD DBC BCA DCA

15. After $j$ we place the $(n - 1)$-combinations of \{j, \ldots, m\} with repetition, so the answer is $\binom{(m+1-j)\times(n-1)}{(n-1)-1}.$ [Jean Borrel, also known as Buteonis, pointed this out on pages 305–309 of his early book Logistica (Lyon: 1560). He tabulated all throws of $n$ dice for $1 \leq n \leq 4$, then used a sum over $j$ to deduce that there are $56 + 35 + 20 + 10 + 4 + 1 = 252$ distinct throws for $n = 5$, and 462 for $n = 6$.]

16. N1. [Initialize.] Set $r \leftarrow n$, $t \leftarrow 0$, and $a_0 \leftarrow 0$.

N2. [Advance.] While $r \geq q$, set $t \leftarrow t + 1$, $a_t \leftarrow q$, and $r \leftarrow r - q$. Then if $r > 0$, set $t \leftarrow t + 1$ and $a_t \leftarrow r$.

N3. [Visit.] Visit the composition $a_1 \ldots a_t$.

N4. [Find $j$.] Set $j \leftarrow t - t - 1, \ldots, \text{ until } a_j \neq 1$. Terminate the algorithm if $j = 0$.

N5. [Decrease $a_j$.] Set $a_j \leftarrow a_j - 1$. $r \leftarrow t - j + 1$, $t \leftarrow j$; return to N2.]

For example, the compositions for $n = 7$ and $q = 3$ are 331, 322, 3211, 313, 3121, 31111, 323, 231, 223, 2221, 2212, 22111, 2131, 2122, 21211, 2113, 21121, 21112, 211111, 133, 1321, 1312, 13111, 13112, 131121, 131111, 131112, 1311111, 1311112, 13111111, 13111112, 131111111, 131111112, 1311111111, 1311111112, 13111111111, 13111111112, 131111111111, 131111111112, 1311111111111, 1311111111112, 13111111111111.

Nārāyana's sūtras 79 and 80 gave essentially this procedure, but with the strings reversed (133, 223, 123, \ldots), because he preferred decreasing colex order. Curiously, he called this a "famous method. told by scholars of old dramatic art." although no references to prior descriptions are currently known except in the case $q = 2$.

17. The number $V_n$ of visits is $F_{n+q-1}^q = \Theta(\alpha_q^n)$; see exercise 5.4.2–7. The number $X_n$ of times step N4 tests $a_j = 1$ satisfies $X_n = X_{n-1} + \cdots + X_{n-q} + 1$, and we find
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\[ X_n = V_0 + \cdots + V_n = (qV_n + (q-1)V_{n-1} + \cdots + V_{n-q}\) + 1)/(q - 1) = \Theta(V_n). \]  
The number \( Y_n \) of times step \( N_2 \) sets \( a_t \mapsto q \) satisfies the same recurrence, and we find \( Y_n = X_{n-q}. \) And the number of times step \( N_2 \) finds \( r = 0 \) turns out to be \( V_{n-q}. \)

18. It was MDCLXVI in Roman numerals, where \( M > D > C > L > X > V > I. \)

19. Lines 329 and 1022. (Puteanus included 139 such verses among his list of 1022.)

20. With ‘tria’ preceding ‘lumina’, there are \( 5! \times 2! \times (11, 12, 12, 16) \) ways having a dactyl in the 1st, 2nd, 3rd, 4th foot, respectively; with ‘lumina’ preceding ‘tria’ there are \( 5! \times 2! \times (16, 12, 12, 11). \) So the total is 24480. [Leibniz considered this problem near the end of his *Dissertatio de Arte Combinatoria*, and came up with the answer 45870; but his argument was fiddled with errors.]

21. (a) The generating function \( 1/((1 -izu - yu^2)(1 - zw - ywz^2)) \) is clearly equal to \( \sum_{p, q, r, s, t \geq 0} f(p, q, r; s, t) w^p v^q w^r z^t y^t. \)

(b) If ‘tibi’ is \( \sim \sim \) and ‘Virgo’ is \( \sim \sim \), the number is \( 3! \cdot 3! \cdot \sum_{k=0}^{3} f(2k + 1, 6 - 2k; 2; 3, 3) + f(2k, 6 - 2k; 2; 2, 2). \) Namely 36 \((7+7)+(9+5)+(10+5)+(14+7)) = 2304. Otherwise ‘tibi’ is \( \sim \sim \), ‘Virgo’ is \( \sim \sim \), and the number is \( 2! \cdot 3! \cdot \sum_{k=0}^{3} f(2k, 5 - 2k; 3, 3, 2) + f(2k, 6 - 2k; 1; 3, 2). \) Namely 12 \((7+6)+(5+4)+(4+3)+(6+1)) = 432.

