COMMUNICATIONS IN STATISTICS
Theory and Methods
Vol. 33, No. 11, pp. 2575–2594, 2004

Computing the Distribution and Expected Value of the Concomitant Rank-Order Statistics

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ABSTRACT

This work gives a new representation of the distribution and expected value of the concomitant rank of order statistics. An advantage of this representation is its ability to extend without any complexity to the multivariate case. Moreover, it gives a new direct approach to compute an approximate formula for the distribution and expected value of the concomitant rank of order statistics. Finally, an upper bound is derived for the confidence level of the tolerance region of the original bivariate (resp., multivariate) d.f., from which the sample is drawn.

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DOI: 10.1081/STA-200037944 0361-0926 (Print); 1532-415X (Online)

Key Words: Order statistics; Concomitants; Ranking; Tolerance region.

Mathematics Subject Classification: Primary 60F05; Secondary 62E15, 62E20, 62G30.

1. INTRODUCTION

Let (X_j, Y_j), $j = 1, 2, \ldots, n$, be independent and identically distributed random vectors. We consider the order statistics $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ of the first component, and we denote the corresponding Y_js by $Y_{[1:n]}, Y_{[2:n]}, \ldots, Y_{[n:n]}$. That is, if $X_j = X_{r:n}$, then $Y_{[r:n]} = Y_j$. The sequence $Y_{[r:n]}$, $1 \le r \le n$ is called the concomitants of order statistics. These concomitants are of interest in selection and prediction problems based on the ranks of the Xs. The foundations of the concomitants of order statistics, which is applications oriented, were laid down in the works of David (1973), David and Galambos (1974), and Bhattacharya (1974). Further works are by O'Connell and David (1976), Sen (1976), David et al. (1977) and Yang (1977). The book by David (1970) has extensive material on the subject. An excellent review of work on concomitants of order statistics is available in David and Nagaraja (1998).

Galambos (1978, Chapter 5), pointed out that in many cases (such as normal distributions), the concomitants of the extremes among the Xs are not extremes among the Ys (with high probability). It is, therefore, an interesting question to investigate the rank $R_{r:n}$ of $Y_{[r:n]}$, which can be defined by $R_{r:n} := \sum_{j=1}^{n} I(Y_{[r:n]} - Y_j)$, where

$$I(x) = \begin{cases} 1, & \text{if } x \ge 0, \\ 0, & \text{if otherwise.} \end{cases}$$

The distribution of $R_{r:n}$ is obtained by David et al. (1977). Namely, let (X, Y) denote a generic (X_j, Y_j) and $F(x, y)$ be the distribution function (d.f.) of (X, Y) Furthermore, let $G(x, y) := P(X \ge x, Y \ge y)$ be the survival function of (X, Y). Finally, let $F_1(x)$ and $F_2(y)$ be the marginals of $F(x, y)$, while $G_1(x)$ and $G_2(y)$ be the marginals of $G(x, y)$. Then $A_{r:n}(s) := P(R_{r:n} = s)$ may be written in the form

$$\begin{aligned} A_{r:n}(s) = & \sum_{k=0\vee(r+s-n-1)}^{r-1\wedge s-1} \frac{n!}{(r-k-1)!k!(s-k-1)!(n-r-s+k+1)!} \\ & \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (F_1(x) - F(x,y))^{r-k-1} F^k(x,y) \\ & \times (F_2(y) - F(x,y))^{s-k-1} G^{n-r-s+k+1}(x,y) f(x,y)\,dx\,dy, \qquad (1) \end{aligned}$$

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where $f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$ is the probability density function (p.d.f.) of $F(x,y)$, $a \vee b =: \max(a,b)$ and $a \wedge b =: \min(a,b)$. From the exact distribution $A_{r:n}(s)$, the expected value of the rank $\mu_n(r) := E(R_{r:n})$ can evidently be computed. However, the evaluation of $A_{r:n}(s)$ and $\mu_n(r)$ is often tedious or even impossible without using some suitable numerical procedures and the aid of a computer.

Páez Borrallo and Zazo (1999) developed a formula to compute the bivariate factorized expected value from the knowledge of the joint cumulative d.f. of any random variables (r.v.s). In the present paper, by using a slight generalization of this formula, a general new representation of $A_{r:n}(s)$, and, consequently, $\mu_n(r)$, is given. This representation enables us to derive some new properties for the rank concomitants order statistics $R_{r:n}$ and its expected value $\mu_n(r)$. Moreover, it gives us a new direct approach to compute approximately $A_{r:n}(s)$ and $\mu_n(r)$ for any given bivariate d.f. which has an explicit dependence function. One of the advantages of this representation is its ability to extend without any complexity to the multivariate case. This extension to the multivariate case will be discussed in Sec. 2.2. Finally, as a noteworthy consequence of this new expression for $A_{r:n}(s)$, we conclude an upper bound (depends on the d.f.s of the ranks of the concomitants) for the confidence level of the tolerance region (constructing by the order statistics) of the d.f. of the random vector (X, Y), with a given tolerance proportion. This consequence is extended to the multivariate case in Theorem 6 and is supplemented with an illustrative example on the bivariate normal distributions with different correlation coefficients. In the rest of this section, we present the results of Páez Borrallo and Zazo (1999), but in a slightly general form, which is needed in what follows.

Let (Z, W) be a random vector distributed as $H(z, w)$. Let (a, c) and (b, d) be the vectors of the left end points and the right end points of $H(z, w)$, respectively. Then for continuous nonconstant and differentiable bivariate function $g(z, w)$, $E(g(Z, W))$ is classically defined as Stieltjes integral:

$$E(g(Z,W)) := \int_a^b \int_c^d g(z,w) d_{zw} H(z,w), \tag{2}$$

where, if the p.d.f. of the vector (Z, W) exists, then $d_{zw}H(z,w) = \frac{\partial^2 H(z,w)}{\partial z \partial w} = h(z,w)dzdw$, and therefore (2) represents the expected value as a Riemann integral. Páez Borrallo and Zazo (1999) presented a closed formula to compute any multivariate factorized expected value from the knowledge of the joint cumulative d.f. of any r.v. The next theorem gives a slight generalization of the result of Páez Borrallo and Zazo to compute the expected value in (2).

Theorem 1. *Let* $g(z, w)$ *be a bivariate function as defined above. Furthermore, let* $g_z(z, w) := \frac{\partial g(z,w)}{\partial z}$, $g_w(z, w) := \frac{\partial g(z,w)}{\partial w}$, *and* $g_{zw}(z, w) := \frac{\partial^2 g(z,w)}{\partial z \partial w}$. *Finally, let* $H_1(z) := H(z, d)$ *and* $H_2(w) := H(b, w)$. *Then*

$$E(g(Z, W)) = g(b, d) - \int_a^b g_z(z, d) H_1(z) dz - \int_c^d g_w(b, w) H_2(w) dw + \int_a^b \int_c^d g_{zw}(z, w) H(z, w) dz dw. \tag{3}$$

Moreover, if $g(b, d) = 1,\ g(a, w) = g(z, c) = g(a, c) = 0\ \forall z, w$, *then*

$$E(g(Z, W)) = \int_a^b \int_c^d g_{zw}(z, w) G(z, w) dz dw, \tag{4}$$

where $G(z, w) := P(Z \geq z, W \geq w) = 1 - H_1(z) - H_2(w) + H(z, w)$.

