Statistics & Probability Letters 7 (1989) 283–286
North-Holland February 1989

MAXIMA OF NORMAL RANDOM VECTORS: BETWEEN INDEPENDENCE AND COMPLETE DEPENDENCE

Jürg HÜSLER

Institut Mathematische Statistik, Universitat Bern, Switzerland

Rolf-Dieter REISS

FB Mathematik, Universität-Gesamthochschule Siegen, FR Germany

Received January 1988
Revised May 1988

Abstract: A new approach to the asymptotic treatment of multivariate sample maxima is suggested and exemplified in the particular case of maxima of normal random vectors. In the limit one obtains a class of multivariate maxstable distributions not considered in literature so far.

AMS 1980 Subject Classification Primary 60F05, 62G30

Keywords: multivariate sample maxima, normal random vectors, correlation coefficient, max-stability

1. Introduction

Let $X=(X_1,\ldots,X_d)$ be a random vector with values in the Euclidean d-space $\mathbb{R}^d$. Let $X_i=(X_{i,1},\ldots,X_{i,d})$ be independent copies of X for $i=1,\ldots,n$. Define the multivariate sample maximum M_n by

$$M_n=(M_{n,1},\ldots,M_{n,d})$$

where $M_{n,j}=\max(X_{1,j},\ldots,X_{n,j})$ is the jth marginal maximum. The theory of multivariate maxima has found an increasing interest in literature since the articles by J. Tiago de Oliveira (1958), J. Geffroy (1958/59) and M. Sibuya (1960). For consolidated representations of the present state of the art we refer to the books by J. Galambos (1987) (1st edition in 1978) and S.I. Resnick (1987). It has been known from the beginning that in prominent cases the marginal maxima $M_{n,1},\ldots,M_{n,d}$ are asymptotically independent. In the particular case of a normal random vector, having all correlation coefficients smaller than one, this was proved by Sibuya (1960).

We believe that from the applied and theoretical point of view the asymptotic theory of multivariate maxima has to be supplemented by that type of results which are presented here in the particular case of normal random vectors. We choose the multivariate normal case because it is historically the classical one. This case is also mathematically very attractive and, as we shall see, helps to clarify the problem.

From asymptotic theory we know that for most of the standard d.f.'s (like the bivariate normal d.f. with correlation coefficient smaller than 1) the marginal maxima are asymptotically independent. But consider the situation where measurements of a certain phenomenon are taken at two separate places that, however, are close together. The asymptotic result should in this case also reflect the possible dependence of the marginal maxima. Thus, the problem arises whether there is another asymptotic formulation which provides a suitable discussion for such cases. Speaking in mathematical terms, it means that we are looking for an asymptotic approach leading to a limit distribution which is a more accurate approximation to the above situation. In Section 2 we suggest an asymptotic formulation where the underlying bivariate normal distribution has a correlation coefficient $\rho \equiv \rho_n$ that varies as the sample size increases. It will be shown that the marginal maxima are neither asymptotically independent nor completely dependent if $(1-\rho(n))\log n$ converges to a positive constant as $n \to \infty$. The extension to the dimension $d \geqslant 2$ will be carried out in Section 3.

Our treatment of multivariate maxima is comparable to that of binomial r.v.'s where according to the condition that the parameter p is fixed or $p \equiv p_n$ satisfies $np_n \to \lambda > 0$ as $n \to \infty$ the normal or Poisson approximation is appropriate.

It is obvious that our results will stimulate further research work. One can e.g. combine the present work with that in Hüsler and Schüpbach (1988) in connection with limit distributions of multivariate maxima for dependent Gaussian sequences.

2. The bivariate case

Because of notational simplicity we first consider a bivariate normal vector $\boldsymbol{X}=(X_1, X_2)$ where w.l.g. X_1 and X_2 are standard normal r.v.'s. Denote by ρ the correlation and by F_ρ the d.f. of $\boldsymbol{X}$. It is well known that F_ρ^n is the d.f. of the maximum $\boldsymbol{M}_n$.