(c) The fifth foot begins with the second syllable of ‘cielo’, ‘dotes’, or ‘Virgo’. Hence the additional number is \( 3! \cdot 3! \cdot \sum_{k=0}^{3} f(2k, 5 - 2k; 2; 3, 2) = 36\cdot(7 + 5 + 4) = 576, \) and the grand total is 2304 + 432 + 576 = 3312.

22. Let \( \alpha \in \{\text{quot. sunt. tot.}\}, \beta \in \{\text{cielo dotes. Virgo}\}, \sigma \in \text{sidera.} \) and \( \tau = \text{tibi.} \) Prestet’s analysis was essentially equivalent to that of Bernoulli, but he forgot to include the 36 cases \( \alpha \sigma \alpha \tau \beta \beta \sigma \beta. \) (In his favor one can say that those cases are poetically sterile; Puteanus found no use for them.) The 1675 edition of Prestet’s book had also omitted all permutations that end with \( \tau \beta \).

Wallis divided the possibilities into 23 types. \( T_1 \cup T_2 \cup \cdots \cup T_{23}. \) He claimed that his types 6 and 7 each yielded 324 verses; but actually \( |T_6| = |T_7| = 252, \) because his variable \( i \) should be \( 7, \) not 9. He also counted many solutions twice: \( |T_3 \cap T_5| = 72, \) \( |T_3 \cap T_7| = |T_3 \cap T_9| = |T_6 \cap T_{10}| = 36, \) and \( |T_{11} \cap T_{14}| = |T_{12} \cap T_{13}| = |T_{14} \cap T_{15}| = 12. \) He missed the 36 possibilities \( \alpha \beta \sigma \alpha \sigma \alpha \beta \beta \sigma \) (19 of which were used by Puteanus). And he also missed all the permutations of exercise 20(c); Puteanus had used 250 of those 576. The Latin edition of Wallis’s book, published in 1693, corrected several typographic errors in this section, but none of the mathematical mistakes.

Whitworth and Hartley omitted all cases with ‘tibi’ = \( \sim \sim \) (see exercise 18), possibly because people’s knowledge of classical hexameter was beginning to fade.

[Speaking of errors. Puteanus actually published only 1020 distinct permutations, not 1022, because lines 592 and 593 in his list were identical to lines 601 and 602. But he would have had no trouble finding two more cases—for example, by changing ‘tot sunt’ to ‘sunt tot’ in lines 252, 345, 511, 548, 659, 663, 678, 693, or 797.]

23. Reading each diagram left-to-right, so that \( 12|345 \leftrightarrow \mathbb{I} \), we get
24. His rule was: For \( k = 0, 1, \ldots, n-1 \) and for each combination \( 0 < j_1 < \cdots < j_k < n \) of \( n-1 \) things taken \( k \) at a time, visit all partitions of \( \{1, \ldots, n-1\} \setminus \{j_1, \ldots, j_k\} \) together with the block \( \{j_1, \ldots, j_k, n\} \). His order for \( n = 5 \) was:

```
1 1 1 1 1
1 1 1 1 1
1 1 1 1 1
1 1 1 1 1
1 1 1 1 1
```

But strictly speaking, the answer to this exercise is “No” — because Honda’s rule is not complete until the order of the combinations is specified. He generated combinations in order (lexicographic on \( j_1 \ldots j_k \)). Lexicographic order on \( j_1 \ldots j_k \) would also be consistent with the list given for \( n = 4 \), but it would put \( \text{A} \) before \( \text{B} \). Reference: T. Hayashi, Tôhoku Math. J. 33 (1931), 332–337.

25. No: \( \{8\} \) misses 141235 (the top-bottom reflection of its second pattern).

26. Let \( a_n \) be the number of indecomposable partitions of \( \{1, \ldots, n\} \), and let \( a'_n \) be the number that are both indecomposable and complete. These sequences begin \( \langle a_1, a_2, \ldots \rangle = \langle 1, 1, 2, 6, 22, 92, 426, \ldots \rangle \) and \( \langle a'_1, a'_2, \ldots \rangle = \langle 0, 1, 1, 3, 9, 33, 135, \ldots \rangle \); and the answer to this exercise is \( a'_n - 1 \) for \( n \geq 2 \). It turns out that \( a_n \) is also the number of symmetric polynomials of degree \( n \) in noncommuting variables. [See M. C. Wolf, Duke Math. J. 2 (1936), 626–637, who also tabulated indecomposable partitions into \( k \) parts.]