Proof. The proof is exactly the same as formulas (9) and (12), in Páez Borrallo and Zazo (1999), only with the obvious modification. □

2. MAIN RESULTS

2.1. Two Variates Case

If in (1) one makes the substitutions $u = F_1(x)$ and $v = F_2(y)$, i.e., if one employs the probability transform, then one obtains $F(x, y) = F(F_1^{-1}(u), F_2^{-1}(v)) := C(u, v)$, where $C = C(u, v)$ is the dependence function of F. The term ‘‘dependence function’’ is used by Galambos (1978, Chapter 5). Sklar (1959) introduced the general notion of this function and used the term ‘‘copula’’, which seems to be more current in the literature than the first term. This is a function that links the two-dimensional d.f. to its one-dimensional margins and is a continuous d.f. (whenever, each margin of F is continuous) on the unit square $[0, 1]^2$, with uniform margins. For example, the dependence function of the Morgenstern and Mardia's dependence functions (see Galambos, 1978) are, respectively, $C = uv(1 + \alpha(1 - u)(1 - v))$, $|\alpha| \leq 1$, and

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$C = u + v - 1 + ((1+u)^{-1} + (1+v)^{-1} - 1)^{-1}$. Now, it is easy to verify that $f_1(u)f_2(v)c(u,v) = f(F_1^{-1}(u), F_2^{-1}(v))$, where $c = c(u,v) := \frac{\partial^2 C(u,v)}{\partial u \partial v}$. Therefore, we get

$$A_{r:n}(s) = \sum_{k=0\vee(r+s-n-1)}^{r-1\wedge s-1} \frac{n!}{(r-k-1)!k!(s-k-1)!(n-r-s+k+1)!} \times \int_0^1 \int_0^1 (u - C(u,v))^{r-k-1} C^k(u,v)(v - C(u,v))^{s-k-1} \times (1 - u - v + C(u,v))^{n-r-s+k+1} c(u,v)dudv. \tag{5}$$

Now, if we envision (U_j, V_j), $j = 1, 2, \ldots, n$, as being identically independent distributed r.v.s with common joint d.f. $C(u,v)$, and write "$\stackrel{d}{=}$" to be read as "has the same d.f. as," then $X_{i:n} \stackrel{d}{=} F_1^{-1}(U_{i:n})$, $Y_{i:n} \stackrel{d}{=} F_1^{-1}(V_{i:n})$, and $(X_{i:n}, Y_{j:n}) \stackrel{d}{=} (F_1^{-1}(U_{i:n}),\ F_2^{-1}(V_{j:n}))$, where $U_{i:n}$ and $V_{i:n}$ are the ith order statistics of $U_1, U_2, \ldots, U_n$ and $V_1, V_2, \ldots, V_n$, respectively. The tools are now available to prove the following theorem.

Theorem 2. *For any bivariate d.f. $F(x,y)$ with dependence function C and for any $2 \le r \le n-1,\ 2 \le s \le n-1$, we have*

$$A_{r:n}(s) = n(E(C(U_{r:n-1}, V_{s:n-1})) - E(C(U_{r-1:n-1}, V_{s:n-1})) - E(C(U_{r:n-1}, V_{s-1:n-1})) + E(C(U_{r-1:n-1}, V_{s-1:n-1}))). \tag{6}$$

(6) *can be written in the form*

$$0A_{r:n}(s) = nE(P(U_{r-1:n-1} < U < U_{r:n-1}, V_{s-1:n-1} < V < V_{s:n-1})) = nE\left(\int_{V_{s-1:n-1}}^{V_{s:n-1}} \int_{U_{r-1:n-1}}^{U_{r:n-1}} c(u,v)dudv\right). \tag{7}$$

Moreover, for all $2 \le r \le n$,

$$A_{r:n}(1) = n(E(C(U_{r:n-1}, V_{1:n-1})) - E(C(U_{r-1:n-1}, V_{1:n-1}))) = nE(P(U_{r-1:n-1} < U < U_{r:n-1}, V < V_{1:n-1})) = nE\left(\int_0^{V_{1:n-1}} \int_{U_{r-1:n-1}}^{U_{r:n-1}} c(u,v)dudv\right) \tag{8}$$

and for all $2 \leq r \leq n-1$

$$
\begin{aligned}
A_{r:n}(n) &= 1 - A_{r:n}(1) - n(E(C(U_{r:n-1}, V_{n-1:n-1})) - E(C(U_{r-1:n-1}, V_{n-1:n-1})) \\
&\quad - E(C(U_{r:n-1}, V_{1:n-1})) + E(C(U_{r-1:n-1}, V_{1:n-1}))) \\
&= 1 - A_{r:n}(1) - nE(P(U_{r-1:n-1} < U < U_{r:n-1}, V_{1:n-1} < V < V_{n-1:n-1})) \\
&= 1 - A_{r:n}(1) - nE\left(\int_{V_{1:n-1}}^{V_{n-1:n-1}} \int_{U_{r-1:n-1}}^{U_{r:n-1}} c(u,v)\,du\,dv\right). \qquad (9)
\end{aligned}
$$

Proof. Let us consider the expectation $E(C(U_{r:n}, V_{s:n}))$. In view of Theorem 1 and (5), we get

$$E(C(U_{r:n}, V_{s:n})) = \int_0^1 \int_0^1 c(u,v) G_{r,s:n}(u,v)\,du\,dv, \qquad (10)$$

where (see, Barakat, 1990, 1998)

$$
\begin{aligned}
G_{r,s:n}(u,v) &:= P(U_{r:n} \geq u, V_{s:n} \geq v) \\
&= \sum_{i=0}^{r-1} \sum_{j=0}^{s-1} \sum_{k=0 \vee (i+j-n)}^{i \wedge j} \frac{n!}{(i-k)!k!(j-k)!(n-i-j+k)!} \\
&\quad \times (u-C)^{i-k} C^k (v-C)^{j-k} (1-u-v+C)^{n-i-j+k} \qquad (11)
\end{aligned}
$$

Therefore, $(n+1)E(C(U_{r:n}, V_{s:n})) = \sum_{i=0}^{r-1} \sum_{j=0}^{s-1} A_{i+1:n+1}(j+1)$, which implies

$$nE(C(U_{r:n-1}, V_{s:n-1})) = \sum_{i=0}^{r-1} \sum_{j=0}^{s-1} A_{i+1:n}(j+1) = \sum_{i=1}^{r} \sum_{j=1}^{s} A_{i:n}(j). \qquad (12)$$

