First Passage under Restart

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First passage under restart has recently emerged as a conceptual framework suitable for the description of a wide range of phenomena, but the endless variety of ways in which restart mechanisms and first passage processes mix and match hindered the identification of unifying principles and general truths. Hope that these exist came from a recently discovered universality displayed by processes under optimal, constant rate, restart—but extensions and generalizations proved challenging as they marry arbitrarily complex processes and restart mechanisms. To address this challenge, we develop a generic approach to first passage under restart. Key features of diffusion under restart—the ultimate poster boy for this wide and diverse class of problems—are then shown to be completely universal.

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A myriad of basic questions and a wide array of applications have turned first passage time (FPT) processes into a long-standing focal point of scientific interest [1,2]. These processes were studied extensively, e.g., in the context of nonequilibrium systems [3], but despite many years of study, paramount discoveries are still being made and exciting applications continue to be found. Recently, several groups have observed that any FPT process imaginable can become subject to restart, i.e., can be stopped and started anew (Fig. 1). This observation has opened a rapidly moving theoretical research front [4–19], and applications to search problems [20–22], the optimization of randomized computer algorithms [23–29], and in the field of biophysics [30,31], have further propelled its expansion. Universality has always been considered a holy grail of the physical sciences, and novel revelations concerning universality in FPT processes have recently taken center stage and attracted considerable attention [32–34]. In contrast, not a lot is known in general about the problem of first passage under restart (FPUR).

Diffusion with resetting to the origin is a quintessential example of FPUR [5]. In this problem, a particle undergoes diffusion but from time to time is also taken and returned to the place from where it started its motion (reset or restart). In addition, at some distance away from the origin a target awaits and one is interested in the time it takes the particle to first get to the target, i.e., in its distribution and corresponding moments. This problem was first studied with restart rates that are constant in time and the surprise came from the fact that restart was able to expedite search and that a carefully chosen (optimal) restart rate could minimize the mean FPT to the target. Further down the road, other restart mechanisms were also studied [12,13,16,19] and it was shown that these may underperform or overpreform when compared to restart at a constant rate [12,19].

Each variant above carried with it some unique and intriguing features, but exhausting the vast combinatorial space of process-restart pairs—one problem at a time—is virtually impossible. Indeed, restart processes may take different shapes and forms and the effect they have on FPT processes other than diffusion [35–42] is also of interest. Moreover, it is often the case—in real life scenarios—that the process under consideration, the restart mechanism that accompanies it, or both are poorly specified or even

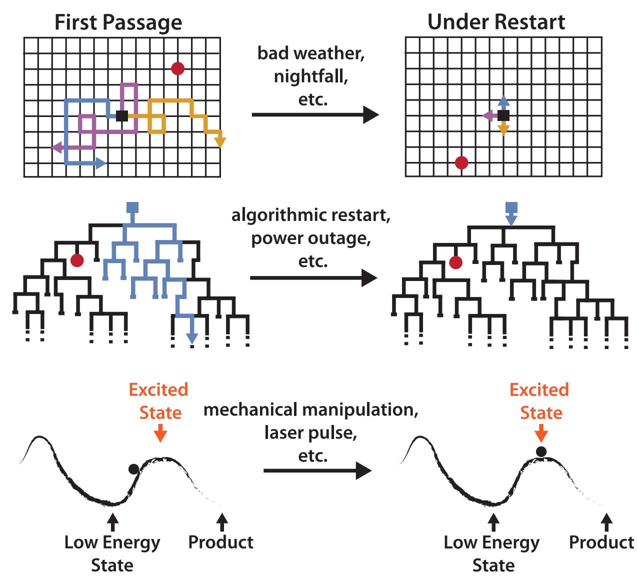

FIG. 1. (Top) Bad weather could force a team of searchers to temporarily cease their efforts and return to base. By the time search is renewed the target may have relocated and search must thus start from scratch. (Middle) A computer algorithm operates as a black box which randomly scans a tree of possibilities in search of a solution. Chance may send the algorithm down the wrong path, but programmed restart could help rescue the search. (Bottom) A molecule that was previously prepared at an excited state decays to a low energy state. A pulse of laser could bring the molecule back to its excited state and restart a chemical or physical reaction. This time, the desired product may be formed.

completely unknown. More general approaches, better suited to deal with partial and missing information and with the need to generalize from specific examples, could then become handy.