Let b_n be defined by the equation

$$b_n = n\phi(b_n) \tag{2.1}$$

where ϕ is the standard normal density. Denote by Φ the standard normal d.f. It is well known that the normalized marginal maxima are asymptotically distributed according to the Gumbel d.f. G_3 defined by

$$G_3(x) = \exp(-e^{-x}). \tag{2.2}$$

More precisely, we have

$$\Phi^n(b_n + x/b_n) \to G_3(x) \quad \text{as } n \to \infty \quad \text{for every real x.} \tag{2.3}$$

Thus it is apparent that in order to obtain a limiting distribution in the bivariate case one has to study the normalized d.f.

$$(x, y) \to F_\rho^n(b_n + x/b_n,\ b_n + y/b_n). \tag{2.4}$$

If X_1 and X_2 are independent (that is, $\rho = 0$) then it is obvious that the limiting d.f., say, H_∞ is given by

$$H_\infty(x, y) = G_3(x)G_3(y). \tag{2.5}$$

If $X_1 = X_2$ (that is, $\rho = 1$) then the limiting d.f. H_0 is given by

$$H_0(x, y) = \exp(-e^{-\min(x,y)}). \tag{2.6}$$

We remark that H_∞ is always the limiting d.f. if $\rho < 1$. For $\lambda \in [0,\infty]$ define

$$H_\lambda(x, y) = \exp\left[-\Phi\left(\lambda + \frac{x-y}{2\lambda}\right)e^{-y} - \Phi\left(\lambda + \frac{y-x}{2\lambda}\right)e^{-x}\right] \tag{2.7}$$

where it is understood that

$$H_0 = \lim_{\lambda \downarrow 0} H_\lambda \quad \text{and} \quad H_\infty = \lim_{\lambda \uparrow \infty} H_\lambda.$$

Notice that for $\lambda = \infty$ and $\lambda = 0$ we get again the d.f.'s in (2.5) and (2.6).

Theorem 1. *If*

$$(1-\rho(n))\log n \to \lambda^2 \in [0,\infty] \quad \text{as } n \to \infty \tag{2.8}$$

then for every x, y,

$$F_{\rho(n)}^n(b_n + x/b_n,\ b_n + y/b_n) \to H_\lambda(x, y) \quad \text{as } n \to \infty. \tag{2.9}$$

Proof. Put $u_n(x) = b_n + x/b_n$. Check that

$$\begin{aligned} F_{\rho(n)}^n&(u_n(x), u_n(y)) \\ &= \exp[-n[1-\Phi(u_n(x)] - n[1-\Phi(u_n(y))] \\ &\qquad + nP\{X_1 > u_n(x),\ X_2 > u_n(y)\}] + o(1) \\ &= \exp[-e^{-x} - e^{-y} \\ &\qquad + nP\{X_1 > u_n(x),\ X_2 > u_n(y)\}] \\ &\quad + o(1). \end{aligned} \tag{2.10}$$

Let $\rho \equiv \rho(n)$. W.l.g. assume that $\rho(n) \in (-1, 1)$; otherwise use in (2.10) the representations $P\{X_1 > u_n(x),\ X_2 > u_n(y)\} = P\{X_1 > \max(u_n(x),\ u_n(y))\}$ if $\rho(n) = 1$, and $P\{X_1 > u_n(x),\ X_2 > u_n(y)\} = P\{u_n(x) < X_1 < -u_n(y)\}$ if $\rho(n) = -1$.

The conditional distribution of X_1 given $X_2 = z$ is the normal distribution $N_{(\rho z, 1-\rho^2)}$ with mean value ρz and variance $1 - \rho^2$. Thus, we get

$$\begin{aligned} &nP\{X_1 > u_n(x),\ X_2 > u_n(y)\} \\ &\quad = n\int_{u_n(y)}^{\infty} \left(1 - N_{(\rho z, 1-\rho^2)}(-\infty,\ u_n(x)]\right)\phi(z)\,\mathrm{d}z \\ &\quad = \int_y^{\infty}\left[1 - \Phi\left(\frac{u_n(x) - \rho u_n(z)}{(1-\rho^2)^{1/2}}\right)\right] \\ &\qquad \times \exp\left[-\left(z + z^2/b_n^2\right)\right]\,\mathrm{d}z. \end{aligned} \tag{2.11}$$