On the other hand, it is easy to show that

$$
\begin{aligned}
A_{r:n}(s) &= \sum_{j=1}^{s} A_{r:n}(j) - \sum_{j=1}^{s-1} A_{r:n}(j) \\
&= \sum_{i=1}^{r} \sum_{j=1}^{s} A_{i:n}(j) - \sum_{i=1}^{r-1} \sum_{j=1}^{s} A_{i:n}(j) \\
&\quad - \sum_{i=1}^{r} \sum_{j=1}^{s-1} A_{i:n}(j) + \sum_{i=1}^{r-1} \sum_{j=1}^{s-1} A_{i:n}(j). \qquad (13)
\end{aligned}
$$

By combining (12) with (13), (6) follows immediately. Now, from (10) and (11), we get

$$E(C(U_{r:n}, V_{1:n})) = \int_0^1 \int_0^1 c(u,v) G_{r,1:n}(u,v)\,du\,dv,$$

where

$$G_{r,1:n}(u,v) = \sum_{i=0}^{r-1} \frac{n!}{i!(n-i)!} (u-C)^i (1-u-v+C)^{n-i}.$$

Therefore,

$$nE(C(U_{r:n-1}, V_{1:n-1})) = \sum_{i=0}^{r-1} A_{i+1:n}(1) = \sum_{i=1}^{r} A_{i:n}(1).$$

Moreover, $A_{r:n}(1) = \sum_{i=1}^{r} A_{i:n}(1) - \sum_{i=1}^{r-1} A_{i:n}(1)$, which implies that

$$A_{r:n}(1) = n(E(C(U_{r:n-1}, V_{1:n-1})) - E(C(U_{r-1:n-1}, V_{1:n-1}))),$$

for all $2 \le r \le n$. This completes the proof of (8). To prove (9), we notice that, for any fixed value of r, $\sum_{s=1}^{n} A_{r:n}(s) = 1$. Therefore, $A_{r:n}(n) = 1 - A_{r:n}(1) - \sum_{s=2}^{n-1} A_{r:n}(s)$. Furthermore, on account of (6) and (8), we get

$$\begin{aligned} A_{r:n}(n) &= 1 - A_{r:n}(1) - nE\left(\int_{U_{r-1:n-1}}^{U_{r:n-1}} \sum_{s=2}^{n-1} \int_{V_{s-1:n-1}}^{V_{s:n-1}} c(u,v)\,du\,dv\right) \\ &= 1 - A_{r:n}(1) - nE\left(\int_{V_{1:n-1}}^{V_{n-1:n-1}} \int_{U_{r-1:n-1}}^{U_{r:n-1}} c(u,v)\,du\,dv\right). \end{aligned}$$

The theorem is established. □

Theorem 3. *For any bivariate d.f. $F(x, y)$ with dependence function C and for any $2 \le r \le n-1$, we have*

$$\begin{aligned} \mu_n(r) &= 1 + n(n-2)(E(C(U_{r:n-1}, V_{n-1:n-1})) \\ &\quad - E(C(U_{r-1:n-1}, V_{n-1:n-1}))) + n\sum_{s=2}^{n-1}(E(C(U_{r-1:n-1}, V_{s-1:n-1})) \\ &\quad - E(C(U_{r:n-1}, V_{s-1:n-1}))) + (n-1)A_{r:n}(n) \\ &= 1 + n\sum_{s=2}^{n-1} E\left(\int_{V_{s-1:n-1}}^{V_{n-1:n-1}} \int_{U_{r-1:n-1}}^{U_{r:n-1}} c(u,v)\,du\,dv\right) + (n-1)A_{r:n}(n). \end{aligned} \tag{14}$$

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Proof. We first observe that

$$\mu_n(r) = \sum_{s=1}^{n} sA_{r:n}(s) = A_{r:n}(1) + nA_{r:n}(n) + \sum_{s=2}^{n-1} sA_{r:n}(s).$$

By using (9), we conclude

$$\mu_n(r) = 1 - nE\left(\int_{V_{1:n-1}}^{V_{n-1:n-1}} \int_{U_{r-1:n-1}}^{U_{r:n-1}} c(u,v)dudv\right) + \sum_{s=2}^{n-1} sA_{r:n}(s) + (n-1)A_{r:n}(n). \tag{15}$$

In addition, from (7), we get

$$\begin{aligned}
\sum_{s=2}^{n-1} sA_{r:n}(s) &= n\sum_{s=2}^{n-1} sE\left(\int_{V_{s-1:n-1}}^{V_{s:n-1}} \int_{U_{r-1:n-1}}^{U_{r:n-1}} c(u,v)dudv\right) \\
&= nE\left(\int_{U_{r-1:n-1}}^{U_{r:n-1}} \left(2\left(\int_{V_{1:n-1}}^{V_{2:n-1}} + \int_{V_{2:n-1}}^{V_{3:n-1}} + \cdots + \int_{V_{n-2:n-1}}^{V_{n-1:n-1}}\right)\right.\right. \\
&\quad + \int_{V_{2:n-1}}^{V_{3:n-1}} + 2\int_{V_{3:n-1}}^{V_{4:n-1}} + 3\int_{V_{4:n-1}}^{V_{5:n-1}} + \cdots + (n-3) \\
&\quad \left.\left.\times \int_{V_{n-2-1:n-1}}^{V_{n-1:n-1}}\right) c(u,v)dudv\right) \\
&= nE\left(\int_{U_{r-1:n-1}}^{U_{r:n-1}} \left(2\left(\int_{V_{1:n-1}}^{V_{2:n-1}} + \cdots + \int_{V_{n-2:n-1}}^{V_{n-1:n-1}}\right)\right.\right. \\
&\quad + \left(\int_{V_{2:n-1}}^{V_{3:n-1}} + \cdots + \int_{V_{n-2:n-1}}^{V_{n-1:n-1}}\right) + \cdots \\
&\quad \left.\left. + \int_{V_{n-2:n-1}}^{V_{n-1:n-1}}\right) c(u,v)dudv\right) \\
&= nE\left(\int_{U_{r-1:n-1}}^{U_{r:n-1}} \left(2\int_{V_{1:n-1}}^{V_{n-1:n-1}} + \int_{V_{2:n-1}}^{V_{n-1:n-1}}\right.\right. \\
&\quad \left.\left. + \int_{V_{3:n-1}}^{V_{n-1:n-1}} + \cdots + \int_{V_{n-2:n-1}}^{V_{n-1:n-1}}\right) c(u,v)dudv\right) \\
&= n\sum_{s=2}^{n-1} E\left(\int_{V_{s-1:n-1}}^{V_{n-1:n-1}} \int_{U_{r-1:n-1}}^{U_{r:n-1}} c(u,v)dudv\right) \\
&\quad + nE\left(\int_{V_{1:n-1}}^{V_{n-1:n-1}} \int_{U_{r-1:n-1}}^{U_{r:n-1}} c(u,v)dudv\right).
\end{aligned} \tag{16}$$

