GIAEVER, Harald Birger, 1927–
OPTIMAL DAIRY COW REPLACEMENT POLICIES.

University of California, Berkeley, Ph.D., 1966
Economics, agricultural

University Microfilms, Inc., Ann Arbor, Michigan

Optimal Dairy Cow Replacement Policies

By

Harald Birger Giaever

Grad. (Agricultural College of Norway) 1954
M.S. (North Carolina State College of Agriculture and Engineering of the University of North Carolina) 1959

DISSERTATION

Submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Agricultural Economics

in the

GRADUATE DIVISION

of the

UNIVERSITY OF CALIFORNIA, BERKELEY

Approved:

Oscar R. Burt

Harold O. Carter

Robert C. Laben

Committee in Charge

Degree conferred..
Date

ACKNOWLEDGMENTS

This study has drawn upon ideas and help from many sources:

The chairman in my dissertation committee, Dr. Oscar R. Burt, conceived the original idea of a dairy cow replacement study and of using Markovian dynamic programming as part of the method. He has been available for numerous discussions of problems met during the progress of the work; has read the manuscript in various editions; and given many helpful suggestions. Dr. Robert C. Laben in the Department of Animal Husbandry supplied the empirical data for the analysis. Without this extremely valuable observation material, painstakingly collected over a large number of years, the empirical part of this study would not have been possible. He has also been very helpful in discussing the many problems in the field of dairy science which had to be faced when formulating a specific model of this kind and when estimating parameters for such a model. Also Dr. Harold O. Carter has been available for a number of discussions; has read the manuscript; and has given many helpful suggestions.

The idea to use a multivariate stochastic model for analyzing milk production data of dairy cows, as a basis for defining a sequence of "production variables" which meets the Markov specifications, and as a basis for deriving transition probabilities in a finite Markov chain, was my own. When developing a multivariate parameter estimation method which corrects for bias due to culling, I received very valuable help from Dr. Henry Scheffé in the Department of Statistics. Also Dr. Ivan Lee furnished help in this part of the study.

Miss Elaine Borkon performed the numerical calculations of multivariate parameters on the computer. Mrs. Suzanne Hjerpe wrote the dynamic programming program which was used, and was responsible for the numerical calculations of optimal solutions. Mrs. Hanna Svensson did a careful job in typing the final version of the dissertation.

In addition to the assistance described above, I have drawn upon many theoretical developments and empirical research findings which are mentioned in footnote references in the next.

The Department of Agricultural Economics at University of California furnished the research assistantship which enabled me to spend almost two and a half most rewarding years at Berkeley and to complete this study. During this time period, I was on a leave of absence from a position at the Agricultural Research Council of Norway, which also gave some economic support. The United States Educational Foundation in Norway provided a travel grant.

I want to express my gratitude to the many persons and institutions mentioned above, who in this way have contributed to making this study possible.

TABLE OF CONTENTS

Chapter		Page
I.	INTRODUCTION	1
II.	RELATION TO ECONOMIC THEORY AND TO PREVIOUS STUDIES .	5
III.	A THEORETICAL FRAMEWORK FOR A DAIRY COW REPLACEMENT MODEL	29
IV.	DERIVATION OF TRANSITION PROBABILITIES WITH RESPECT TO PRODUCTION	65
V.	PHYSICAL RELATIONSHIPS IN DAIRY PRODUCTION, AND THE SPECIFICATION OF A DAIRY COW REPLACEMENT MODEL	96
VI.	ESTIMATION OF PARAMETERS	135
VII.	NUMERICAL REPLACEMENT MODELS	183
VIII.	REPLACEMENT POLICIES AND IMPUTED VALUES	203
IX.	OPTIMAL REPLACEMENT TIME WITHIN A LACTATION	219
X.	SUMMARY AND CONCLUSIONS	229
LITERATURE CITED		245
APPENDIX A		250
APPENDIX B		294

LIST OF TABLES

Number		Page
5.1.	Reasons for Cow Culling	98
5.2.	Milk Revenue less Variable Feed Costs for Quantities of Milk Corresponding to 100 Pounds Milk, 1 Pound Milk Fat, and 100 Pounds FCM, for Different Fat Percentages . . .	103
7.1.	Selected Rows in the Matrix of Discounted Expected Physical Quantities for the ME-herd	202
A.1.	Age-Corrected Herd Averages of Pounds 305 Days Fat-Corrected Milk	251
A.2.	Regression Analysis of Pounds 305 Days FCM: Herd Average Excluded as Explanatory Variable; ME-herd	252
A.3.	Regression Analysis of Pounds 305 Days FCM: Herd Average Included as Explanatory Variable; ME-herd	253
A.4.	Regression Analysis of Pounds 305 Days FCM: Herd Average Excluded as Explanatory Variable; MA-herd	254
A.5.	Regression Analysis of Pounds 305 Days FCM: Herd Average Included as Explanatory Variable; MA-herd	255
A.6.	Estimated Parameters in a Conditional Multivariate Distribution of 305 Days FCM; ME-herd	256
A.7.	Estimated Parameters in a Conditional Multivariate Distribution of 305 Days FCM; MA-herd	257
A.8.	Expected 305 Days FCM and Estimated Coefficients of Variation	258
A.9.	Definition of Production Variables v_f's a Sequence of Which Forms a Markov Process, and Comparison with Variables v_f^*'s Based on Least Squares Regression Coefficients	259
A.10.	Unconditional and Conditional Variances of u_f	260
A.11.	Production Class Intervals	261

A.12. Transition Probabilities with Respect to Production Classes; ME-herd 264

A.13. Transition Probabilities with Respect to Production Classes; MA-herd 266

A.14. Estimated Relationships between 305 Days FCM and FCM for Separate Months when Production is not Influenced by a New Pregnancy; ME-herd 268

A.15. Estimated Regression Coefficients for Monthly FCM Regressed on 305 Days FCM for Months when Production is Influenced by a New Pregnancy; ME-herd, 1st Lactation . 269

A.16. Estimated Regression Coefficients for Monthly FCM Regressed on 305 Days FCM for Months when Production is Influenced by a New Pregnancy; ME-herd, 2nd - 6th Lactation 270

A.17. Assumed Relationship between 305 Days FCM and FCM for Separate Months; for Use in the Replacement Models: 1st Lactation 271

A.18. Assumed Relationship between 305 Days FCM and FCM for Separate Months; for Use in the Replacement Models: 2nd - 6th Lactation 272

A.19. Observed Frequencies of Lactations with Known Calving Intervals 273

A.20. Observed Frequencies of Lactations with Unknown Calving Intervals (Last Lactation of Each Cow) 274

A.21. Estimated Probability Distribution of Calving Intervals in Populations Where All Cows Are Kept at Least Until 625 Days after Last Freshening 275

A.22. Number Observed Lactations and Number Involuntary Removals Classified by Level of Production 276

A.23. Number Observed Lactations and Number Involuntary Removals Classified by Time Period 277

A.24. Number Observed Lactations and Number Involuntary Removals Classified by Lactation Number 278

A.25. Probabilities of Involuntary Removals Estimated as Functions of Time Period and of Lactation Number 279

A.26. Assumed Monthly Probabilities of Involuntary Removals . 279

A.27. Assumed Relationships between Milk Production and Feed Consumption 280

A.28. Alternative Price Sets Used in the Replacement Models . 281

A.29. Expected 305 Days FCM in Pounds for Present and Next Lactation when Production Class is Given; ME-herd . . . 282

A.30. Expected 305 Days FCM in Pounds for Present and Next Lactation when Production Class is Given; MA-herd . . . 283

A.31. Optimal Replacement Decisions and Present Values under an Infinite Planning Horizon: ME-herd; Price Alternative 1 284

A.32. Optimal Replacement Decisions and Present Values under an Infinite Planning Horizon: ME-herd; Price Alternative 2 285

A.33. Optimal Replacement Decisions and Present Values under an Infinite Planning Horizon: ME-herd; Price Alternative 3 286

A.34. Optimal Replacement Decisions and Present Values under an Infinite Planning Horizon: ME-herd; Price Alternative 4 287

A.35. Optimal Replacement Decisions and Present Values under an Infinite Planning Horizon: MA-herd; Price Alternative 1 288

A.36. Present Values under an Infinite Planning Horizon if the Replacement Policy Deviates from the Optimal: ME-herd; Price Alternative 1; no Voluntary Replacement 289

A.37. Replacement Decisions and Present Values under an Infinite Planning Horizon for a Replacement Policy Which Deviates from the Optimal: ME-herd; Price Alternative 1; Intensive Culling . . . 290

A.38. Replacement Decisions and Present Values under an Infinite Planning Horizon for a Replacement Policy Which Deviates from the Optimal: ME-herd; Price Alternative 1; Replacement Policy as for the MA-herd 291

A.39. Replacement Decisions and Present Values under an Infinite Planning Horizon for a Replacement Policy Which Deviates from the Optimal: MA-herd; Price Alternative 1; Replacement Policy as for the ME-herd 292

A.40. Probabilities of Calving, and Probability Distributions of Replacement Causes, under Given Replacement Policies 293

I. INTRODUCTION

The purpose of this study is to analyze the decision problem facing a dairy producer with respect to when, or under which conditions, to replace a dairy cow. This is sought accomplished through the development of an economic decision model for dairy cow replacement, estimation of parameters for this model based on empirical data, and derivation of optimal solutions to the model under alternative price conditions.

The objectives underlying management decisions may be of different kinds. For example, a dairy producer may want to replace a given animal in order to increase the average production of his herd, or in order to maximize the rate of genetic improvement in the herd, regardless of economic considerations. For the majority of dairy producers, however, it is believed that economic considerations are of greatest importance. This study will consider the replacement problem only within the context of profit-maximizing behavior, and disregard the other considerations just mentioned.

Dairy production involves incomes and outlays at different points in time. In order to make economic consequences of different replacement policies comparable, it is proposed to discount future incomes and outlays to present values. Because these future incomes and outlays can be predicted only in probabilistic sense, present value of the differences between expected future incomes and expected future outlays can be used as a criterion by which to compare different replacement policies.

Most replacement theory assumes that rates of income and rates of outlay associated with a durable asset of production are mathematical functions of the age of the asset, and even that the discounted difference between rate of income and rate of outlay decreases monotonically with increasing age after a given age is reached. It is evident that an analysis of the dairy cow replacement problem calls for a much more complicated model.

First, milk production is cyclic by nature. This gives the replacement problem two dimensions: A dairy producer must decide whether or not to replace a cow during a given lactation, and if he has decided to replace it during one lactation, when during the lactation to do it. While this problem preferably should have been solved by some simultaneous procedure, it has been necessary in this study to simplify it by assuming a given time for replacement within the lactation, and concentrate attention on the problem of selecting the best lactation number for replacement. It has been shown that deviations from the assumption make very little difference with respect to selecting the best lactation number for replacement, and that the determination of optimal replacement time within the lactation can be done by very simple methods.

