Miscellany

What Makes for a Beautiful Problem in Science?

. . . It may be logical beauty: Proof that the set of prime numbers cannot be finite—since the product of any set of finite numbers plus one gives a new prime number—is as aesthetically neat in our times as it was in Euclid's. But a problem takes on extra luster if, in addition to its logical elegance, it provides *useful* knowledge. That the shortest distance between two points on a sphere is the arc of a great circle is an agreeable curiosity; that ships on earth actually follow such paths enhances its interest.

By the above test, we must judge Professor Chakravarty's book to be fascinating. India and indeed much of the world has a desperate need to develop economically. Bringing the beautiful tools of optimal control theory to bear upon this vital problem thus cannot help but add to their luster. And Dr. Chakravarty is uniquely suited to perform this task. Along with the zeal of a patriot, his superb natural endowments have been sharpened by economic and mathematical training in Calcutta, Rotterdam, and Cambridge, Massachusetts. And here is a case of water rising above its own source: Interested as much in the dualisms of Tolstoy as in those of linear programming, Dr. Chakravarty is that rare specimen of an almost empty set—namely, the logical intersection of C. P. Snow's two cultures.

Ramsey Economics

This book can be classified as research in the economics of Frank Ramsey. Ramsey, a brilliant Cambridge philosopher and logician, died tragically young in 1930. But not before bequeathing to economics three great legacies—legacies that were for the most part mere by-products of his major interest in the foundations of mathematics and knowledge. The best known of these contributions was Ramsey's 1928 theory of saving. Essentially, this involved a strategically beautiful application of the calculus of variations to define how much an economy should invest

This Introduction is reprinted from *Capital and Development Planning* by Sukhamoy Chakravarty by permission of the M.I.T. Press, Cambridge, Mass.

rather than consume currently in order to maximize an integral of all future utility. Any glance at the journals and monographs of this last decade will show how fruitful for modern growth theory the Ramsey paper has been.

Because reviewers have called attention to various critical remarks in my *Collected Scientific Papers* concerning this Ramsey model, I should like to affirm here an admiration for his contribution. And this is an appropriate place to do so, for Professor Chakravarty has elucidated the two points that worried me about the Ramsey construction: first, I could not agree with the widespread interpretation that Ramseyism leads to a "vast amount of saving." Depending upon the form of the utility function specified, the relative and absolute level of optimal savings can be large or small—as this book demonstrates. Second, as Chakravarty elucidates, maximizing a Ramsey integral rather than some more general functional is equivalent to assuming a kind of cardinal and ordinal "independence" between the marginal utility of one time and that of another. As a young man I was impatient of "restrictive" assumptions, and insufficiently aware of the element of truth in the maxim enunciated by the Nobel biologist, Peter Medawar, that science must deal with that which can be managed, eschewing the intractable. Still, along with Sir John Hicks and Chakravarty, I was right to be concerned with the question of how much of the Ramsey result depends upon this peculiar independence assumption. Since a Foreword is above all the place to look into the future, I should like to mention here that Professor Henry Y. Wan, Jr., of Cornell University, who once honored me by studying at M.I.T., has unpublished research demonstrating that the "turnpike" properties of the Ramsey solutions can still prevail even if we weaken the postulate of strong independence.

Calculus of Variations

The present studies in the theory of development would be classified by the noneconomist as belonging to the modern theory of optimal control. This theory has a fascinating lineage: (1) What is the shortest distance between two points? A straight line. Can you prove it? (2) What about the shortest distance between two points on a curved surface? What begins as a mathematical curiosity culminates in Einstein's theory of general relativity. Indeed, much of theoretical physics can be formulated in terms of variational problems. (3) Light bends between my eye and my toe in the bathtub. How? So as to solve a variational problem; and it reflects in leaving a mirror at the same angle as that at which it has entered, again in order to solve the problem—known to the ancient Greeks and the contemporaries of Fermat—of least elapsed time of arrival. (4) To switch the example, the olive in my martini comes to rest at the bottom of the glass. To act so it need only know the simple Newton-Leibnitz calculus of a single variable, picking the coordinate at which its center of gravity is lowest. (5) What about a flexible rope or chain hanging between two pegs? To make up its mind where to recline, it must determine all the *infinite* coordinates of its snaky self so as to minimize its center of gravity. To fall into a graceful catenary requires solution of a functional problem in the calculus of variations (and modern economic growth theory can solve "consumption catenary problems" by using a lazy rope as an analogue computer!). (6) That apple in Newton's eye, how did it know when to fall? Since the time of Maupertuis, Euler, and Lagrange, Hamilton, Jacobi, and Hertz, we have known that it falls to solve the problem of least action—actually, two such problems simultaneously. (7) Mother Nature, it appears, is a great economist. Her pendulums swing in a path to *minimize* the integral of kinetic potential. But wait, the Lady is a myopic home economist: just as Nature abhors a vacuum only up to 30 inches of mercury, so Nature minimizes Hamilton's integral only over the first half-period of the pendulum. Why this myopia? Just because—just because that's the way the pendulum swings (as Chakravarty's discussion of Jacobi conjugate points can help to illuminate).