Combining (15) with (16), we get the desired result. The proof is completed. □

Example 1. Let X and Y be two bivariate standard normal r.v.s with correlation ρ. Then, in view of Theorem 2, we get

$$A_{r:n}(s) = nE\left(\int_{\Phi^{-1}(V_{s-1:n-1})}^{\Phi^{-1}(V_{s:n-1})}\int_{\Phi^{-1}(U_{r-1:n-1})}^{\Phi^{-1}(U_{r:n-1})} f_\rho(u,v)dudv\right), \quad 2 \le r,\ s \le n-1, \tag{17}$$

$$A_{r:n}(1) = nE\left(\int_{-\infty}^{\Phi^{-1}(V_{1:n-1})}\int_{\Phi^{-1}(U_{r-1:n-1})}^{\Phi^{-1}(U_{r:n-1})} f_\rho(u,v)dudv\right), \quad 2 \le r \le n \tag{18}$$

and

$$A_{r:n}(n) = 1 - A_{r:n}(1) - nE\left(\int_{\Phi^{-1}(V_{1:n-1})}^{\Phi^{-1}(V_{n-1:n-1})}\int_{\Phi^{-1}(U_{r-1:n-1})}^{\Phi^{-1}(U_{r:n-1})} f_\rho(u,v)dudv\right), \quad 2 \le r \le n-1, \tag{19}$$

where $\Phi(\cdot)$ refers to the standard normal d.f., and $f_\rho(u,v)$ denotes the bivariate normal density function with correlation ρ. By using the well-known method of series approximation (see, for instance, Arnold et al., 1992), (17), (18) and (19) provide us an approximate formula for $A_{r:n}(s)$, $1 \le s \le n$, to any desired accuracy. For example, by using the fact that $E(U_{i:n}) = \frac{i}{n+1}$, $E(V_{j:n}) = \frac{j}{n+1}$, $\forall 1 \le i,\ j \le n$, we get from (17), (18) and (19)

$$A_{r:n}(s) \approx \hat{A}^o_{r:n}(s) := n\left(\int_{\Phi^{-1}(\frac{s-1}{n})}^{\Phi^{-1}(\frac{s}{n})}\int_{\Phi^{-1}(\frac{r-1}{n})}^{\Phi^{-1}(\frac{r}{n})} f_\rho(u,v)dudv\right), \quad 2 \le r \le n-1,$$

$$A_{r:n}(1) \approx \hat{A}^o_{r:n}(1) := n\left(\int_{-\infty}^{\Phi^{-1}(\frac{1}{n})}\int_{\Phi^{-1}(\frac{r-1}{n})}^{\Phi^{-1}(\frac{r}{n})} f_\rho(u,v)dudv\right), \quad 2 \le r \le n,$$

and

$$A_{r:n}(n) \approx \hat{A}^o_{r:n}(n) := 1 - \hat{A}^o_{r:n}(1) - n\left(\int_{\Phi^{-1}(\frac{1}{n})}^{\Phi^{-1}(\frac{n-1}{n})}\int_{\Phi^{-1}(\frac{r-1}{n})}^{\Phi^{-1}(\frac{r}{n})} f_\rho(u,v)dudv\right), \quad 2 \le r \le n-1,$$

which are considered primary approximate values of $A_{r:n}(s), 1 \leq s \leq n$. Clearly, more accurate values can be obtained by expanding the function

$$C(u, v) = \int_{-\infty}^{\Phi^{-1}(v)} \int_{-\infty}^{\Phi^{-1}(u)} f_\rho(x, y)dxdy$$

in (7), (8) and (9) in a Taylor series around the point $\left(E(U_{i:m}) = \frac{i}{m+1}, E(V_{j:m}) = \frac{j}{m+1}\right)$, where $(i, j) = (r, s), (r-1, s), (r, s-1), (r-1, s-1)$, $m = n-1$ in (7); $(i, j) = (r, 1), (r-1, 1), m = n$, in (8) and $(i, j) = (r, n-1), (r-1, n-1), (r, 1), (r-1, 1), m = n-1$, in (9). On the other hand, a primary approximate value of $\mu_n(r)$ is given by

$$\hat{\mu}_n^o(r) := 1 + n\sum_{s=2}^{n-1}\left(\int_{\Phi^{-1}\left(\frac{s-1}{n}\right)}^{\Phi^{-1}\left(\frac{n-1}{n}\right)} \int_{\Phi^{-1}\left(\frac{r-1}{n}\right)}^{\Phi^{-1}\left(\frac{r}{n}\right)} f_\rho(u, v)dudv\right) + (n-1)\hat{A}^{o}_{r:n}(n).$$

The Table 1 gives a comparison between some primary approximate values of $A_{r:n}(s)$ $(\hat{A}^o_{r:n}(s))$ and the corresponding actual values of $A_{r:n}(s)$ (denoted by π_{rs}) computed in David et al. (1977). Although we use a primary approximate, this table shows that the suggested approximations

Table 1. Comparison between some values of $\hat{A}^o_{r:n}(s)$ and the corresponding values of π_{rs} for $n = 9$.

		$\rho = 0.1$		$\rho = 0.2$		$\rho = 0.3$		$\rho = 0.7$		$\rho = 0.9$	
r	s	$\hat{A}^o_{r:n}(s)$	π_{rs}	$\hat{A}^o_{r:n}(s)$	π_{rs}	$\hat{A}^o_{r:n}(s)$	π_{rs}	$\hat{A}^o_{r:n}(s)$	π_{rs}	$\hat{A}^o_{r:n}(s)$	π_{rs}
9	8	0.1305	0.1285	0.1494	0.1459	0.1674	0.1631	0.2295	0.2185	0.2250	0.2033
	5	0.1098	0.1100	0.1061	0.1069	0.0999	0.1015	0.0441	0.0546	0.0027	0.0149
	4	0.1044	0.1051	0.0954	0.0975	0.0846	0.0877	0.0234	0.0345	0.0000	0.0059
	1	0.0801	0.0856	0.0549	0.0635	0.0351	0.0451	0.0000	0.0041	0.0000	0.0001
8	8	0.1242	0.1220	0.1350	0.1334	0.1474	0.1458	0.2313	0.2355	0.3825	0.3992
	5	0.1125	0.1109	0.1125	0.1108	0.1125	0.1102	0.0999	0.0941	0.0396	0.0511
	4	0.1089	0.1078	0.1053	0.1042	0.1017	0.0999	0.0657	0.0656	0.0099	0.0233
	1	0.0945	0.0946	0.0756	0.0773	0.0576	0.0612	0.0045	0.0107	0.0000	0.0006
7	7	0.1125	0.1154	0.1170	0.1205	0.1179	0.1266	0.1737	0.1824	0.2835	0.3104
	5	0.1098	0.1114	0.1116	0.1124	0.1134	0.1141	0.1296	0.1253	0.1224	0.1112
	4	0.1080	0.1094	0.1071	0.1082	0.1062	0.1071	0.0990	0.0955	0.0495	0.0573
	1	0.0981	0.1000	0.0846	0.0877	0.0702	0.0747	0.0108	0.0205	0.0000	0.0021
6	6	0.1116	0.1126	0.1134	0.1150	0.1170	0.1188	0.1584	0.1625	0.2547	0.2742
	4	0.1089	0.1107	0.1098	0.1110	0.1107	0.1122	0.1269	0.1234	0.1242	0.1127
	3	0.1080	0.1095	0.1071	0.1081	0.1062	0.1071	0.0990	0.0955	0.0495	0.0573
	1	0.1044	0.1048	0.0954	0.0973	0.0846	0.0877	0.0234	0.0345	0.0000	0.0059
5	5	0.1116	0.1117	0.1134	0.1134	0.1161	0.1165	0.1548	0.1570	0.2484	0.2640

are close to the actual values given by David et al. (1977), especially for small values of ρ. In calculating a double integral of type $\int_a^b \int_c^d f_\rho(u,v)dudv$, required in the evaluation of $\hat{A}^o_{r:n}(s)$, we have used the program Bivar1c.exe of the National Institute of Occupational Health, Denmark (home page: http://www.amidk).