Notice that the integrand is dominated by e^{-z}. Check that

$$\begin{aligned} &\frac{u_n(x) - \rho(n)u_n(z)}{\left(1-\rho(n)^2\right)^{1/2}} \\ &\quad = \lambda(n) + \frac{x - z}{(1+\rho(n))\lambda(n)} + \frac{(1-\rho(n))^{1/2} z}{(1+\rho(n))^{1/2} b_n} \end{aligned}$$

and $\lambda(n)^2 = b_n^2(1 - \rho(n))/(1 + \rho(n)) \to \lambda^2$ as $n \to \infty$ since $b_n^2 \sim 2\log n$. Thus, the dominated convergence theorem implies that the right-hand side of (2.11) converges to

$$\int_y^{\infty}\left[1 - \Phi\left(\lambda + \frac{x - z}{2\lambda}\right)\right]\mathrm{e}^{-z}\,\mathrm{d}z \tag{2.12}$$

with the convention that this term is equal to 0 if $\lambda(n) \to \infty$ and is equal to $\mathrm{e}^{-\max(x,y)}$ if $\lambda(n) \to 0$ as $n \to \infty$. Obviously

$$\int_y^{\infty} \mathrm{e}^{-z}\,\mathrm{d}z = \mathrm{e}^{-y}. \tag{2.13}$$

Moreover, by partial integration we get

$$\begin{aligned} &\int_y^{\infty}\Phi\left(\lambda + \frac{x - z}{2\lambda}\right)\mathrm{e}^{-z}\,\mathrm{d}z \\ &\quad = \Phi\left(\lambda + \frac{x - y}{2\lambda}\right)\mathrm{e}^{-y} \\ &\qquad - \int_y^{\infty}\frac{1}{2\lambda}\phi\left(\lambda + \frac{x - z}{2\lambda}\right)\mathrm{e}^{-z}\,\mathrm{d}z \\ &\quad = \Phi\left(\lambda + \frac{x - y}{2\lambda}\right)\mathrm{e}^{-y} \\ &\qquad - \mathrm{e}^{-x}\int_y^{\infty}\frac{1}{2\lambda}\phi\left(\lambda + \frac{z - x}{2\lambda}\right)\mathrm{d}z \\ &\quad = -\mathrm{e}^{-x} + \left[\Phi\left(\lambda + \frac{x - y}{2\lambda}\right)\mathrm{e}^{-y}\right. \\ &\qquad \left. + \Phi\left(\lambda + \frac{y - x}{2\lambda}\right)\mathrm{e}^{-x}\right]. \end{aligned} \tag{2.14}$$

Combining (2.10)–(2.14) the proof is complete. □

The marginals of the limiting d.f. H_λ are the Gumbel d.f. G_3. Remember that G_3 is max-stable since $G_3^n(x + \log n) = G_3(x)$. This is also true for H_λ; we have

$$H_\lambda^n(x + \log n,\ y + \log n) = H_\lambda(x,\ y). \tag{2.15}$$

In general it is not true that the limit d.f. of multivariate maxima is max-stable if the underlying d.f. varies with the sample size. However, it is known (compare with Resnick (1987, Proposition 5.1)) that the limit d.f. is always max-infinitely divisible.

3. The case $d \geqslant 2$

Throughout $X_1, \ldots, X_d$ are standard normal r.v.'s with correlation matrix $\Sigma = (\rho_{i,j})_{i,j \leqslant d}$. Denote by F_Σ the joint d.f. It is apparent that many different limit situations can occur depending on the behaviour of the correlations $\rho_{i,j} \equiv \rho_{i,j}(n)$. In analogy to (2.8) assume that for $1 \leqslant i, j \leqslant d$:

$$\left(1 - \rho_{i,j}(n)\right)\log n \to \lambda_{i,j}^2 \quad \text{as } n \to \infty. \tag{3.1}$$