Second, different stochastic elements enter the replacement problem and are so important that they can not be neglected in a replacement model for dairy cows. While expected milk production in a population of dairy cows can be taken as a function of age, there is a very substantial individual variation around the expected population values. There is also a substantial variation between production records for different lactations of the same cow, even after corrections are made for the effects of age. Another stochastic variable of great importance for the replacement decision is the degree of breeding efficiency, or as it has been expressed in this study, the length of the calving interval. A third stochastic element which should be considered is the possibility of forced removal due to accidents, serious illness, etc.

No known previous studies have attempted to analyze the replacement problem in dairy herds with consideration of all these factors. The reason may be that the necessary mathematical and stochastic framework for such an analysis has only recently become available. This study deals with the problem within the general framework of dynamic programming with Markov processes. This decision model provides a quite satisfactory framework for the analysis, however some intriguing problems have to be faced before the general model can be used.

Chapter II deals with the replacement problem as a part of general economic theory, discusses some existing replacement models, and deals with the few known attempts to handle the dairy cow replacement problem in an analytical way. Chapter III presents the theoretical framework of Markovian dynamic programming. While existing presentations of the method usually assume that stages or time intervals between each decision are of constant length, it is shown in this chapter that the model can easily be extended to the case where stage length is a variable and may even have a stochastic distribution.

Two difficult problems in applying the general method to the dairy cow replacement case are to satisfy the so-called Markov requirement and to derive transition probabilities with respect to a criterion of level of milk production. Chapter IV shows that this problem can be dealt with within the framework of multivariate analysis, if it is assumed that milk production measured as 305 days production for consecutive lactations has a multivariate normal distribution. Under this assumption, a criterion for level of milk production which satisfies the Markov requirement can be defined, and transition probabilities in a finite Markov process can be derived from the estimated parameters in the multivariate distribution.

One problem which has to be faced regardless of which stochastic model is used for analysis, is that the dairy herds from which we can draw samples have been subject to culling and, therefore, do not supply unbiased samples from the population which we want to sample. An attempt has been made to develop an estimation method which will give unbiased estimates of population parameters in a multivariate normal distribution. This method is presented in an appendix.

Chapter V discusses received dairy science results which are relevant to the specification of a dairy cow replacement model and to the estimation of parameters for such a model. Based on such results, a detailed replacement model is specified. Chapter VI presents results from the estimation of parameters based on records of about 1,350 Jersey cows divided in two different herds. In chapter VII, the final construction of numerical replacement models based on the estimated parameters is discussed. Results of the optimization procedure are presented in chapter VIII. A method for determining optimal replacement time within the lactation is described in chapter IX. Most of the numerical results, both of the estimation of parameters and of the derivation of optimal solutions, are tabulated in appendix A.

II. RELATION TO ECONOMIC THEORY AND TO PREVIOUS STUDIES

A. Economic Theory and the Replacement Problem

The theory of replacement of durable capital items used in production may be considered part of production theory in economics. Evidently, however, only that part of production theory which deals with production over time is relevant to the replacement problem. In the theory of one-period production, replacement problems have no place, since all factors are assumed used up during the current production period.

The replacement problem is only a very special case within production theory, however. Conceptually, it is possible to arrive at a solution to any replacement problem as a part of the general equilibrium solution for a firm. In practice, models giving the general equilibrium solution would often be extremely large and unwieldy, and in order to formulate operationally manageable models, we must often be willing to settle for something less. Thus, literature dealing with the replacement problem has usually been limited to something which may be called suboptimizing or partial equilibrium analysis. Most aspects of the firm's decisions have been left untouched or have been assumed given, and attention has been concentrated on the more detailed problem at hand.

Thus, what the general theory of production over time can supply are some general concepts useful in analyzing a replacement problem. Some of these concepts will be discussed below. To look for operationally manageable models, we must turn to more specialized literature.

1. Multiperiod Production Under Certainty

In order to make problems analytically manageable, the parts of production theory which deal with production over time often divide up the total time span considered into time periods of equal length. Multiperiod production is characterized by a situation where factors of production purchased during one time period will influence levels of output during subsequent time periods. Therefore, the theory uses the concept of a multiperiod production function, where inputs and outputs are dated and there exist functional relationships between variables with different dating.

It is further assumed that there exists a market for money (in the sense of purchasing power) at which money can be borrowed or lent at some given rate of interest. By using this rate of interest, outlays or revenues incurred during different time periods can be made comparable by discounting them to one and the same time period, usually to the present.

The interest rate expresses the cost of borrowing, or income from lending, for a duration of one time period, as a proportion of the amount borrowed or lent. If the interest rate is i, this means that an amount A borrowed on the first day of one time period will be repaid with the amount $A(1 + i)$ on the first day of the next time period, or, if the interest does not change over time, with the amount $A(1 + i)^2$ on the first day of the time period thereafter, or, in general, with the amount $A(1 + i)^t$ on the first day t time periods after the amount was originally borrowed. An amount A payable t time periods from the present can always be exchanged for, and is therefore equivalent to, an amount $A(1 + i)^{-t}$ at the present, since a person or a firm which expects to receive an amount A, t time periods into the future, can borrow the amount $A(1 + i)^{-t}$ at the present time and t time periods later can repay it with the amount

$A(1 + i)^{-t}(1 + i)^{t} = A$. The <u>discount factor</u> by which a future amount should be multiplied in order to give its present value is thus $(1 + i)^{-t}$.

Discounting furnishes a method for transferring a stream of future incomes, outlays, or the differences between the two, to a single number which is called the <u>present value</u> of the future stream. For example, if an entrepreneur expects to receive incomes $R_1, R_2, \ldots, R_T$ during time periods 1, 2, . . . , T, respectively, and the corresponding discount factors are $\beta_1, \beta_2, \ldots, \beta_T$, the present value of this revenue stream is

$$R = \sum_{k=1}^{T} \beta_k R_k$$

While one-period production theory assumes that an entrepreneur will maximize his profit during the given period, the corresponding assumption in multiperiod theory is that he will maximize the difference between present value of future incomes and present value of future outlays. It is evident that it gives the same result whether we first compute present values of incomes and outlays separately and then take the difference between the two, or we compute the differences between incomes and outlays for each time period first and then take the present value of the differences.

So far, we have assumed perfect knowledge both with respect to the multiperiod production function and with respect to input and output prices at any time period in the future. Modifications in the conceptual apparatus to allow for deviations from perfect knowledge will be discussed below. At this point, it is convenient to introduce the concept of the firm's <u>planning horizon</u>. A firm's planning horizon is the number of future time periods for which it determines the values of input and output variables. The introduction of a limited planning horizon may have different purposes. Each input and each output for each time period within the planning horizon is a separate variable, and by limiting the planning horizon we can limit the number of variables we have to handle in an operational model. We may also want to introduce a limited planning horizon as a way to allow for the fact that knowledge about future events is really less than perfect and that we know less about parameter values for time periods far ahead than about those close ahead.

If the planning horizon is less than infinite, the objective is restated as maximization of present value of the differences between incomes and outlays within the planning horizon plus the discounted value of the firm at the end of the planning horizon.

2. Delineation of the Replacement Problem

Within the conceptual framework described above, we will try to delineate the replacement problem from the total planning problem of the firm. The total planning problem consists in determining the value of all input and output variables for all time periods within the planning horizon. The replacement problem is only concerned with determining values of input variables for those variables which represent assets of durability more than one production period and, furthermore, only with cases where one durable asset is replacing another already in operation, instead of simply being added to the total number of such assets. For example, if a farmer decides to add one dairy cow to those already in production at some given time, this is not a replacement situation but one of increasing the size of the dairy operation. On the other hand, it has been argued that for a replacement situation to exist, it is not necessary that the replaced asset is removed from the firm, only that it is transferred from the class of service it has rendered so far.[1/]

Further, it has been argued that a replacement situation exists only when the level of any output is not changed as an integral part of the action.[2/] According to this view, only outlays and not revenues will be influenced by a replacement action, and the purpose of the action should be to minimize the present value of the outlay stream for given levels of output. A case where an entrepreneur replaces an old capital item with a new one and at the same time adjusts the level of output would be looked upon as a combination of a replacement action and a change in scale of operation.

While such a restrictive definition of the concept of "replacement" may be justified in a more general theory of the role of capital in production, it appears to be extremely difficult to maintain this distinction between "replacement" and "scale adjustment" in an operational model. The reason can be expressed in this way: the short-run equilibrium position for a firm is where the short-run marginal costs for each output equals the corresponding marginal revenues. By substituting a new capital item for an old one, however, we will go from one short-run situation to another, and short-run marginal costs for given levels of output will often change. If the firm was in short-run equilibrium before the substitution, it may be in a state of disequilibrium after the substitution

---

[1/] George Terborgh, Dynamic Equipment Policy (New York: McGraw-Hill Book Co., 1949), pp. 23-24.

[2/] Vernon L. Smith, Investment and Production (Cambridge, Mass.: Harvard University Press, 1961), pp. 131-134.

if level of output is not adjusted. The economic result of the substitution should be analyzed by comparing the short-run equilibrium position before the substitution with the short-run equilibrium position after the substitution.

As an example, consider the replacement of an old cow with low productive ability with a new cow with higher productive ability. Be feeding the new cow a less-than-optimal feed ration, it might be possible to maintain milk production at the same level as before; however, the economic result from such a policy would not be a proper basis for studying the economic effect of the replacement. Again, it may be argued that we could maintain the distinction between "replacement" (not changing the level of output) and "scale increase" (changing the level of output) by, for example, considering the substitution of four new cows with higher productive ability for five old cows with lower productive ability, thus keeping output constant. Such a substitution would, with the restrictive definition of replacement, be a true replacement, while adding a fifth new cow would be an increase in scale. Thus the distinction can be maintained in theory but in practice would be extremely impractical to maintain in an operational model, and especially when indivisibilities of large capital items come into the picture. Thus, it is hardly practical to consider replacement of one old truck with four-fifths of a new one.

For the purpose of this study, the term "replacement" will be taken to mean any case where one durable capital item is substituted for another one, whether the replacement is accompanied by a change in output or not. To classify a problem as a "replacement problem" will imply that only a part of the firm's planning problem will be studied, however, so that some of the variables over which the entrepreneur has control will be assumed given. Exactly what will be assumed given should be decided in the individual case.

3. Considerations of Risk and Uncertainty

The analytical treatment of the replacement problem is often simplified by assuming perfect knowledge of input-output relationships and of prices within the firm's planning horizon. The introduction of risk and/or uncertainty into the conceptual apparatus will complicate the problem but may, from a practical point of view, be justified when the added realism is considered. Following Knight, Marschak, and Tintner, we will define a risk situation aa a situation where the outcome of a future event is not known but where we know the probability distribution of the future event. A situation where we know the probability distribution of alternative probability distritutions is classified as an uncertainty situation.[1/]

It is evident that objective risk situations are seldom encountered in the real world; however, to treat a real planning problem *as if* we know the probability distributions of future events is still likely to

---

1/ The concepts of risk and uncertainty in this sense were developed by Knight, formalized by Marschak, and elaborated by Tintner in the literature cited below. Tintner used the terms "subjective risk" and "subjective uncertainty" to emphasize that neither the "probability distribution of future events," nor "the probability distribution of alternative probability distributions" are usually known, but the individual may act *as if he knows them*.

Frank H. Knight, *Risk, Uncertainty and Profit* (Boston: Houghton Mifflin Co., 1921).