Corollary 1. *Using the fact that X and Y are independent if and only if* $C(u,v) = uv$ *(see Nelsen,* 1999*), one can see that if X and Y are independent, then* $A_{r:n}(1) = \frac{1}{(n+1)^2}$, $A_{r:n}(s) = \frac{1}{n}$, $2 \le r$, $s \le n-1$, *and* $A_{r:n}(n) = 1 - \frac{1}{(n+1)^2} - \frac{1}{n}$. *Moreover,* $\mu_n(r) = \frac{3}{2}n - \frac{5}{2} + o\left(\frac{1}{n}\right)$.

Corollary 2 (Universal Bounds for $A_{r:n}(s)$ and $\mu_n(r)$)**.** *Using the well-known Fréchet bounds* $x + y - 1 \vee 0 \le C(x,y) \le x \wedge y$, *valid for all dependent functions and arbitrary* $0 \le x$, $y \le 1$, *one can derive lower bounds* $\underline{A}_{r:n}(s)$ *and* $\underline{\mu}_n(r)$ *as well as upper bounds* $\overline{A}_{r:n}(s)$ *and* $\overline{\mu}_n(r)$ *for* $A_{r:n}(s)$, $1 \le s \le n$ *and* $\mu_n(r)$, $2 \le r \le n-1$, *respectively. For example,* $\underline{\mu}_n(r) \le \mu_n(r) \le \overline{\mu}_n(r)$, $2 \le r \le n-1$, *where*

$$\begin{aligned}
\underline{\mu}_n(r) &= 1 + n(n-2)E(U_{r:n-1} + V_{n-1:n-1} - 1 \vee 0) \\
&\quad - n(n-2)E(U_{r-1:n-1} \wedge V_{n-1:n-1}) \\
&\quad + n\sum_{s=2}^{n-1}(E(U_{r-1:n-1} + V_{s-1:n-1} - 1 \vee 0) \\
&\quad - E(U_{r:n-1} \wedge V_{s-1:n-1})) + (n-1)\underline{A}_{r:n}(n),
\end{aligned}$$

$$\begin{aligned}
\overline{\mu}_n(r) &= 1 + n(n-2)E(U_{r:n-1} \wedge V_{n-1:n-1}) \\
&\quad - n(n-2)E(U_{r-1:n-1} + V_{n-1:n-1} - 1 \vee 0) \\
&\quad + n\sum_{s=2}^{n-1}(E(U_{r-1:n-1} \wedge V_{s-1:n-1}) \\
&\quad - E(U_{r:n-1} + V_{s-1:n-1} - 1 \vee 0)) + (n-1)\overline{A}_{r:n}(n),
\end{aligned}$$

$$\begin{aligned}
\underline{A}_{r:n}(n) &= 1 - \overline{A}_{r:n}(1) - n(E(U_{r:n-1} \wedge V_{n-1:n-1}) \\
&\quad - E(U_{r-1:n-1} + V_{n-1:n-1} - 1 \vee 0) \\
&\quad - E(U_{r:n-1} + V_{1:n-1} - 1 \vee 0) + E(U_{r-1:n-1} \wedge V_{1:n-1})),
\end{aligned}$$

$$\begin{aligned}
\overline{A}_{r:n}(n) &= 1 - \underline{A}_{r:n}(1) - n(E(U_{r:n-1} + V_{n-1:n-1} - 1 \vee 0) \\
&\quad - E(U_{r-1:n-1} \wedge V_{n-1:n-1}) - E(U_{r:n-1} \wedge V_{1:n-1}) \\
&\quad + E(U_{r-1:n-1} + V_{1:n-1} - 1 \vee 0)), \\
\underline{A}_{r:n}(1) &= E(U_{r:n} + V_{1:n} - 1 \vee 0) - E(U_{r-1:n} \wedge V_{1:n})
\end{aligned}$$

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and

$$\overline{A}_{r:n}(1) = E(U_{r:n} \wedge V_{1:n}) - E(U_{r-1:n} + V_{1:n} - 1 \vee 0).$$

In the above-stated relations, given the possibility that $\underline{A}_{r:n}(t) < 0$ *and* $\overline{A}_{r:n}(t) > 1, t = 1, n$; *in this case, we must take* $\underline{A}_{r:n}(t) = 0$ *and* $\bar{A}_{r:n}(t) = 1$, *respectively.*

Many facts and useful properties of $A_{r:n}(s)$ [and, consequently, $\mu_n(r)$] can be directly derived by using the representation given by (6) through (9). In the sequel, we assume that X and Y are r.v.s with continuous joint d.f. $F(x, y)$, p.d.f. $f(x)$, and (unique) dependence function $C(u, v)$. Moreover, we will use the symbols $A_{r:n}^{(X,Y)}(s)$ and $\mu_n^{(X,Y)}(r)$ instead of $A_{r:n}(s)$ and $\mu_n(r)$, if we need to emphasize the role of the random vector (X, Y).

Remark 1 (See Relations 1 and 2 of David, 1977)**.** On account of the known properties of the dependence function, and by using Theorems 2.2 and 2.4, we deduce the following

(1) If X and Y are exchangeable, i.e., $F(x, y) = F(y, x)$, $\forall x, y$, then $A_{r:n}(s) = A_{s:n}(r)$ and $\mu_n(r) = \mu_n(s)$.
(2) If $f(x, y) = f(-x, -y)$, then $A_{r:n}(s) = A_{n-r+1:n}(n - s + 1)$.

Remark 2. By using Theorem 2 and Theorem 2 of Schweizer and Wolff (1981), we deduce that, if $Y = \phi(X)$ a.s., where ϕ is strictly increasing (resp., decreasing) a.s. on the range of X, then $A_{r:n}(s), 1 \le s \le n$, and $\mu_n(r)$ are given by (6) through (8) and (14), respectively, by replacing the function $C(u, v)$ by $u \wedge v$ (resp. $u + v - 1 \vee 0$).