Recently, two attempts to unify treatment were made. In Ref. [18], an approach suitable to the description of a generic FPT process under constant rate restart was presented. The approach was utilized to show that when restart is optimal, the relative fluctuation in the FPT of the restarted process is always unity. This result holds true regardless of the underlying process, be it diffusion or something else, but is no longer valid for time-dependent restart rates as these were not covered by the approach to begin with. Restart rates with arbitrary time dependence were considered in Ref. [12], but analysis there was limited to diffusion and did not cover other FPT processes. Here, we will be interested in merging the two approaches in an attempt to get the best of both worlds. To this end, we consider a generic FPT process that has further become subject to a generic restart mechanism. This setting is extremely general and captures, as special cases, the overwhelming majority of models that have already appeared in the literature. We analyze this scheme to attain, and concisely describe, several broad scope results that unravel universal features of this wide class of problems. In what follows, we use $f_Z(t)$, $\langle Z\rangle$, $\sigma^2(Z)$, and $\tilde{Z}(s) \equiv \langle e^{-sZ}\rangle$ to denote, respectively, the probability density function, expectation, variance, and Laplace transform of a real-valued random variable Z.

Mean FPT under restart.—Consider a generic process that starts at time zero and, if allowed to take place without interruptions, ends after a random time T. The process is, however, restarted at some random time R. Thus, if the process is completed prior to restart, the story there ends. Otherwise, the process will start from scratch and begin completely anew. This procedure repeats itself until the process reaches completion. Denoting the random completion time of the restarted process by T_R, it can be seen that

$$T_R = \begin{cases} T & \text{if } T < R \\ R + T'_R & \text{if } R \le T, \end{cases} \tag{1}$$

where T'_R is an independent and identically distributed copy of T_R.

A scheme similar to the one described in Eq. (1) was analyzed in Ref. [18]. There, no assumptions were made on the distribution of the time T which governs the completion of the underlying process, but the restart time R was assumed to be exponentially distributed with rate parameter r. This means that restart is conducted at a constant rate r, i.e., that for any given time point the probability that restart will occur at the next infinitesimal time interval dt is rdt. Here, we relax this assumption allowing for generally distributed restart times or, equivalently, for restart rates with arbitrary time dependence. Letting $r(t)$ denote the restart rate at time t, we note that the two perspectives are related via (Fig. 2)

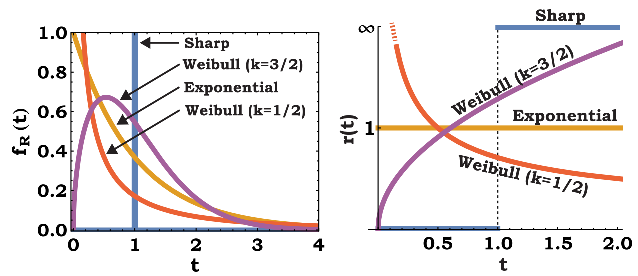

FIG. 2. A few examples of restart time distributions (left) and the restart rates they induce (right). Below $\delta(x)$ is the Dirac delta function, $\Gamma(x)$ is the Gamma function, and $\langle R\rangle = 1$ in all plots. (i) Sharp (deterministic) restart $f_R(t) = \delta(t - \langle R\rangle)$. Restart rate jumps abruptly from zero to infinity at $t = \langle R\rangle$. (ii) Exponentially distributed restart $f_R(t) = \langle R\rangle^{-1} e^{-t/\langle R\rangle}$. Restart rate is constant: $r(t) = 1/\langle R\rangle$, (iii) and (iv) Weibull distributed restart $f_R(t) = k/\lambda(t/\lambda)^{k-1} e^{-(t/\lambda)^k}$. Restart rate is given by $r(t) = kt^{k-1}/\lambda^k$ and could monotonically decrease (e.g., $k = 1/2$, $\lambda = \langle R\rangle/2$) or increase [e.g., $k = 3/2$, $\lambda = \langle R\rangle/\Gamma(5/3)$] with time.

$$\Pr(R \le t) = 1 - \exp\left(-\int_0^t r(x)dx\right), \tag{2}$$

where $\Pr(R \le t)$ is the probability that $R \le t$ [43].

Equation (1) could be used to provide a simple formula for the mean FPT of a stochastic process under restart. Indeed, noting that it can also be written as $T_R = \min(T, R) + I\{R \le T\}T'_R$, where $\min(T, R)$ is the minimum of T and R and $I\{R \le T\}$ is an indicator random variable that takes the value 1 when $R \le T$ and zero otherwise, we take expectations to find

$$\langle T_R\rangle = \frac{\langle \min(T, R)\rangle}{\Pr(T < R)}. \tag{3}$$

The right-hand side of Eq. (3) can then be computed given the distributions of T and R if one also recalls that the cumulative distribution function of $\min(T, R)$ is given by $\Pr(\min(T, R) \le t) = 1 - \Pr(T > t)\Pr(R > t)$.