J. Marschak, "Money and the Theory of Assets," *Econometrica*, VI (October, 1938), p. 324.

Gerhard Tintner, "A Contribution to the Nonstatic Theory of Production," *Studies in Mathematical Economics and Econometrics*, ed. O. Lange, F. McIntyre, and T. O. Yntema (Chicago: University of Chicago Press, 1942), pp. 92-109.

give more satisfactory solutions than to treat it as if we know the exact outcomes. Instead of including the true probability distributions in the analytical treatment, we can include the best available estimates of these probability distributions and proceed with the analysis as if these estimates are the true distributions. The practical problem lies in the added complexity of an analytical model which is able to include risk situations.

The introduction of risk or uncertainty in a replacement problem requires a restatement of the objective criterion. Under multiperiod production but perfect knowledge, maximization of "present value of the differences between future incomes and outlays within the planning horizon plus discounted value of the firm at the end of the planning horizon" appears to be a reasonable criterion. The closest alternative under risk is maximization of "present value of expected differences between future incomes and outlays plus discounted expected value of the firm at the end of the planning horizon".

Under risk or uncertainty, however, we may think of a number of alternative objective functions which are not relevant under certainty, and it is far from obvious which one should be preferred. Some entrepreneurs might actually prefer a production alternative with high risks (one giving higher probabilities for very high earnings in return for higher probabilities for low earnings) for one with low risk, while the opposite may be true for most entrepreneurs. Some may want to specify a lower limit for income under which it should not be allowed to fall, even under the most unfavorable circumstances, but want to maximize expected income subject to this restriction, etc. Since there is important literature dealing with these considerations, and the present study does not intend to explore the question further, the problem will be left here.

In a dairy herd, some cows will give lower returns than anticipated, while others will give higher returns, and as an average over a herd with, say, 20 cows or more, the realized return for a given replacement policy is not likely to deviate very much from the expected. Nor does it seem likely that one replacement policy will give much more spread in realized results than another. It appears, therefore, that in our specific case we may fairly safely assume maximization of expected present value to be the objective criterion. This is not necessarily the case in all replacement problems involving risk.

4. Continuous Production Over Time

It is often convenient to conceive of a production process as continuous over time. The continuous case can simply be thought of as the limiting case for multiperiod production, as each time period is made smaller and smaller until it approaches zero. In the continuous case, divisible inputs and outputs will be measured as instantaneous rates where the rate at any given point in time is given as a function of time. Indivisible inputs and output will be given as units at a given point in time; however, their value may well be a continuous function of time. For example, the salvage value of a durable asset may be thought of as decreasing continuously with increasing age.

When we are dealing with continuous production, it is also convenient to redefine discounting to be a continuous process. In the multiperiod case, interest was conceived as being added to the amount borrowed or lent as it was earned for each time period. The discount factor by which a future could be converted to present value was thus defined as $\beta = (1 + i)^{-t}$, where t was an integer number of time periods and i was given as a proportion (of amount borrowed or lent) per time period.

In the continuous case, i can be thought of as an instantaneous rate of discounting, t as a continuous variable, and interest can be thought of as being added continuously to capital. It can be shown that if i is constant over time, the discount factor will transform to

$$\beta = e^{-it}$$

where e is the base of the system of natural logarithms (e = 2.71828 apx), i is the instantaneous rate with the dimension "proportion (of capital) per time unit", and i and t are measured in regard to the same time units (for example, years).[1/]

The actual difference in results between the two methods for transforming a future amount to present value, or a present amount to future value, is not very large. For example, if the interest rate is given as .06 per year, an amount of $1,000.00 borrowed today should be repaid by $1,060.00 after one year if interest is added to capital once each year, and by $1,061.84 if we use instantaneous discounting. If we want to use instantaneous discounting, but avoid the more rapid growth in value of the amount borrowed or lent, we can simply use a slightly lower nominal interest rate.

The present value of a continuous stream of future income between now and time T can be defined as

$$\int_{0}^{T} R(t)e^{-it}dt$$

where R(t) is the rate of income as a function of time. Present value of a continuous outlay stream is defined analogous.

1/ R. G. D. Allen, Mathematical Analysis for Economists (London: Macmillan & Co., 1960), pp. 228-232.

B. Some Replacement Models Relevant to this Study

1. A Survey of Some Relevant Models

It was pointed out in the introductory remarks to this chapter that economic models dealing with the replacement problem usually are models for suboptimization. Many of the variables which would enter a complete production planning problem are assumed given or are ignored. Such simplifications are often necessary to keep the problem within manageable dimensions.

Thus, it is often assumed that the levels of all variable inputs and outputs associated with the use of one durable asset have been determined before we set out to find the optimal replacement policy for that asset. We may assume that these levels have been determined as a part of a short-run optimization procedure under the assumption that the given asset is present. When input and output prices are also assumed given, we will know the incomes and the outlays associated with that piece of asset for any age of the asset. These incomes and outlays can be given as periodic amounts in the multiperiod case or as instantaneous rates in the continuous case.

A "family" of replacement models which are of special interest to our specific problem will be mentioned here. These models all rest on the same assumptions:

(1) We have a "chain" of replacement, where one asset is always replaced by another of identical or similar type. We may have several parallel "chains" of assets in operation at the same time, but attention is concentrated on one chain only, and variation in the number of chains is not considered.

(2) Variation in intensity of use of the asset is not considered. Thus, these models do not consider the case where the life-span of one asset depends on how heavily it is being used.

(3) Income per time unit, outlays per time unit, and salvage value of the old asset all are known mathematical functions of age of the asset. These functions may be discontinuous (as when production is conceived of as periodic) or continuous.

(4) There is no capital rationing. Capital is available at a given rate of interest.

(5) The objective criterion is maximization of the present value of incomes, less outlays, or of its annualized equivalent.

A correct solution to the optimization problem is attributed to the German forester, Faustman, who wrote a treatise on this subject as early as 1849.[1/] The first modern treatment of the problem seems to be one by Preinreich.[2/] If we use continuous functions and continuous discounting, Preinreich's principle says that if we have a chain consisting of n links, we should select values of t_j so as to maximize V_{j-1} in the recursive formula

$$V_{j-1} = \int_0^{t_j} [R(t) - E(t)] e^{-it} dt + S(t_j) e^{-it_j} + V_j e^{-it_j} - A \qquad (2.1)$$

where,

t_j ($j = 1, \ldots, n$) = the replacement age for the jth asset in the chain.

$R(t)$ = rate of income as a function of age.

$E(t)$ = rate of outlays as a function of age.

1/ Martin Faustman, "Berechnung des Wertes, welchen Waldboden, sowie noch nicht haubare Holzbestände für die Waldwirtschaft besitzen," Allg. Forst und Jagd Zeitung, XXV (1849), pp. 441-455. Quoted from: M. Mason Gaffney, Concepts of Financial Maturity of Timber and Other Assets, North Carolina State College, A. E. Information Series No. 62 (Raleigh, N.C., 1957).

2/ Gabriel A. D. Preinreich, "The Economic Life of Industrial Equipment," Econometrica, VIII (January, 1940), pp. 12-44.

$S(t)$ = salvage value as a function of age.

A = acquistition outlay for each new asset.

In words, the principle says that we should select replacement age so as to maximize the present value of the whole future difference between income and outlay streams, where the present value is the present value associated with the existing asset, plus the present value of all future replacements at the time of first replacement, discounted to today.

We find the first-order maximum condition by taking the first derivative of V_{j-1} with respect to t_j and setting it equal to zero. The result is:

$$R(t_j) - E(t_j) + \frac{d}{dt} S(t_j) = i[S(t_j) + V_j] \tag{2.2}$$

In words, the result says that we should replace when rate of income from the asset, plus rate of increase in salvage value, less rate of outlay, equal the interest rate times the sum of salvage value and present value (at the time of the replacement) of the next asset in the chain.

We may look at this as a special case of the general marginal principle. The left-hand side of equation (2.2) expresses "marginal net revenue per time unit," while the right-hand side expresses the marginal cost of using time in production. The marginal cost is in this case an opportunity cost; namely, the foregone income from the next asset in the chain, plus the opportunity cost of the capital tied up in the salvage value of the old asset. The existing asset must compete with its subsequent replacements for its "space" in the enterprise. This is an important feature with this principle, which was often overlooked in earlier treatises on the replacement problem. The principle generally leads to earlier replacement than when this feature is not considered.

The principle is valid for assets which appreciate over time (like a crop of forest trees) as well as for assets which depreciate or have a constant salvage value regardless of time. Preinreich discussed only industrial equipment and simplified the formula by assuming salvage value constant; however, by letting the salvage value be a function of age, the formulae are valid for typically appreciating assets like forest trees and growth animals as well.

Further, the principle is relevant if we either have a given number of links left in the replacement chain (n is given) or we want to maximize under an infinite planning horizon. The last case seems to be most relevant for practical decision making. <u>If</u> we want to consider a finite planning horizon, this will usually consist of a given number of years rather than of a given number of links in the chain. This could be taken care of by introducing a restriction on time:

$\sum_{j=1}^{n} t_j = T$, and maximizing for V. The solution would be much more complicated, however, since we would have to find all t-values simultaneously.

With n given, we can find successive values of t_j by working backwards from $j = n$. V_n must be assumed given. By inserting this value in (2.2), we can find t_n, and this value inserted in (2.1) will give V_{n-1}, which again can be inserted in (2.2) to find t_{n-1}, and so on. As the chain is lengthened, a limit eventually emerges where $V_j = V_{j-1}$ and $t_j = t_{j-1}$. That is the case of the infinite chain, in which the replacement age of all assets in the chain is the same.[1]

---

[1] Ibid., p. 17.

This replacement age, and the corresponding present value at the time for replacement, can be found from (2.1) by dropping the subscripts j and $j-1$, solving for V, and then maximizing.

Replacement rules equivalent to the principles described here are found in literature in many different forms. They may vary somewhat according to whether we use continuous or discontinuous functions and continuous or discontinuous discounting and according to whether the objective maximized is present value or its annualized equivalent. Faustman's formula seems to have been based on the annualized equivalent of present value, and the same is the case with two applications of special interest to agriculture. M. Mason Gaffney has dealt with the optimal harvest age of timber.[1/] J. Edwin Faris has discussed more generally biological production cycles involving time, like growth animals, forest trees, and fruit trees.[2/] Winder and Trant have discussed some of Faris's replacement rules and provided a more mathematical discussion of the principles.[3/]

Faris also discussed the case where the interest rate was set equal to zero. In this case, a reasonable objective criterion is maximization of average net return per time unit. It seems intuitively likely, though

---

1/ M. Mason Gaffney, Concepts of Financial Maturity of Timber and Other Assets, North Carolina State College, A.E. Information Series No. 62 (Raleigh, 1957).

2/ J. Edwin Faris, "Analytical Techniques Used in Determining the Optimum Replacement Pattern," Journal of Farm Economics, XLII (November, 1960), pp. 755-766.

3/ J. W. L. Winder and G. I. Trant, "Comments on 'Determining the Optimum Replacement Pattern,'" Journal of Farm Economics, XLIII (November, 1961), pp. 939-951.

no attempt will be made to demonstrate it here, that for assets where the optimal replacement age is very low (for example, pigs and broilers), the replacement age which maximizes present value (with a positive interest rate) will deviate very little from the replacement age which maximizes average net return per time unit.