Theorem 4. *Let X and Y be as in the above-mentioned remarks. Then*

(1) *If* ϕ_1 *and* ϕ_2 *are strictly increasing a.s. on Range X and Range Y, respectively, then* $A_{r:n}^{(\phi_1(X),\phi_2(Y))}(s) = A_{r:n}^{(X,Y)}(s)$, $1 \le s \le n$, *and* $\mu_n^{(\phi_1(X),\phi_2(Y))}(r) = \mu_n^{(X,Y)}(r)$.
(2) *If* ϕ_1 *and* ϕ_2 *are strictly decreasing a.s. on Range X and Range Y, respectively, then, for all* $2 \le s \le n-1$, $A_{r:n}^{(\phi_1(X),Y)}(s)$, $A_{r:n}^{(X,\phi_2(Y))}(s)$, *and* $A_{r:n}^{(\phi_1(X),\phi_2(Y))}(s)$ *are given by* (6) *by replacing the function* $C(u, v)$ *by* $-C(1-u, v)$, $-C(u, 1-v)$, *and* $C(1-u, 1-v)$, *respectively. Moreover,*

$$A_{r:n}^{(\phi_1(X),Y)}(1) = -E(C(1-U_{r:n}, V_{1:n})) + E(C(1-U_{r-1:n}, V_{1:n})),$$

$$A_{r:n}^{(X,\phi_2(Y))}(1) = \frac{1}{n+1} - E(C(U_{r:n}, 1-V_{1:n})) + E(C(U_{r-1:n}, 1-V_{1:n}))$$

and

$$A_{r:n}^{(\phi_1(X),\phi_2(Y))}(1) = \frac{1}{n+1} - E(C(1-U_{r:n}, 1-V_{1:n})) + E(C(1-U_{r-1:n}, 1-V_{1:n})).$$

Finally, $A_{r:n}^{(\phi_1(X),Y)}(n)$, $A_{r:n}^{(X,\phi_2(Y))}(n)$, *and* $A_{r:n}^{(\phi_1(X),\phi_2(Y))}(n)$ *are given by* (9) *at first by replacing* $A_{r:n}^{(X,Y)}(1)$, *in* (9) *by* $A_{r:n}^{(\phi_1(X),Y)}(1)$, $A_{r:n}^{(X,\phi_2(Y))}(1)$ *and* $A_{r:n}^{(\phi_1(X),\phi_2(Y))}(1)$, *respectively, and then by replacing the function* $C(u,v)$ *by* $-C(1-u,v), u-C(u,1-v)$, *and* $u+C(1-u,1-v)$, *respectively.*

(3) *If* L_1 *and* L_2 *are any given continuous distributions, then* $A_{r:n}^{(L_1^{-1}(F_1(X)),L_2^{-1}(F_2(Y)))}(s) = A_{r:n}^{(X,Y)}(s)$, $1 \le s \le n$ *and* $\mu_n^{(L_1^{-1}(F_1(X)),L_2^{-1}(F_2(Y)))}(r) = \mu_n^{(X,Y)}(r)$.

Proof. Using Theorems 2 and 3, the proofs of (1), (2), and (3) are followed from (i), (ii), and (iii) of Theorem 3 of Schweizer and Wolff (1981), respectively, and a series of straightforward verifications. □

2.2. Multivariate Extension

Suppose that associated with each X there are m variates $Y_j (j = 1, 2, \ldots, m)$, i.e., we have n independent sets of variates $(X_i, Y_{1i}, \ldots, Y_{im})$, with common joint d.f. $F(x, y_1, \ldots, y_m)$ and dependence function $C(u, v_1, v_2, \ldots, v_m)$. Triggered by a problem in hydrology, this situation has recently been intensively studied, especially when the $m+1$ variates have a multivariate normal distribution (see David and Nagaraja, 1998). Let $Y_{j[r:n]}$ denote Y_{ji}, which is paired with $X_{r:n}$. Our aim in this subsection is to investigate the joint d.f. $A_{r:n}(s_1, s_2, \ldots, s_m) := P(R_{1r:n} = s_1, R_{2r:n} = s_2, \ldots, R_{mr:n} = s_m)$. In principal, $A_{r:n}(s_1, s_2, \ldots, s_m)$ may be derived by using the same method of David et al. (1977) by which they derived $A_{r:n}(s)$. However, in this case, the situation becomes tedious and complicated, for example, let $\lambda(X_i)$ denote the rank of X_i among the nXs, with a similar meaning for $\lambda(Y_{ji})$, $j = 1, 2$, we have for $r, s = 1, 2, \ldots, n$,

$$A_{r:n}(s_1, s_2) = \sum_{i=1}^{n} P(\lambda(X_i) = r, \lambda(Y_{1i}) = s_1, \lambda(Y_{2i}) = s_2) = nP(\lambda(X_n) = r, \lambda(Y_{1n}) = s_1, \lambda(Y_{2n}) = s_2),$$

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where the subscript is taken to be n for definiteness. Let $F(x, y_1, y_2) := P(X < x, Y_1 < y_1, Y_2 < y_2)$ and $f(x, y_1, y_2)$ be the p.d.f. of $F(x, y_1, y_2)$. Let $F_1(x)$, $F_2(y_1)$, and $F_3(y_2)$ be the marginals of $F(x, y_1, y_2)$. For convenience, let $\underline{x} := (x_1, x_2, x_3) := (x, y_1, y_2)$. Then, $F(\underline{x}) := F(x, y_1, y_2)$ and $f(\underline{x}) := f(x, y_1, y_2)$. Finally, for $1 \le i_1 \ne i_2 \ne i_3 \le 3$, let $F_{i_1 i_2}(x_{i_1}, x_{i_2}) = \lim_{x_{i_3}\to\infty} F(\underline{x})$. It can now be shown that after some routine calculations,

$$A_{r:n}(s_1, s_2) = \sum_{j_1=0}^{s_1-1\wedge s_2-1} \sum_{j_2=0}^{r-1\wedge s_2-1} \sum_{j_3=0}^{r-1\wedge s_1-1} \sum_{l=\underline{\ell}}^{\overline{\ell}} \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{j_1,j_2,j_3;l:n}(\underline{x}) f(\underline{x}) dx_1 dx_2 dx_3,$$

where $\underline{\ell} := \max(0; j_2 + j_3 - r + 1; j_1 + j_3 - s_1 + 1; j_1 + j_2 - s_2 + 1)$, $\overline{\ell} := \min(j_1; j_2; j_3; j_1 + j_2 + j_3 - r - s_1 - s_2 + n + 2)$, and

$$K_{j_1,j_2,j_3;l:n}(\underline{x}) = \frac{n! F^l(\underline{x}) \left(1 - \sum_{i=1}^{3} F_i(x_i) + \sum\sum_{1\le i<j\le 3} F_{ij}(x_i, x_j) - F(\underline{x})\right)^{n-r-s_1-s_2+j_1+j_2+j_3-l+2}}{l!(n - r - s_1 - s_2 + j_1 + j_2 + j_3 - l + 2)!}$$

$$\times \prod_{\substack{t=1 \\ 1\le t_1<t_2\le 3 \\ t_1,t_2\ne t}}^{3} \frac{(F_{t_1 t_2}(x_{t_1}, x_{t_2}) - F(\underline{x}))^{j_t - r} \times (F_t(x_t) - F_{tt_1}(x_t, x_{t_1}) - F_{tt_2}(x_t, x_{t_1}) + F(\underline{x}))^{i_t - j_{t_1} - j_{t_2} + l - 1}}{(j_t - l)!(i_t - j_{t_1} - j_{t_2} + l - 1)!},$$