A hallmark of restart is its ability to minimize (optimize) mean FPTs. For example, when the restart rate $r(t) = r$ is constant, it is straightforward to show that Eq. (3) reduces to $\langle T_R\rangle = [1 - \tilde{T}(r)]/[r\tilde{T}(r)]$, where $\tilde{T}(r)$ is the Laplace transform of T evaluated at r. One could then seek an optimal rate r^*, which brings $\langle T_R\rangle$ to a minimum, derive general conditions for this rate to be strictly larger than zero, and further discuss universal properties of the optimal rate itself [15,30]. Clearly, this line of inquiry is not limited to the case of exponentially distributed restart times and could also be extended to other parametric distributions. Various optimization questions could then be addressed directly, but we would now like to consider a broader optimization question. Specifically, we ask if within the vast space of stochastic restart strategies, and irrespective of the underlying process being restarted, there is a single winning strategy that could not be beat.

Sharp restart is a dominant strategy.—Consider a particle "searching" for a stationary target via one-dimensional

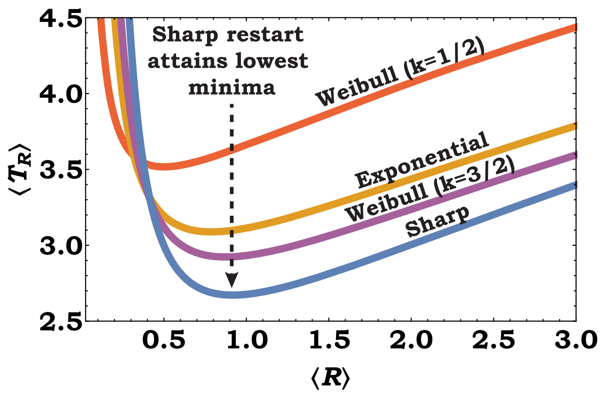

FIG. 3. Mean FPT for diffusion mediated search with restart versus the mean restart time, for various restart time distributions taken from Fig. 2.

diffusion. The particle starts at the origin, the initial distance between the particle and the target is L, and the diffusion coefficient of the particle is D. Denoting the particle's FPT to the target with T, the latter is known to come from the Lévy-Smirnov distribution $f_T(t)=\sqrt{L^2/4D\pi t^3}e^{-L^2/4Dt}$ [1]. Considering the same problem under restart, we take $D = 1/2$ and $L = 1$, and utilize Eq. (3) to plot $\langle T_R \rangle$ as a function of $\langle R \rangle$ for various restart time distributions (Fig. 3). As can be seen, a minimum of $\langle T_R \rangle$ is always attained, and while the values taken by the different minima and their positions clearly depend on the distribution of the restart time, it is sharp restart that attains the lowest of minima. A similar observation was made in the past, and it was consequently conjectured that in the case of diffusion mediated search sharp restart is the optimal restart strategy [12,19]. Strikingly, this is also true in general.

Consider, for the sake of simplicity, a random restart time R characterized by a proper density $f_R(t)$ and note that $\langle \min(T,R)\rangle = \int_0^\infty f_R(t)\langle \min(T,R)|R=t\rangle dt = \int_0^\infty f_R(t)\langle \min(T,t)\rangle dt$, which then implies

$$\langle T_R \rangle = \int_0^\infty \frac{f_R(t)\Pr(T<t)}{\Pr(T<R)} \frac{\langle \min(T,t)\rangle}{\Pr(T<t)} dt.$$

However, $\int_0^\infty [f_R(t)\Pr(T<t)dt]/[\Pr(T<R)] = 1$, and $\langle \min(T,t)\rangle/\Pr(T<t)$ is simply the mean completion time of a process that is restarted sharply after t units of time. Thus, if there exists some t^* such that $\langle \min(T,t^*)\rangle/\Pr(T<t^*) \le \langle \min(T,t)\rangle/\Pr(T<t)$ for all $t \ge 0$, sharp restart at t^* will also beat any random restart time that is governed by a proper density. Moreover, the law of total expectation implies $\langle \min(T,R)\rangle = \langle\langle \min(T,R)|R\rangle_T\rangle_R$ and steps similar to those taken above assert that [see Supplemental Material (SM) [44]]

$$\frac{\langle \min(T,t^*)\rangle}{\Pr(T<t^*)} \le \frac{\langle \min(T,R)\rangle}{\Pr(T<R)}, \tag{4}$$

for any random restart time R regardless of its distribution. Equation (4) thus asserts that sharp restart is optimal among all possible stochastic restart strategies in continuous time, and we refer the reader to Luby *et al.* for a complementary, algorithm oriented, discussion on the discrete time case [23].