Just as some people live interesting lives, some subjects have fascinating biographies. The memoirs of the calculus of variations are especially juicy. When Queen Dido received as a dowry as much land as a bull's hide would surround, she naturally cut it into a long rope. But what shape should be the perimeter of her domain? Of all triangles, the equilateral proved the best. Of all rectangles, the perfect square. Symmetry considerations, even without the calculus, suggested a regular polygon of equal sides; and the more sides the better.[1] Eureka! A circle, the limiting form of regular polygons, was seen to be the solution to this first isoperimetric problem in the calculus of variations (where one integral is maximized subject to another integral's being constant).

The Bernoulli's were great mathematicians but rotten family men. To show up his brother James, John Bernoulli challenged the mathematicians of the world to solve an old problem of Galileo. How should we bend a wire between two points so that a bead will fall smoothly along it from one point to the other in the least time? John himself solved this brachistochrone problem by imagining that the bead was a light ray going through media of different densities so as to solve Fermat's problem of least time. James developed a general method for solving such problems, and the

[1] This brings to mind a related but different kind of maximum problem. What form should market areas take on a homogeneous plane if the mean trip to town is to be at a minimum? Symmetry suggests that any polygons shall be regular, but the triangle, square, and hexagon are the only regular polygons that "cover the plane." Since the hexagon is the *nearest* one to a circle (and that the circle itself will not do is known to every baker of cookies who always has dough left over from circular stampings), the hexagonal shapes of Lösch and Isard are optimal.

calculus of variations was born. (Newton,[2] no longer young, solved the problem after a long day at the Mint. When John saw his solution, he said, "I recognize the paw of the lion.") It remained only for Euler to give the general differential equation that is still associated with his name. (Euler said, in effect: As a rope, I have too many degrees of freedom to worry about in deciding how to recline; so I will pretend I am a chain with only a *finite* number of links; the simple calculus tells me how to arrange each link; letting the number of my links grow indefinitely large, I discern the differential equation that the finite-difference conditions of the discrete variables are approaching.)

All this is good enough for the brilliant eighteenth century. But by the nineteenth it was a scandal that a rigorous mathematical theory was still not known. Thanks to Jacobi, and above all to Weierstrass, the logical gaps were closed. Gauss, Riemann, Levi-Civita, and other geometers worked out the geodesics on general surfaces. In our own time, the school of Bolza and Bliss codified the classical variational theory. Like Latin, it became a dead subject.

But just at this time the sunshine of application brought the dead back to life. Richard Bellman in the United States began to tackle the general problems of dynamic programming. In Russia, the great blind mathematician, Pontryagin, initiated a whole school of investigators. Classical methods of the great Carathéodory and of Hamilton-Jacobi were disinterred and seen to have relevance. Students of Bliss-Valentine, McShane, Hestenes—independently contributed to the foundations of optimal control theory. New problems involving *inequalities* characterize this modern phase.

(1) What is the longest distance between two points? Surely, no answer is possible. (2) But consider two points a mile apart on a horizontal line in the plane. Suppose that the admissible paths cannot have an absolute slope exceeding one million. Then it is not hard to envisage that the longest distance is not a bit over a million miles. But it takes the methods of Pontryagin and not those of the classical theory to handle such

[2] Perhaps the first variational problem in the modern era was posed and solved by Newton—I think in the *Principia*. What shape shall a projectile take to minimize frictional air resistance? Here the master was down one for one: his solution is actually wrong, since it provides only a "weak" minimum that is dominated by a discontinuous solution. In connection with the Brachistochrone problem, the general impression among scholars was long that John's solution while more ingenious than his brother James's less elegant one, provided less profound insight. But Professor Chakravarty has unearthed for me the remarks made by the great Carathéodory at the Harvard 1936 Tercentennary, in which it is shown that John had anticipated much of the differential geometry of Gauss and Riemann. And it was the same unpleasant John who, for a tutorial stipend, sold to affluent noblemen many of the celebrated theorems in mathematics which go under their names. Which brings to mind a conversation I once had with my colleague Harold A. Freeman: "Would you," I asked, "sell your soul, Faust-like, to the Devil in return for a theorem?" "No," he replied thoughtfully, "but I would for an *inequality*."