If \( A(z) = \sum_n a_n z^n \), and if \( B(z) = \sum_n w_n z^n \) is the non-exponential generating function for Bell numbers, we have \( A(z)B(z) = B(z) - 1 \), hence \( A(z) = 1 - 1/B(z) \). And the result of exercise 7.2.1.3–35 implies that \( \sum_n a'_n z^n = zA(z)/(1+z-A(z)) = z(B(z)-1)/(1+zB(z)) \). Unfortunately \( B(z) \) has no especially nice closed form. Notice that indecomposable set partitions with \( n > 1 \) correspond to vacillating tableau loops with no three consecutive \( \lambda \)s equal to zero (see exercise 7.2.1.5–27).

27. The problem is ambiguous because genji-ko diagrams are not well defined. Let’s require all vertical lines of a block to have the same height; then, for example, 145236 has no single-crossing diagram because \( \text{A} \) is not allowed.

The number of partitions with no crossing is \( C_n \) (see exercise 7.2.1.6–26). For one crossing, the elements of the two blocks that cross must appear within the restricted growth sequence as either \( x^iy^jy^k \) or \( x^iy^jx^iy^k \) or \( x^iy^jx^iy^kx^l \), where \( i, j, k, l > 0 \).

Suppose the pattern is \( x^iy^jx^k \). The number of such partitions is

\[
[z^{n-i-j-k-1}]C(z)^{i+j+k+2} = C_{(n-i-j-k-1)}C_{(n-i-j-k+1)}
\]

by Eq. 7.2.1.6–(24). Summing on \( k \) gives \( C_{(n-i-j-2)}C_{(n+1)} \); then summing on \( j \) and \( i \) gives \( C_{(n-4)}C_{(n+3)} \).

Similarly, the other two patterns contribute \( C_{(n-5)}C_{(n+3)} \) and \( C_{(n-5)}C_{(n+4)} \). The total number of single-crossing partitions is therefore \( C_{(n-5)}C_{(n+3)} + C_{(n-4)}C_{(n+4)} \).

28. Order the divisors of \( cbbaa \) by their number of prime factors and then lexicographically: \( 1 < a < b < c < aa < bb < ca < cb < aab < bba < caa < bba < baa < cbba \). For every such divisor \( d \), in decreasing order, let \( d \) be the first factor; recursively append all factorizations of \( cbbaa/d \) whose first factor is \( \leq d \).
If the divisors had been ordered lexicographically (namely \(1 < a < aa < aaa < b < ba < \cdots < ebbaa < ebbaa\)). Wallis’s algorithm would have been equivalent to Algorithm 7.2.1.5M with \((n_1, n_2, n_3) = (1, 2, 3)\). He probably chose his more complicated ordering of the divisors because it tends to agree more closely with ordinary numerical order when \(a \approx b \approx c\); for example, his ordering is precisely numerical when \((a, b, c) = (7, 11, 13)\). By generating the divisors according to his somewhat complex scheme, Wallis was essentially generating multiset combinations, which we noted in Section 7.2.1.3 are equivalent to bounded compositions. [Reference: A Discourse of Combinations (1685), 126-128, with two typographic errors corrected.]

29. The factorizations \(edcba, edcb\cdot a, edca\cdot b, \ldots, e\cdot d\cdot c\cdot b\cdot a\) correspond respectively to

\[
\begin{array}{ccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
\]

30. The coefficient is zero unless \(i_1 + 2i_2 + \cdots = n\); in that case it is \(\binom{m}{k} a_0^{m-k}(a_{i_1} a_{i_2} \cdots)^k\) where \(k = i_1 + i_2 + \cdots\). (Consider \((a_0z)^m\) times \((1 + (a_1/a_0)z + (a_2/a_0)z^2 + \cdots)^m\).)

31. The order produced by that algorithm is decreasing lexicographic; the reverse of \((31)\), if we assume that partitions \(a_1 \cdots a_k\) have \(a_1 \geq \cdots \geq a_k\); de Moivre’s was increasing \(\text{colexicographic}\).

32. \(20 \cdot 1 = 7 + 13 \cdot 1 = 2 \cdot 7 + 6 \cdot 1 = 10 + 10 \cdot 1 = 10 + 7 + 3 \cdot 1 = 2 \cdot 10\). In general, Boscovich suggested starting with \(n-1\) and computing the successor of \(a \cdot 10 + b \cdot 7 + c \cdot 1\) as follows: If \(c \geq 7\), the successor is \(a \cdot 10 + (b+1) \cdot 7 + (c-7) \cdot 1\); otherwise if \(c + 7b \geq 10\), the successor is \((a+1) \cdot 10 + (c + 7b - 10) \cdot 1\); otherwise stop.
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