To avoid technical difficulties and to concentrate our attention to the most interesting case we assume that

$$\lambda_{i,j} \in (0,\ \infty) \quad \text{for } 1 \leqslant i, j \leqslant d \quad \text{with } i \neq j. \tag{3.2}$$

Put

$$\Lambda = \left(\lambda_{i,j}\right)_{i,j \leqslant d}. \tag{3.3}$$

Moreover, for $2 \leqslant k \leqslant d$ and $\boldsymbol{m}' = (m_1, \ldots, m_k)$ with $1 \leqslant m_1 < m_2 < \cdots < m_k \leqslant d$ define

$$\Gamma_{k,\boldsymbol{m}} = \left(2\left(\lambda_{m_i,m_k}^2 + \lambda_{m_j,m_k}^2 - \lambda_{m_i,m_j}^2\right)\right)_{i,j \leqslant k-1}. \tag{3.4}$$

Furthermore, let $S(\cdot \mid \Gamma)$ denote the survivor function of a normal random vector with mean vector $\mathbf{0}$ and covariance matrix Γ. For Λ as in (3.3) define

$$H_\Lambda(\boldsymbol{x}) = \exp\left(\sum_{k=1}^{d} (-1)^k \Sigma^{(k)} h_{k,\boldsymbol{m}}(x_{m_1}, \ldots, x_{m_k}) \right) \tag{3.5}$$

with

$$h_{k,\boldsymbol{m}}(y) = \int_{y_k}^{\infty} S\left(\left(y_i - z + 2\lambda^2_{m_i,m_k} \right)_{i=1}^{k-1} \mid \Gamma_{k,\boldsymbol{m}} \right) \mathrm{e}^{-z} \, \mathrm{d}z \tag{3.6}$$

for $2 \leqslant k \leqslant d$ where $\Sigma^{(k)}$ means summation over all k-vectors $\boldsymbol{m} = (m_1, \ldots, m_k)$ with $1 \leqslant m_1 < \cdots < m_k \leqslant d$. Morover, $h_{1,m}(y) = \mathrm{e}^{-y}$ for $m = 1, \ldots, d$.

Theorem 2. *If Σ_n are non-singular matrices such that* (3.1) *and* (3.2) *hold then*

$$F^n_{\Sigma_n}(b_n + x_1/b_n, \ldots, b_n + x_d/b_n) \to H_\Lambda(\boldsymbol{x}) \quad \text{as } n \to \infty \tag{3.7}$$

for every $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

Proof. Runs along the lines of the proof to Theorem 1 by using Theorem 5.3.1 in Galambos (1987) in place of (2.10), by conditioning according to the last component of the normal vectors in analogy to (2.11) and by taking into account weak convergence of multivariate normal distributions. □

Again one can check that H_Λ is max-stable. Moreover notice that $h_{2,(i,j)}$ can be written as

$$h_{2,(i,j)}(x, y) = \left[1 - \Phi\left(\lambda_{i,j} + \frac{x-y}{2\lambda_{i,j}}\right)\right] \mathrm{e}^{-y} + \left[1 - \Phi\left(\lambda_{i,j} + \frac{y-x}{2\lambda_{i,j}}\right)\right] \mathrm{e}^{-x}. \tag{3.8}$$

References

Galambos, J (1987), *The Asymptotic Theory of Extreme Order Statistics*, 2nd Edition (Krieger, Florida)

Geffroy, J (1958/59), Contributions à la theorie des valeurs extremes, *Publ Inst Statist Univ Paris* **7/8**, 37–185.

Husler, J. and M Schüpbach (1988), Limit results for maxima in non-stationary multivariate Gaussian sequences, *Stoch Proc Appl* **27**, 91–99.

Resnick, S.I. (1987), *Extreme Values, Regular Variation, and Point Processes* (Springer, New York).

Sibuya, M (1960), Bivariate extreme statistics. *Ann. Inst. Statist Math* **11**, 195–210.

Tiago de Oliveira, J. (1958), Extremal distributions, Revista da Fac. Ciencias, Univ Lisboa **A7**, 215–227.