It should be mentioned here that the principle of dynamic programming in many cases will provide a convenient analytical and computational framework for dealing with the replacement problem under the given assumptions.1/ In fact, formula (2.1) and the method Preinreich suggests for finding the optimal solution is just a special case of the general principle of dynamic programming later described by Bellman.

All models referred to above assume constant technology. Terborgh has proposed principles for dealing with problems where assets become obsolete because of developments in technology.2/ For this purpose, Terborgh introduces the concept of an "inferiority gradient" which measures the growth in inferiority of an old asset as compared to the newest asset available. This inferiority is assumed to consist partly of obsolescence and partly of physical deterioration. Dreyfus has dealt with the same problem within a dynamic programming framework.3/

The models mentioned above all assume perfect knowledge with respect to all relations considered in the model. Burt has developed a model for

---

1/ Richard Bellman, Dynamic Programming (Princeton: Princeton University Press, 1959).

2/ Terborgh, op. cit.

3/ Stuart E. Dreyfus, "A Generalized Equipment Replacement Study," Journal Society for Industrial and Applied Mathematics, VIII (Sept., 1960).

cases where the asset considered is subject to chance destruction or failure in such a way that there is no corrective action except replacement.[1/] We may extend the problem to cases where there also is a stochastic variation in other relationships. The method of "dynamic programming with Markov processes," as developed by Howard, provides a general framework for dealing with such cases; however, it requires that time and asset characteristics (like age and productivity) be defined as discrete variables.[2/]

To this author's knowledge, no model is yet developed to deal with cases where both obsolescence and stochastic variation in variables are present.

2. Validity of Assumptions

The line of theoretical developments described above is relevant to an actual replacement problem only under given conditions. As pointed out by Preinreich, the general theory of replacement is simply the theory of maxima and minima. The actual replacement rule which should be applied depends on "scarcities" in the given system.[3/]

In the developments described above, there is assumed unlimited access to capital at a given market rate of interest and perfectly elastic demand and supply conditions. If we introduce a rigid restriction in output per time unit and still assume a given number of assets (and

---

1/ Oscar R. Burt, "Optimal Replacement Under Risk," Journal of Farm Economics, XLVII (May, 1965), pp. 324-346.

2/ Ronald A. Howard, Dynamic Programming and Markov Processes (2d print.; Cambridge, Massachusetts: The Massachusetts Institute of Technology Press, 1960).

3/ Preinreich, op. cit., p. 35.

therefore a given output per asset per time unit), maximization of "present value of incomes less outlays" is equivalent to minimization of present value of outlays. The models can very easily be reformulated to handle this.

More complicated decision models may result if we put a rigid restriction on output; assume, as before, that output per asset may vary as a function of age; and allow the number of assets simultaneously employed to vary. Other modifications in basic assumptions may be to introduce downward sloping demand curves, capital restrictions, etc.

With a rigid restriction on capital, maximization of "internal rate of return" may appear to be a better objective criterion than maximization of "present value." This would require a theoretical development different from what will be followed in this thesis. An alternative way to handle replacement problems under capital rationing may be to maintain the general framework of maximization of present value but substitute the market rate of interest with an internal rate of interest assumed to be the opportunity cost of capital within the firm.

It seems justified to say that conditions in agriculture and in forestry are quite often such that the "family" of models mentioned above represent the economic replacement problem in a fairly realistic way. From the point of view of the individual firm, supply elasticities in factor markets and demand elasticities in product markets are quite often perfectly elastic. The assumption that capital is available at a given rate of interest may more often be detrimental to the realism of the model. As long as we can assume a given rate of interest (either the market rate or an internal rate), the optimal replacement policy in a "chain" of durable assets can be studied independently of the number of parallel chains. However, if a rigid restriction is put on capital, the problem will become more complicated. In this case, the entrepreneur must consider whether he should employ a replacement policy which, on the average, will tie up a large amount of capital per chain, or he should use his capital to increase the number of chains. For ecample, a forest owner could use his limited capital to invest in a large forest acreage and employ a replacement policy which, on the average, would tie up little capital per acre, or he could invest in a smaller acreage and, on the average, more heavy stocking per acre. In this case, the "internal rate of return" on capital is not given but must be determined as an integral part of the maximization procedure.

Fortunately for the practical usefulness of our models, the "size of operation" measured as numbers of parallel chains is often either given or will be considered only at time intervals much longer than the life span of a single asset. We may say that determination of size of operations enters only into long-run production decisions, while determination of optimal replacement patterns often enters also into production decisions concerning much shorter time spans. The cases where we have to determine "size" simultaneously with "replacement policy" are therefore relatively infrequent; the cases where we can take size as given and concentrate our attention on replacement are more frequent. It may be easier to arrive at reasonable estimates of "internal rate of return" in these short-run situations, since we do not have to consider the competition for capital from investments in size.

C. Replacement Studies Related to Dairy Production

1. Economic Studies

Very few known studies have dealt explicitly with the economics of dairy cow replacement. The only known study where the problem of dairy cow replacement has been formulated explicitly as a problem of maximization or minimization is published by Jenkins and Halter.[1/2/] Their method is based on dynamic programming principles and maximization of present value. Two different stochastic elements are introduced in the model. First, it is assumed that for each production period there is a given probability of what the authors call "failure," or that the cow will die or have to be replaced because of sickness, accidents, etc. This probability of failure is allowed to vary as a function of age. Second, the model considers certain probebilities that the dairy producer will be able to find at the market a replacement animal of given age. Production is assumed to be a deterministic function of age, and variations in the length of each lactation cycle are not considered. The study must be considered a methodological study in the field of dynamic programming, and the application to the dairy cow replacement problem is more an illustration of given principles than an attempt to solve a realistic replacement problem.

1/ Keith B. Jenkins and Albert N. Halter, A Multi-State Stochastic Replacement Decision Model, Oregon State University, Agricultural Experiment Station Technical Bul. 67 (Corvallis, 1963).

2/ It is known that Dr. Robert Hutton at Pennsylvania State University is working on a study of dairy production where the replacement problem is considered as part of the problem. Dr. Hutton is using a simulation technique. More detailed information is not available.

2. Dairy Science Studies Concerned with the Replacement Problem

Several research projects in the subject field of dairy science have studied current replacement patterns and disposal causes; however, no literature in this subject field known by the author has tried to answer the question of what the optimal replacement pattern should be.

More often, attention has been focused on a somewhat related problem - namely, the efficiency of production records of different length as a basis for genetic selection. The answer to this may conceivably influence the replacement decision, since the desire to secure a reliable measure of a cow's genetic ability may induce a dairy producer to keep his cows longer than he would otherwise have done. However, several authors have found that a cow's first lactation record alone is almost as efficient as a basis for selection as the average of several lactation records, and that even production records for the first few months of the first lactation are fairly efficient as a selection basis. Since it will seldom be necessary, for economic reasons, to replace a cow before the latter part of the first lactation, it appears that this consideration can be safely ignored in an economic replacement model. References to some of the studies mentioned will be made in a later chapter of this thesis.

Skjervold has brought attention to a somewhat different economic implication of replacement policy. Skjervold argues that the replacement policy followed influences the rate of genetic progress in a population through its effect on selection intensity, average generation interval, etc, and that the value of such genetic progress must be considered in an economic replacement model. No attempt has been made to quantify the size of this effect.[1/]

D. The Dairy Cow Replacement Problem as Related to Existing Replacement Models

Dairy production exhibits some special characteristics which, if possible, should be considered in a dairy cow replacement model. Such characteristics as probability of involuntary replacement, periodicity of production, stochastic variation in productivity, stochastic variation in length of the production periods, and effects of genetic advance can be mentioned.

For any time period in a cow's life, there is some probability that the dairy producer will have to replace her involuntarily because of such reasons as accidents, poor health, or sterility. Empirical results suggest that the probability for such "involuntary replacement" during a given year may range between 0.05 and 0.15.

Milk production is by nature periodical. Daily milk production normally rises rapidly during the very first weeks after a new freshening, reaches a maximum, and then starts declining until it normally reaches zero some time before the next freshening. The replacement decision is, therefore, really a two-step procedure. The dairy producer must decide when during a given lactation to replace a cow <u>if</u> she is going to be replaced during that lactation, and whether to replace her during that lactation at all. Since each of these two decisions will influence the optimal value of the other one, they should preferably be made simultaneously.

1/ Harald Skjervold, Agricultural College of Norway, Vollebekk, Norway. Personal communication.

There is a considerable stochastic variation in milk production for a given cow, and this variation is such, by nature, that knowledge of production during previous time periods will enable us to make better predictions about production during later time periods.

There is also a considerable stochastic variation in the length of each production cycle. Since a longer calving interval means a longer time period with low production, the length of the calving interval is also one of the factors which, preferably, should be considered in a replacement model.

We can expect that in cow populations where rational breeding programs are carried out the average genetic composition will improve in such a way that average production under the same environmental conditions will increase over time. Empirical studies of some cow populations have suggested an annual increase in production due to genetic change of 0.7 to 1.0 percent per year.[1/] On the average, therefore, we can expect each year's batch of heifers to be able to produce somewhat more than cows of the same age born the previous year. The effect for the replacement decision will, in principle, be the same as the effect of obsolescence for replacement of machinery.

Even if the stochastic nature of some variables is ignored, the periodic nature of dairy production makes the use of the continuous-time equation less practical.[2/] If we are willing to assume a deterministic

---

1/ Clive W. Arave, "A Study of Genetic Change in Twelve California Dairy Herds" (unpublished Ph.D. dissertation in Genetics, Graduate Division, University of California, Davis, 1962).

2/ See formula (2.1) and (2.2), pp. 16-17.

relationship between time and production, this formula can be used to obtain the optimal time for replacement within a given lactation, under the assumption that we know the present value of the replacement animal and all her successive replacements.

The Markovian type of dynamic programming models seems to be the only type of models which is able to include consideration of most of the factors mentioned above. This model requires that the stochastic relationships can be expressed in terms of a Markov process. It will be shown later that this is possible if we take care to define variables in a proper way.

We have mentioned the two-step nature of the dairy cow replacement problem. This problem could, in principle, have been handled in a Markovian type dynamic programming model by defining each stage in the process as a quite small time interval—for example, as one month. This would have increased the size of the model beyond manageable limits, however, and the problem will have to be solved by assuming the time for replacement <u>within</u> the lactation cycle as given and concentrating attention on finding during which lactation number replacement should take place.

Earlier theoretical development in dynamic programming with Markov processes assumes that each stage in the process is of equal length. It will be shown that the model can fairly easily be modified to handle cases where the stage length varies in a stochastic manner.

The effect of genetic advance or "obsolescence" can not easily be considered within this framework. It seems possible to allow for consideration of this effect by making some additional assumptions which will be discussed in a later part; however, such considerations will not be included in the empirical part of this study.