where in the above formula we adopt $(i_1, i_2, i_3) := (r, s_1, s_2)$. Obviously, the complexity of the preceding formula retards any expected benefit resulting from it. However, by using the same argument applied in Theorem 2 we can represent it in usable form. Namely, we have

$$E(C(U_{r:n}, V_{1,s_1:n}, V_{2,s_2:n})) = \int_0^1\int_0^1\int_0^1 c(u, v_1, v_2) G_{r,s_1,s_2:n}(u, v_1, v_2) du dv_1 dv_2,$$

where $c(u, v_1, v_2) = \frac{\partial^3 C(u,v_1,v_2)}{\partial u \partial v_1 \partial v_2}$, $C(u, v_1, v_2)$ is the dependence function of $F(x, y_1, y_2)$; $X_{r:n} \underset{=}{d} F_1^{-1}(U_{r:n})$; $Y_{t-1,s_{t-1}:n} \underset{=}{d} F_t^{-1}(V_{t-1,s_{t-1}:n})$, $t = 2, 3$; $Y_{t,s_t:n}$ is the s_tth order statistic of $(Y_{1t}, Y_{2t}, \ldots, Y_{nt}), t = 1, 2,$ respectively; and

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finally (see Barakat et al., 2004),

$$G_{r,s_1,s_2:n}(u,v_1,v_2) = P\big(U_{r:n} \geq u, V_{1,s_1:n} \geq v_1, V_{2,s_2:n} \geq v_2\big) = \sum_{i_1=0}^{r-1}\sum_{i_2=0}^{s_1-1}\sum_{i_3=0}^{s_2-1} I_{i_1,i_2,i_3:n}(u,v_1,v_2),$$

where

$$I_{i_1,i_2,i_3:n}(u,v_1,v_2) := \sum_{j_1=0}^{i_2\wedge i_3}\sum_{j_2=0}^{i_3\wedge i_1}\sum_{j_3=0}^{i_1\wedge i_2}\sum_{l=\underline{\ell}'}^{\overline{\ell}'} K^*_{i_1,i_2,i_3;j_1,j_2,j_3;l:n}(u,v_1,v_2),$$

and

$$K^*_{i_1,i_2,i_3;j_1,j_2,j_3;l:n}(u,v_1,v_2) = \frac{n!F^r(\underline{x})\Big(1-\sum_{i=1}^{3}F_i(x_i) + \sum\sum_{1\leq i<j\leq 3}F_{ij}(x_i,x_j) - F(\underline{x})\Big)^{n-i_1-i_2-i_3+j_1+j_2+j_3-l}}{l!(n-i_1-i_2-i_3+j_1+j_2+j_3-l)!} \times \prod_{\substack{t=1\\ 1\leq t_1<t_2\leq 3\\ t_1,t_2\neq t}}^{3} \frac{(F_{t_1t_2}(x_{t_1},x_{t_2})-F(\underline{x}))^{j_t-l} \times (F_{t(x_t)} - F_{tt_1}(x_t,x_{t_1}) - F_{tt_2}(x_t,x_{t_1}) + F(\underline{x}))^{i_t-j_{t_1}-j_{t_2}+l}}{(j_t-l)!(i_t-j_{t_1}-j_{t_2}+l)!},$$

where in the above-mentioned formula we adopt $(x_1,x_2,x_3) := (u,v_1,v_2)$. Therefore,

$$(n+1)E(C(U_{r:n-1},V_{1,s_1:n-1},V_{2,s_2:n-1})) = \sum_{i_1=0}^{r-1}\sum_{i_2=0}^{s_1-1}\sum_{i_3=0}^{s_2-1} A_{i_1+1:n+1}(i_2+1,i_3+1),$$

which, after some routine calculations, implies that

$$A_{r:n}(s_1,s_2) = nE\bigg(\int_{V_{2,s_2-1:n-1}}^{V_{2,s_2:n-1}}\int_{V_{1,s_1-1:n-1}}^{V_{1,s_1:n-1}}\int_{U_{r-1:n-1}}^{U_{r:n-1}} c(u,v_1,v_2)\,du\,dv_1\,dv_2\bigg), \tag{20}$$

for all $2 \leq r, s_1, s_2 \leq n-1$.

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The representation (20) can be extended to $m+1$ variates, as follows;

$$A_{r:n}(s_1, s_2, \ldots, s_m) = nE\bigg(\int_{V_{m,s_m-1:n-1}}^{V_{m,s_m:n-1}} \int_{V_{m-1,s_{m-1}-1:n-1}}^{V_{m-1,s_{m-1}:n-1}} \cdots \int_{V_{1,s_1-1:n-1}}^{V_{1,s_1:n-1}} \times \int_{U_{r-1:n-1}}^{U_{r:n-1}} c(u, v_1, v_2, \ldots, v_m)\, du dv_1 dv_2 \cdots dv_m \bigg), \quad (21)$$

for all $2 \le r,\ s_1, s_2, \ldots, s_m \le n-1$, where $c(u, v_1, v_2, \ldots, v_m) = \frac{\partial^{m+1} C(u,v_1,v_2,\ldots,v_m)}{\partial u \partial v_1 \partial v_2 \cdots \partial v_m}$, $X_{r:n} \underline{\underline{d}} F_1^{-1}(U_{r:n})$; $Y_{t-1,s_{t-1}:n} \underline{\underline{d}} F_t^{-1}(V_{t-1,s_{t-1}:n})$; $t = 2, \ldots, m+1$ and $Y_{t,s_t:n}$ is the s_tth order statistic of $(Y_{1t}, Y_{2t}, \ldots, Y_{nt})$, $t = 1, 2, \ldots, m$, respectively.

By using the method of series approximation, (21) provides us an approximate formula for $A_{r:n}(s_1, s_2, \ldots, s_m), 2 \le r, s_1, \ldots, s_m \le n-1$, to any desired accuracy. For example,

$$A_{r:n}(s_1, s_2, \ldots, s_m) \approx \widehat{A}^{o}_{r:n}(s_1, s_2, \ldots, s_m)$$
$$:= n \int_{\frac{s_{m-1}}{n}}^{\frac{s_m}{n}} \int_{\frac{s_{m-1}-1}{n}}^{\frac{s_{m-1}}{n}} \cdots \int_{\frac{s_1-1}{n}}^{\frac{s_1}{n}} \int_{\frac{r-1}{n}}^{\frac{r}{n}} c(u, v_1, v_2, \cdots, v_m)\, du dv_1 dv_2 \cdots dv_m$$

is a primary approximate value of $A_{r:n}(s_1, s_2, \ldots, s_m)$, $2 \le r, s_1, s_2, \ldots, s_m \le n$.