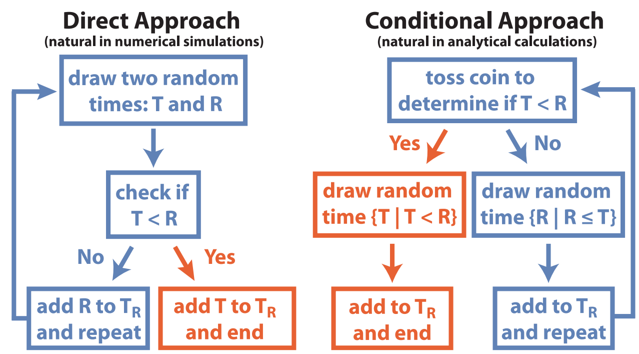

FIG. 4. Two approaches to first passage under restart.

Distribution of FPT under restart.—So far, we have only been concerned with the *mean* FPT of a restarted process, but we will now move on to discuss the full distribution of T_R. The scheme described in Eq. (1) suggests a direct approach for numerical simulation of FPUR (Fig. 4, left). In this approach, one draws two random times from the distributions of T and R, and only then—based on the outcome of that draw—decides which of the two, restart or completion, happened first. An equivalent approach would operate in reversed order. A coin with probability $\Pr(T<R)$ will first be tossed to determine if completion preceded restart (or vice versa) and only then, given that information, the appropriate—conditional—random time will be drawn (Fig. 4, right). This approach is somewhat awkward and indirect for the purpose of numerical simulations, but is actually quite natural when coming to compute expectations and Laplace transforms where one usually starts by conditioning on the occurrence of an event of interest. Indeed, analytical formulas could be simplified with the aid of two auxiliary random variables: $R_{\min} \equiv \{R|R=\min(R,T)\}$ and $T_{\min} \equiv \{T|T=\min(R,T)\}$. In words, $R_{\min}$ is the random restart time given that restart occurred prior to completion, and $T_{\min}$ is defined in a similar manner. Conditioning on whether $T<R$ and applying the law of total expectation to $\tilde{T}_R(s) = \langle e^{-sT_R}\rangle$, we obtain (see SM [44])

$$\tilde{T}_R(s) = \frac{\Pr(T<R)\tilde{T}_{\min}(s)}{1-\Pr(R\le T)\tilde{R}_{\min}(s)}. \tag{5}$$

Equation (5) allows one to explicitly compute the distribution of T_R in Laplace space. For example, when T and R are correspondingly governed by probability densities $f_T(t)$ and $f_R(t)$, we have $\Pr(T<R) = \int_0^\infty f_T(t)[\int_t^\infty f_R(t')dt']dt$ and the probability densities governing $T_{\min}$ and $R_{\min}$ are similarly given by

$$f_{T_{\min}}(t) = f_T(t)\int_t^\infty f_R(t')dt'/\Pr(T<R),$$

$$f_{R_{\min}}(t) = f_R(t)\int_t^\infty f_T(t')dt'/\Pr(R\le T). \tag{6}$$

Plugging in concrete probability distributions, explicit formulas can be obtained, e.g., for exponentially distributed restart $f_R(t) = re^{-rt}$, and one could readily show that $\tilde{T}_R(s) = \tilde{T}(s+r)/[s/(s+r) + (r/(s+r))\tilde{T}(s+r)]$ (see SM [44]), as was previously obtained in Ref. [18] by other means.

Fluctuations in FPT under optimal sharp restart obey a universal inequality.—Given Eq. (5), one could utilize the known relation between moments and the Laplace transform [41] to find (see SM [44])

$$\langle T_R^2 \rangle = \frac{\langle \min(T,R)^2 \rangle}{\Pr(T < R)} + \frac{2\Pr(R \le T)\langle R_{\min} \rangle \langle \min(T,R) \rangle}{\Pr(T < R)^2}. \tag{7}$$

A special case of this result was used to show that the relative fluctuation $\sigma(T_R)/\langle T_R \rangle$ is always unity when a process is restarted at a constant rate $r^* > 0$, which brings $\langle T_R \rangle$ to a minimum. Optimal sharp restart could lower the mean FPT $\langle T_R \rangle$ well below the value it attains for optimal constant rate restart, but unless the resulting fold reduction is also matched or exceeded by a fold reduction in $\sigma(T_R)$, the relative fluctuation in the FPT would surely increase. It is thus possible that the ability of the sharp restart strategy to attain lower mean FPTs comes at the expense of higher relative fluctuations—and, hence, greater uncertainty—in the FPT itself. However, when Eq. (7) was utilized to examine diffusion and other case studies (Fig. 5), we consistently found

$$\sigma(T_{t^*})/\langle T_{t^*} \rangle \le 1, \tag{8}$$

for the relative fluctuation at the optimal restart time t^*.