III. A THEORETICAL FRAMEWORK FOR A DAIRY COW REPLACEMENT MODEL

A. Markov Processes and Markov Chains

We will consider a sequence of experiments or of observations, where each link in the sequence can be called a stage. The outcome of one stage is assumed to depend on the outcomes of the previous stages in a probabilistic sense, such that the probability distribution of outcomes at a particular stage is known when the actual outcome of all previous stages are known. Such a sequence is called a stochastic process.[1/]

A Markov process is a stochastic process where the probability distribution of outcomes at any given stage depends only on the actual outcome at the last preceding stage. If we have a Markov process and we know the outcome of the last experiment or observation, we can neglect completely any information we have about previous experiments or observations in predicting the future.

Here, we will only be dealing with finite Markov processes. That a Markov process is finite means that there is a finite number of possible outcomes for each stage. Each possible outcome is called a state. The number of states in a Markov process may conceivably be different for different stages; however, we will here usually be dealing with cases where the number of states is constant over stages. For convenience, each state can be given a number i $(i = 1, 2, \ldots, I)$ or j $(j = 1, 2, \ldots, J)$.

1/ We are referring here to "discrete time" cases of stochastic processes, in which time (or location within the sequence) is given as a discrete variable. There are other stochastic processes in which time is treated as a continuous variable. Markov processes can also be "continuous-time Markov processes."

We will assume here that the state of the process is known on the first day of each stage. The process will stay in the given stage during the whole stage and then either go to another state or remain in the same state as it transfers from one stage to the next. There are given probabilities that a process which is in the $\underline{i}$th state at the $\underline{h-1}$th stage will go to the $\underline{j}$th state as it transfers to the $\underline{h}$th stage. These probabilities are called transition probabilities, and will be denoted $p_{ij(h)}$.

It is convenient to write all transition probabilities for a given stage in a Markov process in the form of an IxJ- dimensional matrix which will be denoted $\underline{P}_{(h)}$:

$$\underline{P}_{(h)} = \left\{p_{ij(h)}\right\} = \begin{bmatrix} p_{11(h)} & p_{12(h)} & \cdots\cdots & p_{1J(h)} \\ p_{21(h)} & p_{22(h)} & \cdots\cdots & p_{2J(h)} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ p_{I1(h)} & p_{I2(h)} & \cdots\cdots & p_{IJ(h)} \end{bmatrix}$$

Since at each stage the process must be in one and only one of the alternative states, all rows in the transition probability matrix must sum to 1: $\sum_j p_{ij(h)} = 1$ (all i and h).

Following the terminology used by Kemeny and Snell, a Markov chain is a Markov process such that the transition probabilities $p_{ij(h)}$ do not depend on h.[1/][2/] The transition probabilities in a Markov chain

---

1/ John G. Kemeny and J. Laurie Snell, Finite Markov Chains (Princeton, New Jersey: D. Van Nostrand Company, 1960), pp. 24-25.

2/ Different authors are using different terminology. What are called "finite Markov processes" here are called "Markov chains," and can therefore simply be denoted p_{ij}, and the transition probability matrix, denoted $\underline{P}$, will always be square since the number of states must be the same for each stage.

As an example of a stochastic process close to the problem studied in this thesis, consider the observation of milk production by a dairy cow through succeeding lactations. Each lactation may be conceived of as a stage, and the "outcome" we want to study is milk production for the first 305 days of each lactation. We will assume that the probability distribution of 305 days production for each lactation is known when actual 305 days production for all previous lactations are known. We may thus look at this as a stochastic process.

This stochastic process is a Markov process if knowledge about 305 days production for the $\underline{h-1}$th lactation completely specifies the probability distribution of 305 days production for the $\underline{h}$th lactation. Common knowledge about milk production suggests that this is not the case. It will be shown later, however, that it is possible to redefine the variables by which milk production is measured in such a way that we will get a Markov process. For pure illustrative purposes, we will assume here that a sequence of observations of lactation records for a dairy cow satisfies the condition for being a Markov process.

Since milk production is a continuous variable, we will have to divide it up in some finite number of class intervals in order to get a finite Markov process. Assume for simplicity that production is classified as "high" if 305 days production is above 400 pounds butterfat, and

---

what are called "Markov chains" here are called "homogeneous Markov chains," in: Emanuel Parzen, Stochastic Processes (San Francisco: Holden-Day Inc., 1962), p. 193.

as "low" otherwise. In this case, I = J = 2. State No. 1 can be taken to represent high production and state No. 2 to represent low production. We have assumed that the state is known on the first day of each stage. If each stage is defined to run from the day of one freshening to the day of the next freshening, then the state will be determined by production during the previous lactation.

Assume, for example, that the transition probability matrix for transition from the second to the third lactation is the following:

$$\begin{bmatrix} .76 & .24 \\ .36 & .64 \end{bmatrix}$$

Since the state during the second lactation is actually determined by production during the first lactation and the state during the third lactation is determined by production during the second lactation, the matrix really gives the stochastic relationship between production during the first and the second lactation. In words, it says that if production was classified as "high" during the first lactation, the probability that it will be classified as "high" during the second lactation is 0.76 and the probability that it will be classified as "low" is 0.24. If it was classified as "low" during the first lactation, the probabilities that it will be classified as "high" or "low" during the second lactation are 0.36 and 0.64, respectively.

The Markov process representing milk production for a dairy cow would be a Markov chain if exactly the same stockastic relations existed between production during the second and the third lactation, between production during the third and fourth lactation, and so on. Even if this is not the case, we will show later that we can redefine the states in such a way that we will get a true Markov chain. Under this, we will introduce additional variables to characterize the states, and the number of states will be much increased. In order to get a simple example to work with, it will be assumed in the following part of this chapter that the simple two-state Markov process we have described here is a Markov chain, so that the same transition probability matrix will be valid for all stages. In order to allow this chain to continue indefinitely, we may assume that the chain represents a dairy cow and her successive replacements.

If we have a Markov process and we know all transition probability matrices and also the probability distribution of outcomes at the initial stage, we can find the probability distribution of outcomes at any later stage. For simplicity, consider a Markov process with only two states. The <u>h</u>th transition probability matrix is

$$\underline{P}_{(h)} = \begin{bmatrix} p_{11(h)} & p_{12(h)} \\ p_{21(h)} & p_{22(h)} \end{bmatrix}$$

Say that there is an initial probability $\Pi_{1(o)}$ that the process is in state 1 and an initial probability $\Pi_{2(o)}$ that it is in state 2. The probability that the process will be in state 1 at the next stage is therefore

$$\Pi_{1(1)} = \Pi_{1(o)}p_{11(1)} + \Pi_{2(o)}p_{21(1)}$$

Likewise, the probability that it will be in state 2 is

$$\Pi_{2(1)} = \Pi_{1(o)}p_{12(1)} + \Pi_{2(o)}p_{22(1)}$$

These results can be written more condensed in matrix notation.

Denote as $\underline{\pi}_{(h)}$ an I-dimensional row vector, the $\underline{i}$th element of which is $\pi_{i(h)}$. We can now write

$$\underline{\pi}_{(1)} = [\pi_{1(1)} \quad \pi_{2(1)}] = [\pi_{1(o)} \quad \pi_{2(o)}] \cdot \begin{bmatrix} p_{11(1)} & p_{12(1)} \\ p_{21(1)} & p_{22(1)} \end{bmatrix}$$

$$= \quad \underline{\pi}_{(o)} \quad \cdot \quad \underline{P}_{(1)}$$

and in general:

$$\underline{\pi}_{(h)} = \underline{\pi}_{(h-1)}\underline{P}_{(h)} \tag{3.1}$$

The result above is easily generalized and is valid for any finite number of states in a Markov process.

If we follow a Markov process through several stages, we see that

$$\underline{\pi}_{(2)} = \underline{\pi}_{(1)}\underline{P}_{(2)} = \underline{\pi}_{(o)}\underline{P}_{(1)}\underline{P}_{(2)}$$

$$\underline{\pi}_{(3)} = \underline{\pi}_{(2)}\underline{P}_{(3)} = \underline{\pi}_{(o)}\underline{P}_{(1)}\underline{P}_{(2)}\underline{P}_{(3)}$$

etc.

In the special case of a Markov chain, the same transition probability matrix $\underline{P}$ is valid for all stages, and we get

$$\underline{\pi}_{(2)} = \underline{\pi}_{(o)}\underline{P}\cdot\underline{P} = \underline{\pi}_{(o)}\underline{P}^2$$

$$\underline{\pi}_{(3)} = \underline{\pi}_{(o)}\underline{P}\cdot\underline{P}\cdot\underline{P} = \underline{\pi}_{(o)}\underline{P}^3$$

and in general:

$$\underline{\pi}_{(h)} = \underline{\pi}_{(o)}\underline{P}^h \tag{3.2}$$

Markov chains are classified in different classes according to the nature of the transition probability matrix.[1] We will in this thesis only be concerned with <u>ergodic chains</u>, and of all ergodic chains, only

---

[1] Kemeny and Snell, <u>op. cit.</u>, pp. 35-38.

with <u>regular chains</u>. An ergodic (and finite) Markov chain is a Markov chain in which it is possible to go from every state to every other state. The transition from one state to another state must not necessarily be in one step for the Markov chain to be ergodic. As an example, consider the transition probability matrix

$$\begin{bmatrix} .6 & 0 & .4 \\ .3 & .7 & 0 \\ 0 & .6 & .4 \end{bmatrix}$$

The process can not go directly from state 1 to state 2, but can go from state 1 to state 3 and from there to state 2. Likewise, it will take two steps to go from state 2 to state 3 and from state 3 to state 1. The chain is ergodic.

Ergodic chains can be cyclic or regular. A cyclic chain is an ergodic chain in which each state can only be entered at certain periodic intervals. In a chain with the transition probability matrix

$$\begin{bmatrix} 0 & .4 & .6 \\ 1.0 & 0 & 0 \\ 1.0 & 0 & 0 \end{bmatrix}$$

state 1 can only be entered for each second stage, while states 2 and 3 can only be entered for the stages in between. Ergodic chains which are not cyclic are regular.

It is of interest to note what will happen with $\underline{\Pi}_{(h)}$ in equation (3.2) if, in a regular Markov chain, h goes to infinity. It can be proved that regardless of the initial position which is characterized by

$\underline{\Pi}_{(o)}$, $\underline{\Pi}_{(h)}$ will go to a limit where all elements are positive constants which depend only on the transition probability matrix $\underline{P}$ and not on $\underline{\Pi}_{(o)}$.[1/]

As an example, consider our previous example of a dairy cow, and say that we start out with a cow classified as a "high" producer. This can be represented by the initial vector

$$\underline{\Pi}_{(o)} = [1 \quad 0]$$

The following probability vectors will be:

$$\underline{\Pi}_{(1)} = \underline{\Pi}_{(o)}\underline{P} = [0.76 \quad 0.24]$$
$$\underline{\Pi}_{(2)} = \underline{\Pi}_{(1)}\underline{P} = \underline{\Pi}_{(o)}\underline{P}^2 = [0.664 \quad 0.336]$$
$$\underline{\Pi}_{(3)} = \underline{\Pi}_{(2)}\underline{P} = \underline{\Pi}_{(o)}\underline{P}^3 = [0.6256 \quad 0.3744]$$
$$\underline{\Pi}_{(4)} = \underline{\Pi}_{(3)}\underline{P} = \underline{\Pi}_{(o)}\underline{P}^4 = [0.61024 \quad 0.38976]$$
$$\underline{\Pi}_{(5)} = \underline{\Pi}_{(4)}\underline{P} = \underline{\Pi}_{(o)}\underline{P}^5 = [0.604096 \quad 0.395904]$$

The limiting state probabilities in this case are 0.6 and 0.4, and we can see that the actual state probabilities are fairly close to the limits after only five stages. We would have reached about the same degree of approximation had we started in state 2. How many stages will be required in order to reach a certain degree of approximation depends on the transition probability matrix, and no conclusions should be drawn from the rapid convergence found in this case.