2.3. Upper Bound of the Confidence Level of the Tolerance Region

A tolerance region for the continuous d.f. $F(x, y)$ with tolerance coefficient γ is a random region such that the probability is γ that this random region covers or includes at least a specified percentage (100β) of the distribution. If this region is a rectangle for which its vertices are order statistics of a random sample of size n, the tolerance region (rectangle) $R_{r_1,s_1;r_2,s_2:n} := \{(X_{r_1:n}, X_{s_1:n}), (Y_{r_2:n}, Y_{s_2:n})\}$, $r_1 < s_1$ and $r_2 < s_2$, symbolically satisfies the condition

$$T_{r_1,s_1;r_2,s_2:n} := P(P(X_{r_1:n} < X < X_{s_1:n}, Y_{r_2:n} < Y < Y_{s_2:n}) \ge \beta) = \gamma.$$

The following theorem gives an upper bound for $T_{r_1,s_1;r_2,s_2:n}$, $1 \le r_1 < s_1 \le n, 1 \le r_2 < s_2 \le n$, in terms of the d.f.s of the concomitant ranks order statistics.

Theorem 5. *For any* $1 \le r_1 < s_1 \le n$ *and* $1 \le r_2 < s_2 \le n$, *we have*

$$T_{r_1,s_1;r_2,s_2:n} \le \frac{1}{(n+1)\beta} \sum_{i=r_1+1}^{s_1} \sum_{j=r_2+1}^{s_2} A_{i:n+1}(j).$$

Proof. Clearly, we have

$$T_{r_1,s_1;r_2,s_2:n} = P\left(\int_{X_{r_1:n}}^{X_{s_1:n}} \int_{Y_{r_2:n}}^{Y_{s_2:n}} f(x,y)dxdy \ge \beta \right). \tag{22}$$

If in (22) we make the substitutions $u = F_1(x)$ and $v = F_2(y)$, we obtain

$$\begin{aligned} T_{r_1,s_1;r_2,s_2:n} &= P\left(\int_{F_1^{-1}(X_{r_1:n})}^{F_1^{-1}(X_{s_1:n})} \int_{F_2^{-1}(Y_{r_2:n})}^{F_2^{-1}(Y_{s_2:n})} c(u,v)dudv \ge \beta \right) \\ &= P\left(\int_{U_{r_1:n}}^{U_{s_1:n}} \int_{V_{r_2:n}}^{V_{s_2:n}} c(u,v)dudv \ge \beta \right). \end{aligned}$$

Therefore, we get

$$T_{r_1,s_1;r_2,s_2:n} = \sum_{i=r_1+1}^{s_1} \sum_{j=r_2+1}^{s_2} T_{i-1,i;j-1,j:n},$$

where

$$\begin{aligned} T_{i-1,i;j-1,j:n} = P\big(&C(U_{i:n}, V_{j:n}) - C(U_{i-1:n}, V_{j:n}) - C(U_{i:n}, V_{j-1:n}) \\ &+ C(U_{i-1:n}, V_{j-1:n}) \ge \beta\big). \end{aligned}$$

On the other hand, because the r.v. $W := C(U_{i:n}, V_{j:n}) - C(U_{i-1:n}, V_{j:n}) - C(U_{i:n}, V_{j-1:n}) + C(U_{i-1:n}, V_{j-1:n})$ is nonnegative, an appeal to the well-known Markov's inequality yields

$$P(W \ge \beta) \le \frac{E(W)}{\beta} = \frac{(n+1)E(W)}{(n+1)\beta} = \frac{A_{i:n+1}(j)}{(n+1)\beta},$$

which was to be proved. □

Example 2. Table 2 gives the upper bound of $T_{r_1,s_1;r_2,s_2:n}$, for $\beta = 0.70$, 0.80, 0.85, 0.90, 0.98, 0.99, $r_1 = r_2 = 1$ and $s_1 = s_2 = n = 8$, where

Table 2. The upper bound of $T_{r_1,s_1;r_2,s_2:n}$ for $r_1 = r_2 = 1$ and $s_1 = s_2 = n = 8$.

ρ	$\beta = 0.99$	$\beta = 0.98$	$\beta = 0.90$	$\beta = 0.85$	$\beta = 0.80$	$\beta = 0.70$
0.1	0.6117	0.6179	0.6729	0.7124	0.7570	0.8651
0.2	0.6146	0.6209	0.6761	0.7159	0.7606	0.8693
0.3	0.6192	0.6255	0.6811	0.7212	0.7662	0.8757
0.4	0.6257	0.6321	0.6883	0.7288	0.7743	0.8849
0.5	0.6346	0.6411	0.6980	0.7391	0.7853	0.8975
0.6	0.6460	0.6526	0.7106	0.7524	0.7995	0.9137
0.7	0.6587	0.6654	0.7246	0.7672	0.8152	0.9316
0.8	0.6806	0.6875	0.7486	0.7926	0.8422	0.9625
0.9	0.7155	0.7228	0.7870	0.8333	0.8854	1
0.95	0.7297	0.7372	0.8027	0.8500	0.9030	1

$F(x, y)$ is a standard bivariate normal distribution with correlation coefficients $\rho = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95$. The different values of $A_{i:9}(j), 1 \le i, j \le 8$ are extracted from a table in David et al. (1977).

Table 2 reveals an interesting and, at the same time, expectant fact that the percentage of the standard bivariate normal distribution, which is covered by a random sample, will increase as the coefficient of correlation increases. This fact clearly implies that the size of the random sample required to cover at least a specified percentage of the standard bivariate normal distribution will decrease with an increasing coefficient of correlation.

Clearly, in view of the results of Sec. 2.2, one can easily derive the following extension of Theorem 5.

Theorem 6. *For any* $1 \le r_i < s_i \le n$, $i = 1, 2, 3, \cdots, m+1$, *we have*

$$T_{r_1,s_1;\cdots;r_{m+1},s_{m+1}:n} \le \frac{1}{(n+1)\beta} \sum_{i=r_1+1}^{s_1} \sum_{j_1=r_2+1}^{s_2} \cdots \sum_{j_m=r_{m+1}+1}^{s_{m+1}} A_{i:n+1}(j_1, j_2, \cdots, j_m),$$

where

$$T_{r_1,s_1;\cdots;r_{m+1},s_{m+1}:n} := P(P(X_{r_1:n} < X < X_{s_1:n},\ Y_{r_2:n} < Y_1 < Y_{s_2:n},\ \ldots,\ Y_{r_{m+1}:n} < Y_m < Y_{s_{m+1}:n}) \ge \beta)$$

and $R_{r_1,s_1;\cdots;r_{m+1},s_{m+1}:n} := \{(X_{r_1:n}, X_{s_1:n}),\ (Y_{r_2:n} < Y_1 < Y_{s_2:n}), \ldots, (Y_{r_m:n} < Y_m < Y_{s_m:n})\}$ *is the* $100\beta\%$ *tolerance region* $[(m+1)$*-dimensional* *rectangle*] *for the d.f.* $F(x, y_1, \cdots, y_m)$, *with tolerance coefficient* $T_{r_1,s_1;\cdots;r_{m+1},s_{m+1}:n}$.

ACKNOWLEDGMENTS

The authors are grateful to the anonymous referee for valuable comments and helpful suggestions that contributed to the quality of the paper presentation.

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