a problem in which the control variables are constrained within a *closed* set. (3) What is the shortest distance between me and my lady love on the other side of the lake? Not being Byron, I follow a straight line until it touches the lakeshore tangentially. Then I race along that curved shore, until I encounter the tangential straight line that runs into the object of my heart's desire. Bolza and Valentine tell how to handle the inequality, "Don't go in the water," and the two tangencies are critical to an optimal solution. (4) I chase the inedible fox, but must not turn my horse in an accelerating turn that will unseat me. What now the optimum? Here the velocity variables as well as those of state are subject to inequalities, and new methods are needed. (5) In order not to neglect economics, consider the planning problem: Begin with durable hammers and robots that can make new hammers or robots with the same symmetrical input requirements for the one as for the other. Each input is subject to familiar diminishing returns; but increasing both inputs proportionally leads to constant returns to scale (e.g. the production functions are the minimum of the inputs). I begin with 10 robots and 5 hammers and must end up in minimum time with (at least) 1,000 robots and 5,000 hammers. How should I proceed? Surely, I begin by producing only hammers, since the redundant robots are subject to diminishing returns. When I reach 10 of each input, I shift gears to produce equal amounts of both, proceeding along what is known as a von Neumann—DOSSO turnpike. You might think I go along it until 1,000 of each are attained, thereafter producing only the 4,000 more hammers. And you might be right. But you might be wrong. Depending upon the exact technology, it might pay to produce more of both along the turnpike, ceasing to produce more robots only after they are somewhere between 1,000 and 5,000 in number. We end up with more than we needed of robots. Irrational, since we don't want them and they cost something? Not necessarily, since they may have been worthwhile for their help in producing the extra hammers we do want. Even though the convex technologies of Professor Chakravarty's book and this example do not permit Jacobi conjugate points to arise, here we have something like that phenomenon and again the problem of Mother Economist's possible myopia.

Cabbages, Kings, and Turnpikes

A scientific problem gains in aesthetic interest if you can explain it to your neighbor at a dinner party. That no map requires more than four colors is such a problem. Can the turnpikes of Professor Chakravarty pass this test? In a system in which everything produces everything, a critical set of proportions at which balanced growth is most rapid seems reasonable. So the general notion of the von Neumann turnpike is conveyed. But suppose we begin away from this highway, and also are to end up away from it. Still, if the journey is to be a sufficiently long one,[3] as seen we shall do well to move in the beginning toward that fast highway; catch a fast ride along it, staying so long there as to leave behind in all directions any rival who sticks to any slower path; then, toward the end of our journey, we can swing off the turnpike to our final rendezvous. The same kind of reasoning used on this production-terminal turnpike can be used for the consumption turnpike theorem. Suppose the best steady-state or golden-age level is called bliss. If we begin anywhere away from it, and are to end somewhere away from it, provided that our journey is to be long enough, surely by going to the bliss level and staying a long, long time in it, we will pile up more well-being than will some rival who stays definitely away from it during that long interval. Hence, the result: In a sufficiently long journey, stay, most of the time, indefinitely near to the turnpike.

A Foreword, like an aperitif, merely whets the appetite for the main course. To Professor Chakravarty's readers, I say, "*Bon appetit!*"

PAUL A. SAMUELSON

Massachusetts Institute of Technology

[3] Planning for a very long time can introduce new complexities and also new simplifications. Here is an example of the latter. As is well known, the Fisher "rate of return over cost" or Boulding "internal rate of return or yield" of an investment project may have multiple roots. Fascinating reswitching problems aside, presumably economists will reject all imaginary roots, but how shall one choose among alternative roots that are real and positive? The following asymptotic theorem has at one time or another been proved independently by Robert Solow, David Gale, and me: If one can reinvest continuously the proceeds of the investment process in similar divisible processes, then asymptotically your capital will approach a rate of growth equal to the largest positive root.

The extra complexities introduced by an infinite time horizon are much the concern of Professor Chakravarty. Ramsey had cleverly introduced satiation assumptions that made the divergence from "bliss" form a convergent infinite time integral. But if utility of consumption is unbounded, growing, say, as the square root of consumption, and if diminishing returns is absent for capital, the more saved the better without limit. The maximand is infinite for all feasible paths; but as every tot knows when pitting his infinity against his playmate's, some infinities seem better than others. Path A may, like Wonderbread, be two ways better; it gives all that Path B gives and more. This is related to the sophisticated Phelps-Koopmans concept of "permanent inefficiency." We convict a system of violating that concept if it stays permanently with more capital than in the golden rule state. But can we at any finite date, however distant, convict them of the stated crime? This is also reminiscent of an improper maximum problem, such as: What is the maximum negative real number? Mathematicians in an earlier century had overlooked these difficulties, and it was not until the time of Weierstrass and Hilbert that it was realized that there might be no minimizing solution to a Dirichlet problem in physics. Nature imitates art and savants catch up, finally, with soap bubbles.

To the paradoxes of the infinite there is no end, so I must break off without discussing competitive equilibria which fail to be Pareto optimal in an infinite-time program. Karl Shell has used in this connection Gamow's example of the fully-occupied hotel with infinite bedrooms. No applicant is ever turned away since he can be given Room 1 on condition that its occupant be given Room 2, and so forth.