[1/] Kemeny and Snell, op. cit., p. 70.

B. Markov Processes with Economic Returns[1/]

Assume that a given Markov process represents some economic activity and that given incomes and outlays are associated with each stage in the process. We will assume that we know the difference between incomes and outlays for each possible transition in the process. We want to find the expected present value of the process when there is a given rate of interest, a given number of stages left under the planning horizon, and the process initially is in the ith state.

We could do this by first finding the expected return for each of the remaining stages and afterwards discounting this stream of expected returns to present value. Another, and computationally more convenient, method is to follow an iterative procedure whereby we will start at the end of the planning horizon and work backwards in time until we reach the present stage.

It is now convenient to renumber each stage. n will denote the number of remaining stages under the planning horizon, $n = 0$ denotes the end of the planning horizon, and $\underline{P}_{(n)}$ is the transition probability matrix when the process transfers from the nth to the n-1th state. As before, the state is assumed known on the first day of each stage.

Denote the difference between incomes and outlays during one stage when the process goes from state i to state j as r_{ij}. The r_{ij}'s will be assumed invariable over stages.[2/] They can be arranged to form a matrix of economic returns:

1/ A more complete treatment is found in Howard, op. cit.

2/ We could allow the r_{ij}'s to vary over stages in cases where the number of remaining stages is finite. Such an assumption would have complicated the procedure only slightly.

$$\underline{R} = \{r_{ij}\}$$

The elements r_{ij} will be called "immediate economic returns," and it is assumed that they are incurred on the first day of each stage. If the elements actually represent the difference between an income stream and an outlay stream distributed over the whole stage length, or they represent point incomes and point outlays at later points in time during the stage, then it will be assumed that these future amounts are already discounted to the first day of the stage.

We may start by computing the expected immediate economic returns for each stage and state in the process. The expected immediate economic return for the ith state and the nth stage will be denoted $q_{i(n)}$, and is simply:

$$q_{i(n)} = \sum_j p_{ij(n)} r_{ij} \qquad (i = 1, 2, \ldots, I)$$

Denote as $f_{i(0)}$ the value of the process at the end of the planning horizon if it is in the ith state at that time, and denote as $f_{i(n)}$ the expected present value of the process when there are n remaining stages and the process is in the ith state. Denote as β the discount factor for discounting an amount from the first day of one stage to the first day of the previous stage.

The expected present value when there is one stage left is evidently the sum of the expected immediate return for the last stage and the discounted expected terminal value:

$$f_{i(1)} = q_{i(1)} + \beta \sum_j p_{ij(1)} f_{j(0)} \qquad (i = 1, 2, \ldots, I)$$

Again, the expected present value when there are two stages left under the planning horizon is the sum of the expected immediate return for the next last stage and the discounted expected present value with one remaining stage:

$$f_{i(2)} = q_{i(2)} + \beta \sum_j p_{ij(2)} f_{j(1)} \qquad (i = 1, 2, \ldots, I)$$

In general:

$$f_{i(n)} = q_{i(n)} + \beta \sum_j p_{ij(n)} f_{j(n-1)} \qquad (3.3)$$
$$(i = 1, 2, \ldots, I)$$

The results above can be written in a more condensed form by using matrix notation. Denote as $\underline{q}_{(n)}$ an I-dimensional column vector the ith element of which is $q_{i(n)}$, and denote as $\underline{f}_{(n)}$ an I-dimensional column vector the ith element of which is $f_{i(n)}$. Equation (3.3) can now be written as

$$\underline{f}_{(n)} = \underline{q}_{(n)} + \beta \underline{P}_{(n)} \underline{f}_{(n-1)} \qquad (3.4)$$

In the special case where the Markov process is a Markov chain, $\underline{P}$ is invariable over stages, $\underline{q}_{(n)}$ will therefore also be invariable over stages and can be denoted simply $\underline{q}$, and equation (3.4) simplifies to:

$$\underline{f}_{(n)} = \underline{q} + \beta \underline{P} \cdot \underline{f}_{(n-1)} \qquad (3.5)$$

Expanding the right-hand side of equation (3.5) gives:

$$\begin{aligned}
\underline{f}_{(n)} &= \underline{q} + \beta \underline{P} \cdot \underline{q} + \beta^2 \underline{P}^2 \underline{f}_{(n-2)} \\
&= \underline{q} + \beta \underline{P} \cdot \underline{q} + \beta^2 \underline{P}^2 \underline{q} + \beta^3 \underline{P}^3 \underline{f}_{(n-3)} \\
&\cdots\cdots\cdots\cdots \\
&= (\underline{I} + \beta \underline{P} + \beta^2 \underline{P}^2 + \ldots + \beta^{n-1} \underline{P}^{n-1}) \underline{q} + \beta^n \underline{P}^n \underline{f}_{(0)}
\end{aligned}$$

where $\underline{I}$ is the IxI identity matrix.

It can be shown that:

$$\lim_{n \to \infty} (\underline{I} + \beta \underline{P} + \beta^2 \underline{P}^2 + \ldots + \beta^{n-1} \underline{P}^{n-1}) = (\underline{I} - \beta \underline{P})^{-1}$$

and:

$$\lim_{n\to\infty} \beta^n \underline{P}^n = \underline{0}$$

where $\underline{0}$ is the IxI null matrix. Therefore,

$$\lim_{n\to\infty} \underline{f}(n) = (\underline{I} - \beta\underline{P})^{-1}\underline{q} \tag{3.6}$$

Proof: Define a matrix $\underline{A}$

$$\underline{A} = (\underline{I} + \beta\underline{P} + \beta^2\underline{P}^2 + \ldots + \beta^{n-1}\underline{P}^{n-1})$$

Premultiply both sides with $(\underline{I} - \beta\underline{P})$:

$$\begin{aligned}(\underline{I} - \beta\underline{P})\underline{A} = \underline{I} &+ \beta\underline{P} + \beta^2\underline{P}^2 + \ldots + \beta^{n-1}\underline{P}^{n-1} \\ &- \beta\underline{P} - \beta^2\underline{P}^2 - \ldots - \beta^{n-1}\underline{P}^{n-1} - \beta^n\underline{P}^n \\ = \underline{I} &- \beta^n\underline{P}^n\end{aligned}$$

We can write

$$\beta^n\underline{P}^n = (\beta\underline{P})^n$$

The matrix $\beta\underline{P}$ is a matrix consisting of nonnegative elements, the sum of the elements of each row being less than one. It can be shown that in this case

$$\lim_{n\to\infty} (\beta\underline{P})^n = \underline{0}$$ [1]

We have now

$$\lim_{n\to\infty} (\underline{I} - \beta\underline{P})\underline{A} = \underline{I}$$

and therefore,

$$\lim_{n\to\infty} \underline{A} = (\underline{I} - \beta\underline{P})^{-1}$$

---

[1] G. Hadley, Linear Algebra (Reading, Mass.: Addison-Wesley Publishing Co., 1961), p. 119.

Thus, if we have a Markov chain with economic returns which are invariable over stages, the expected present value under an infinite planning horizon of any state in the chain can be found directly from equation (3.6).

The well-known method for capitalization of an infinite stream of future incomes under certainty can be considered a special case of equation (3.6). We may consider this as a Markov chain with economic returns, where the number of states is 1 and the transition probability matrix therefore also is 1. Equation (3.6) simplifies to

$$\lim_{n\to\infty} f_{(n)} = (1 - \beta\cdot 1)^{-1} q = \frac{1}{1-\beta} q$$

which is one of the well-known formula for capitalization of a periodic income.

As an example, consider the highly simplified dairy cow example considered previously. In order to allow the chain to continue indefinitely, we must assume that the chain represents one dairy cow and all her successive replacements. Say that the expected immediate return is \$500 if the chain is in state 1 and \$ 100 if it is in state 2. The terminal value, which in this case may be the salvage value of the cow, is \$120, regardless of state. Using equation (3.5), we can compute the present value with one remaining stage as:

$$\underline{f}_{(1)} = \begin{bmatrix} f_{1(1)} \\ f_{2(1)} \end{bmatrix} = \begin{bmatrix} 500 \\ 100 \end{bmatrix} + \frac{1}{1.05} \cdot \begin{bmatrix} .76 & .24 \\ .36 & .64 \end{bmatrix} \cdot \begin{bmatrix} 120.00 \\ 120.00 \end{bmatrix} = \begin{bmatrix} 614.29 \\ 214.29 \end{bmatrix}$$

The present values with two remaining stages are:

$$\underline{f}(2) = \begin{bmatrix} f_1(2) \\ f_2(2) \end{bmatrix} = \begin{bmatrix} 500 \\ 100 \end{bmatrix} + \frac{1}{1.05} \cdot \begin{bmatrix} .76 & .24 \\ .36 & .64 \end{bmatrix} \cdot \begin{bmatrix} 614.29 \\ 214.29 \end{bmatrix} = \begin{bmatrix} 993.61 \\ 441.23 \end{bmatrix}$$

and so on.

We might carry on these iterative computations until, for some high value of n, there is little change in present values from one stage to the next. However, to find the present values under an infinite planning horizon, it is usually more convenient to use equation (3.6):

$$\lim_{n \to \infty} \underline{f}(n) = \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{1.05} \begin{bmatrix} .76 & .24 \\ .36 & .64 \end{bmatrix} \right]^{-1} \cdot \begin{bmatrix} 500 \\ 100 \end{bmatrix} =$$

$$= \begin{bmatrix} .276190 & -.228571 \\ -.342857 & .390476 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 500 \\ 100 \end{bmatrix}$$

$$= \begin{bmatrix} 13.2462 & 7.7538 \\ 11.6308 & 9.3692 \end{bmatrix} \cdot \begin{bmatrix} 500 \\ 100 \end{bmatrix} = \begin{bmatrix} 7{,}398.48 \\ 6{,}752.32 \end{bmatrix}$$

C. Decision-Determined Markov Processes with Rewards

1. The Problem

We will now consider the case where the transition probabilities as well as the economic returns in a Markov process are determined by decisions made by the entrepreneur. For each state and each stage, the entrepreneur may choose between a number of alternative actions. For the ith state, the nth stage, and the kth action, there are given probabilities, $p_{ij(n)}^{k}$, that the process will be in the *j*th state during the next stage, and the expected immediate economic return is $q_{i(n)}^{k}$.

We will define a *policy* as a rule which for each state prescribes a given action. The policy may conceivably be different for different stages, so we will have to specify a separate policy for each stage in the system. The policy for a given stage determines the transition probability matrix for the transition from that stage to the next one, and it also determines the vector of expected immediate economic returns.