Equation (8) is universal. To see this, we assume by contradiction that there exists a FPT process for which $\sigma(T_{t^*})/\langle T_{t^*} \rangle > 1$, and consider a restart strategy R_{mix} in which this process is restarted at a low constant rate $r \ll 1$ in addition to being sharply restarted whenever a time t^* passes from the previous restart (or start) epoch. Applying this restart strategy is equivalent to augmenting the process under sharp restart with an additional restart mechanism that restarts it with rate r. However, if the relative fluctuation in the FPT of a process is larger than unity, restart at a low constant rate will surely lower its mean FPT (and vice versa). This is true regardless of the underlying process, and can be seen by examining $\langle T_R \rangle$ for general T, and R which is exponentially distributed with rate r [see formula for $\langle T_R \rangle$ below Eq. (3)]. Utilizing the moment representation of the Laplace transform, one can then show that $[d\langle T_R \rangle/dr]|_{r=0} < 0$ whenever $\sigma(T)/\langle T \rangle > 1$ (see SM [44]). Denoting the mean FPT under R_{mix} by $\langle T_{R_{\mathrm{mix}}} \rangle$, and letting T_{t^*} take T's place above, it follows that $\langle T_{t^*} \rangle > \langle T_{R_{\mathrm{mix}}} \rangle$. We have thus found a nonsharp restart strategy that lowers the mean FPT beyond that attained for optimal sharp restart. However, this finding must be false as it stands in contradiction to the proven dominance of optimal sharp restart (discussion above), and Eq. (8) then follows immediately. More generally, an equation similar to Eq. (8) must hold for every restart strategy R which attains a FPT that cannot be lowered further by introducing an additional restart rate $r \ll 1$, and Eqs. (3) and (7) could then be utilized to comprehensively characterize this set of strategies (see SM [44]):

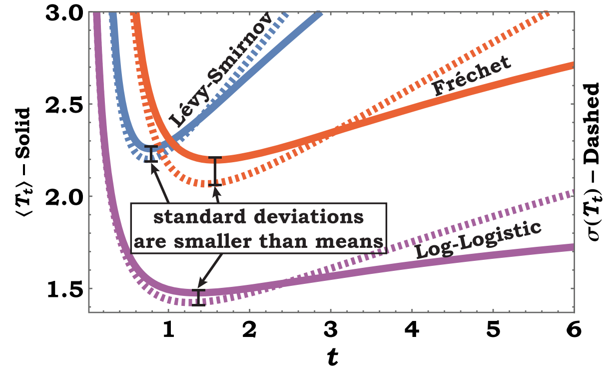

FIG. 5. The mean (solid lines) and standard deviation (dashed lines) of the restarted FPT T_t versus the sharp restart time t, for various distributions of the underlying FPT T (see SM for details [44]).

$$\sigma(T_R)/\langle T_R \rangle \le 1 \Leftrightarrow \langle T_{\min} \rangle \ge \frac{1}{2}\frac{\langle \min(T,R)^2 \rangle}{\langle \min(T,R) \rangle}. \tag{9}$$

A probabilistic interpretation of this result and discussion with examples are given in the SM [44].

Conclusions and outlook.—In this Letter we developed a theoretical framework for first passage under restart. With its aid, we showed how simple observations made for diffusion under restart can be elevated to the level of generic statements which capture fundamental aspects of the phenomena. The universal dominance of sharp restart over other restart strategies is noteworthy. However, while this strategy can be readily applied in some settings, its realization in others may require going to extremes. Particularly, in biophysical settings the generation of tight time distributions relies on the concatenation of irreversible molecular transitions. Restart plays a role in such systems [30,31], but the energetic cost associated with creating an (almost) irreversible transition, and the infinitely many required for mathematically sharp restart, would surely give rise to interesting trade-offs. The incorporation of such thermodynamic considerations into the framework presented herein in the manner of Refs. [46,47], and the identification of those nearly optimal strategies (nonsharp but punctual) [48], which perform best under energy consumption constraints, is yet a future challenge.

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