We may say that the entrepreneur now has the choice between a number of alternative Markov processes where each alternative process is determined by a set of policies, one policy for each stage. We will assume that he wants to select among all alternative Markov processes that process which for any present state maximizes expected present value.

Theoretically, this could be done by computing the expected present value for all alternative processes by one of the procedures suggested above and then selecting the one which gives the highest values. The number of alternative processes is in most cases very large, however. If there are I states in each process and there are K_i alternative actions for the *i*th state, then there will be $\prod_{i=1}^{I} K_i$ alternative policies for each stage. If there are N stages, then the total number of alternative processes will be $\left(\prod_{i=1}^{I} K_i\right)^N$. A more convenient method for selecting the optimal Markov process is required.

2. The Value-Iteration Method[1/]

This method applies to the case where there are a limited number of stages left under the planning horizon. It can also be used but is computationally more burdensome when we want to optimize under an infinite planning horizon.

The method requires us to look at the last stage of the planning horizon and work back to the present stage. Doing this, we are utilizing the dynamic programming principle of optimality as defined by Bellman:[2/] "An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."

In order to simplify notation somewhat, we will assume that the transition probabilities for a given state i and action k are independent of the stage number and can therefore be denoted p_{ij}^{k}. We will start by considering the decisions which may have to be made when there is one stage left under the planning horizon. By that time, the process can possibly be in any of the I different states. For each state, therefore, we must select the action (i.e., the value of k) which maximizes the expression

$$q_i^k + \beta \sum_j p_{ij}^k f_{j(o)}$$

We will denote as $\overline{f}_{i(n)}$ the highest attainable expected present value of the ith state when there are n remaining stages under the

---

1/ This and the following method are described more fully by Howard. See Howard, op. cit.

2/ Richard Bellman, Dynamic Programming (Princeton: Princeton University Press, 1957), p. 83.

planning horizon. Therefore,

$$\bar{f}_{i(1)} = \underset{k}{\text{Max}} \left[q_i^k + \beta \sum_j p_{ij}^k f_{j(o)} \right] \qquad (i = 1, 2, \ldots, I)$$

When the optimal actions and the values of $\bar{f}_{i(n)}$ are determined for all states for $n = 1$, we can go backwards in time and determine optimal actions and maximum expected present values for $n = 2$. These are given by the equation:

$$\bar{f}_{i(2)} = \underset{k}{\text{Max}} \left[q_i^k + \beta \sum_j p_{ij}^k \bar{f}_{j(1)} \right] \qquad (i = 1, 2, \ldots, I)$$

Following the same procedure, we arrive at the general recursive relationship:

$$\bar{f}_{i(n)} = \underset{k}{\text{Max}} \left[q_i^k + \beta \sum_j p_{ij}^k \bar{f}_j(n-1) \right] \qquad (3.7)$$
$$(i = 1, 2, \ldots, I)$$

Equation (3.7) provides a complete algorithm for solving numerical problems when there is a finite and known number of remaining stages under the firm's planning horizon. Working backwards for $n = 1$, $n = 2, \ldots,$ $n = N$, the recursive relation in (3.7) will supply for each stage a list specifying the optimal decisions for each of the I states and therefore identify the optimal policy (or policies) for that stage. Among the many alternative Markov processes, it selects the one(s) which maximizes expected present values for all states. It gives at the same time the expected present values for all stages and states.

Equation (3.7) may select different actions for the same state under different values for n. It can be shown, however, that as n gets large, the policies which are optimal for each given stage converge to a constant policy. In other words, when n is sufficiently large, the optimal action depends only on the state and not on the stage in which the process is found. Convergence in the general case of dynamic programming is shown by Bellman.[1/] The proof is relatively complicated and will not be reproduced here.

The optimal policy towards which the results converge as n gets large is the same as the optimal policy under an infinite planning horizon. Computationally, it may sometimes be convenient to find the optimal policy under an infinite planning horizon by carrying out so many iterations that we feel sure convergence has taken place. In fact, this is the method which will be employed in the empirical part of this study. On the other hand, there might be other cases where it is convenient to optimize under an infinite planning horizon even if what we really want is the optimal policy for some finite but large value of n.

3. The Policy-Iteration Method

The policy-iteration method applies to optimization under an infinite planning horizon. The optimal policy is constant over stages in this case, and our problem is to find this optimal policy. We may start by assuming that any selected policy will be kept constant over stages. Since the same policy for all stages means the same transition probability matrix for all stages, we will have not only a Markov process but a Markov chain and can use equation (3.6) to find the vector of present values under an infinite planning horizon.

If the policy is denoted s, let $\underline{P}^{(s)}$ denote the transition probability matrix, $\underline{q}^{s}$ denote the vector of expected immediate economic

---

1/ Ibid., p. 121.

returns, and let $\underline{g}^s$ denote the vector of present values under an infinite planning horizon. We can find $\underline{g}^s$ as

$$\underline{g}^s = \lim_{n\to\infty} \underline{f}^s(n) = (\underline{I} - \beta \underline{P}^{(s)})^{-1} \underline{q}^s \tag{3.8}$$

Our objective is to find the policy (or policies) which maximizes all elements in $\underline{g}^s$.

The policy-iteration method as suggested by Howard starts by selecting an arbitrary policy and finding the value of each element in $\underline{g}^s$ under this policy from equation (3.8). We can then ask the following question: If we decide to follow the selected policy for the next stage and all following stages, but are free to select any action for each state for the first stage, what will the optimal decisions and present values then be? This can be found easily by substituting the present values under the selected policy as the $f_{j(n-1)}$ values in equation (3.7) and finding the values of k which maximize the expression. Assume that this leads to a policy for the first stage which is different from those for the following stages. If it pays to deviate from the selected policy for the first stage, it will also pay to do so for the following. We may therefore select the policy arrived at for the first stage as one to be kept constant over all stages, find the present value vector under this new policy, and repeat the procedure until no changes in policy occur.

4. The Value-Iteration Method Under an Infinite Planning Horizon

If no computer program for the policy-iteration method has been written but a program for the value-iteration method is available, we may use this to find the optimal policy under an infinite planning horizon simply by carrying out so many iterations that we feel sure convergence has taken place.[1/]

In this case, we would want some criterion for whether convergence aotually has taken place. We might feel confident that this is the case if a substantial number of iterations have occurred without any change in policy; however, this is no absolute safe criterion. A procedure suggested by Burt is to use the value-iteration procedure until the policy has stayed constant over a number of stages, then use equation (3.8) to find the present value over an infinite planning horizon with this policy and again use equation (3.7) in the same way as under the policy-iteration method to check for optimality. If the check shows that true convergence has not yet taken place, the value-iteration method can be continued for a number of new iterations until a new policy has stayed constant over a number of iterations, and the check for optimality can be repeated.[2/]

In the empirical part of this investigation, the value-iteration method was used to find the optimal policy under an infinite planning horizon because this was the only method for which a computer program was available. The check for convergence suggested by Burt was not

---

[1/] When making a choice of computer program to have written, the situation may be such that we may prefer a program for the value-iteration method. Such program is more generally applicable because it can be used both for optimizing under a limited planning horizon and, with some reservations, for optimizing under an infinite planning horizon. Under a limited planning horizon, it can handle cases where the transition probabilities, or the vectors of expected immediate returns, or both, depend on stage number. It is also more economic of limited computer capacity, particularly when many of the elements in the transition probability matrices are zero.

[2/] Another description of this method is in Oscar R. Burt and John R. Allison, "Farm Management Decisions with Dynamic Programming," Journal of Farm Economics, XLV (February, 1963), pp. 121-136.

used, however, because a computer program for the check was not developed. Instead, a large number of iterations were computed after the policy first started to repeat itself. Even if in this case there is a theoretical possibility that even more iterations finally would result in a different policy, the practical importance of this is likely to be very small. In such a case, it is highly likely that the really optimal policy and the policy already arrived at would differ very little in economic results.

5. Transformation to a Linear Programming Problem

It can be shown that a Markovian dynamic programming problem under an infinite planning horizon and with discounting can be transformed to a linear programming problem whereby the usual Simplex procedure (or any other linear programming procedure) can be used to arrive at the optimal solution.[1/] The theoretical development is relatively complicated; however, the following derivation may give a more intuitive demonstration of the relationship between the two problems.[2/]

A linear programming problem can be stated in matrix form as:

$$\max_{\underline{x}_o} \underline{c}_o' \underline{x}_o \tag{3.9}$$

subject to the constraints

$$\underline{b}_o \geq \underline{Ax}_o \tag{3.10}$$

$$x_o \geq \underline{o} \tag{3.11}$$

1/ F. d'Epenoux, "A Probabilistic Production and Inventory Problem," Management Science, X (October, 1963), pp. 98-108.

Another treatment of the problem is found in G. Hadley, Nonlinear and Dynamic Programming (Reading, Mass.: Addison-Wesley Publishing Co., 1964), pp. 464-472.

2/ The primal problem in the derivation below is by d'Epenoux treated as the dual problem.

Following the Simplex procedure, condition (3.10) is transformed to a strict equality by adding to the coefficient matrix $\underline{A}$ a number of "slack vectors," one slack vector for each inequality. The problem can now be restated as

$$\underset{\underline{x}}{\text{Max}}\ \underline{c}'\underline{x} \tag{3.12}$$

subject to the constraints

$$\underline{b}_o = \underline{B}\underline{x} \tag{3.13}$$

$$\underline{x} \geq \underline{o} \tag{3.14}$$

where $\underline{B}$ is an nxm-dimensional matrix arrived at by adding the required number of slack vectors to the original matrix $\underline{A}$, and $\underline{c}$ and $\underline{x}$ are vectors arrived at by adding the corresponding number of elements to the vectors $\underline{c}_o$ and $\underline{x}_o$.

The Simplex procedure proceeds by selecting arbitrarily n of the m column vectors in $\underline{B}$, finding a "basic solution" where only the elements in the x-vector corresponding to these n vectors in basis have nonzero values, and then improving this basic solution step by step by substituting one of the vectors not in the basis for one of those in the basis, so that at each step there are n column vectors in the basis. This procedure is repeated until a given criterion for a maximum is attained. This procedure is well described in literature on linear programming and will not be repeated here.[1/]

Thus, the B-matrix is partitioned in an nxn-dimensional matrix, $\underline{B}_1$, and an nx(m-n)-dimensional matrix, $\underline{B}_2$. Correspondingly, the c-

---

[1/] One of the most complete treatments may be found in George B. Dantzig, Linear Programming and Extensions (Princeton: Princeton University Press, 1963).

vector is partitioned in an n-dimensional vector, $\underline{c}_1$, where the $\underline{i}$th element corresponds to the $\underline{i}$th column vector in $\underline{B}_1$ and an (m-n)-dimensional vector, $\underline{c}_2$, where each element in the same way corresponds to a column vector in $\underline{B}_2$. The x-vector is partitioned in the same way in an n-dimensional vector, $\underline{x}_1$, and an (m-n)-dimensional vector, $\underline{x}_2$. The "basic solution" must satisfy

$$\underline{x}_2 = \underline{0}$$

$$\underline{b}_o = \underline{B}_1 \underline{x}_1$$

Assuming $\underline{B}_1$ has an inverse, the last equation gives the values for $\underline{x}_1$

$$\underline{x}_1 = \underline{B}_1^{-1} \underline{b}_o \tag{3.15}$$

and therefore the value of the objective function is

$$\underline{c}'\underline{x} = \underline{c}_1'\underline{x}_1 + \underline{c}_2'\underline{x}_2$$

$$\underline{c}'\underline{x} = \underline{c}_1'\underline{B}_1^{-1}\underline{b}_o \tag{3.16}$$

We are used to considering the Simplex procedure a procedure for maximizing c'x. As equation (3.16) shows, it may just as well be considered a procedure for selecting from the total matrix $\underline{B}$ such a submatrix $\underline{B}_1$ that the right-hand side of equation (3.16) is maximized. Knowledge of this submatrix also gives the optimal value of the x-vector, since $\underline{x}_2 = \underline{0}$ and $\underline{x}_1$ is given from (3.15).

In Markovian dynamic programming under an infinite planning horizon, we want to select the policy which maximizes each element in the vector of present values:

$$\operatorname*{Max}_{s} \underline{g}^s = [\underline{I} - \beta \underline{P}^{(s)}]^{-1} \underline{q}^s \tag{3.17}$$

It can be shown that if a change in policy increases one or more elements in the vector of present values, it can not at the same time decrease any other elements in this vector.[1/] Therefore, if we form a linear combination, $\underline{d}'\underline{g}^s$, of the elements in $\underline{g}^s$, where each element in the vector $\underline{d}$ is an arbitrary positive constant and we maximize the value of this linear combination, we will at the same time maximize the value of each element in $\underline{g}^s$.

We can therefore restate (3.17) as:

$$\max_s \underline{d}'\underline{g}^s = \underline{d}'[\underline{I} - \beta \underline{P}^{(s)}]^{-1}\underline{q}^s \qquad (3.18)$$

$$= \underline{q}^{s'}\left[[\underline{I} - \beta \underline{P}^{(s)}]'\right]^{-1}\underline{d}$$

The similarity between this last expression and expression (3.16) is easily recognized. To the c_1-vector in linear programming corresponds the vector of "expected immediate economic returns," $\underline{q}^s$; to the resource vector, $\underline{b}_o$, in linear programming corresponds the vector of arbitrary positive constants, $\underline{d}$; and to the submatrix of vectors in basis $\underline{B}_1$ in linear programming corresponds the matrix $[\underline{I} - \beta \underline{P}^{(s)}]'$. The problem we want to solve is to select from all possible matrices $[\underline{I} - \beta \underline{P}^{(s)}]'$ the one which maximizes expression (3.18). We can do this by formulating the problem as a linear programming problem. As the B-matrix, we will form an $I x(\sum_{i=1}^{I} K_i)$-dimensional matrix, where I is the number of states in the system and K_i is the number of alternative actions for the $\underline{i}$th state. In this matrix, the column vector representing the $\underline{i}$th state and the $\underline{k}$th action will have the form

---

1/ Howard, op. cit., pp. 42-43.

$$\begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 1 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} - \beta \begin{bmatrix} p_{i1}^k \\ p_{i2}^k \\ \\ \\ p_{ii}^k \\ \\ \\ p_{iI}^k \end{bmatrix}$$

where the first vector from which the difference is formed is a unit vector with the element 1 in the $\underline{i}$th position. The corresponding element in the c-vector will be q_i^k. The resource vector $\underline{b}_o$ in this case consists of arbitrarily selected positive constants. The problem we want to solve is to select from the total number of $\sum_i K_i$ column vectors in $\underline{B}$ a number of I vectors, one for each state, so that the objective function is maximized. This can be done by following the regular simplex procedure. The set of vectors in the optimal solution defines the optimal policy, and the "shadow prices" which result from the optimal solution will give the present values for each of the I states.

It seems like this method in many cases would be the most practical way of arriving at optimal solutions to the Markovian dynamic programming problem under an infinite planning horizon—among other things, because computer programs for linear programming usually already are available. One possible obstacle in using this method may be large rounding errors resulting from small determinant values. If the discount factor β is close to 1, then the matrix $[\underline{I} - \beta \underline{P}^{(s)}]$ will have a determinant close to zero, and this may possibly result in large rounding errors.

6. A Numerical Example of Different Optimization Methods

We will employ the same simplified dairy cow example as has been used previously; however, we will introduce an option of actions for each state. Say that for both states the entrepreneur can choose between keeping the present cow and replacing it with another cow. In the last case, there will be a probability of 0.90 that production during the current stage will be "high" and a probability of 0.10 that it will be "low". However, because of the immediate outlays associated with replacement, the immediate expected return will be minus $150, the same for both states.

The basic data are now:

For state 1

Action	p_{11}^k	p_{12}^k	q_1^k
1	.76	.24	500
2	.90	.10	-150

For state 2

Action	p_{21}^k	p_{22}^k	q_2^k
1	.36	.64	100
2	.90	.10	-150

Values at the end of the planning horizon are still assumed to be $120, the same for both states.

The Value-Iteration Method

Under this method we will compute the present values with one remaining stage for each state and action. They are:

$$f^1_{1(1)} = 500 + \frac{1}{1.05} [.76 \times 120.00 + .24 \times 120.00] = \underline{614.29}$$

$$f^2_{1(1)} = -150 + \frac{1}{1.05} [.90 \times 120.00 + .10 \times 120.00] = -35.71$$

$$f^1_{2(1)} = 100 + \frac{1}{1.05} [.36 \times 120.00 + .64 \times 120.00] = \underline{214.29}$$

$$f^2_{2(1)} = f^2_{1(1)} = -35.71$$

The maximum present value for each state is underlined. These are the values $\overline{f}_{i(1)}$ to be used in the next iteration. We see that the first action is optimal both for state 1 and for state 2, and these actions define the policy to be followed when there is one remaining stage $(n = 1)$. When deriving the optimal actions for stage $n = 2$, we will assume that this optimal policy will be followed for stage $n = 1$. We can now go on to compute:

$$f^1_{1(2)} = 500 + \frac{1}{1.05} [.76 \times 614.29 + .24 \times 214.29] = \underline{993.61}$$

$$f^2_{1(2)} = -150 + \frac{1}{1.05} [.90 \times 614.29 + .10 \times 214.29] = 396.94$$

$$f^1_{2(2)} = 100 + \frac{1}{1.05} [.36 \times 614.29 + .64 \times 214.29] = \underline{441.23}$$

$$f^2_{2(2)} = f^2_{1(2)} = 396.94$$

Also with two remaining stages, the first action is optimal for both states, and the present values under these actions are used for the further computations:

$$f^1_{1(3)} = 500 + \frac{1}{1.05} [.76 \times 993.61 + .24 \times 441.23] = \underline{1,230.04}$$

$$f^2_{1(3)} = -150 + \frac{1}{1.05} [.90 \times 993.61 + .10 \times 441.23] = 743.69$$

$$f^1_{2(3)} = 100 + \frac{1}{1.05} [.36 \times 993.61 + .64 \times 441.23] = 709.61$$

$$f^2_{2(3)} = f^2_{1(3)} = \underline{743.69}$$

With three remaining stages, the first action is optimal if the process is in the first state, and the second action is optimal if it is in the second state. The corresponding present values are the values of $\bar{f}_{i(3)}$ to be used in the next iteration. If we repeat this procedure for a large number of stages, we will find that this policy is optimal for any higher number of n and therefore is the optimal policy also under an infinite planning horizon.

Expressed with words, the set of optimal policies arrived at says that a cow for which production during the last lactation is classified as "high" should always be kept in the herd, while a cow for which production during the last lactation is classified as "low" should be kept only if there are two stages or less left under the planning horizon, and replaced otherwise.

The Policy-Iteration Method

Say that we start by arbitrarily selecting action 1 for both states. We will compute the present values under an infinite planning horizon for this policy. These values have been derived previously and are:[1/]

$$\underline{g} = \lim_{n \to \infty} \underline{f}(n) = \begin{bmatrix} 7,398.48 \\ 6,752.32 \end{bmatrix}$$

Assuming that this policy will be followed from the next stage but that we are free to select another policy for the current stage, we can compute new expected present values under the alternative actions:

[1/] See p. 42.

$$f^1_{1(\infty)} = 500 + \frac{1}{1.05}[.76 \times 7{,}398.48 + 0.24 \times 6{,}752.32] = \underline{7{,}398.48}$$

$$f^2_{1(\infty)} = -150 + \frac{1}{1.05}[.90 \times 7{,}398.48 + .10 \times 6{,}752.32] = 6{,}834.63$$

$$f^1_{2(\infty)} = 100 + \frac{1}{1.05}[.36 \times 7{,}398.48 + .64 \times 6{,}752.32] = 6{,}752.32$$

$$f^2_{2(\infty)} = f^2_{1(\infty)} = \underline{6{,}834.63}$$

The results show that it would pay to change from action 1 to action 2 in state 2. Computing expected present values under this new policy and repeating the procedure above, we would see that this policy is optimal.

Linear Programming

Arranged in a traditional way, the "initial tableau" in a linear programming formulation of the optimization problem would be:

	$c_j \rightarrow$	500	-150	100	-150
$\underline{b}_o$		$\underline{b}_1$	$\underline{b}_2$	$\underline{b}_3$	$\underline{b}_4$
1		.276190	.142857	-.342857	-.857143
1		-.228571	-.095238	.390476	.904762

Here,

$\underline{b}_1$ represents state 1, action 1

$\underline{b}_2$ represents state 1, action 2

$\underline{b}_3$ represents state 2, action 1

$\underline{b}_4$ represents state 2, action 2

Each element in the "resource vector" is arbitrarily set equal to 1, and both restrictions are strict equalities. The vectors in the B-matrix are derived as explained on page 53. For example,

$$b_{11} = 1 - \beta^1 p_{11}^1$$

$$= 1 - \frac{1}{1.05} 0.76 = .276190$$

The optimization procedure yields the following solution:

$$x_1 = 32.6471$$

$$x_2 = 0$$

$$x_3 = 0$$

$$x_4 = 9.3529$$

$$\sum x_i c_i = 14{,}920.6135$$

The "shadow prices" are

$$u_1 = 7{,}747.07$$

$$u_2 = 7{,}173.54$$

Since x_1 and x_4 have positive values, this tells us that the optimal policy consists of action 1 for state 1 and action 2 for state 2. The shadow prices give the expected present values for state 1 and state 2, respectively. The value of the objective function is here simply the sum of the expected present values over all states, since both the positive constants in the linear combination of present values were set equal to 1.

D. Extension to the Case Where Stage Length is a Stochastic Variable

So far, each stage has been assumed to be of constant length. The dairy cow replacement problem requires a model allowing for stochastic variation in stage length. If we want to maximize under an infinite planning horizon, this can be achieved without much further complications.

As before, we assume that for any given state the kth action determines a row vector of transition probabilities

$$[p_{i1}^k \quad p_{i2}^k \cdots p_{iJ}^k]$$

where

$$\sum_{j=1}^{J} p_{ij}^k = 1$$

For each transition from the ith state to the jth state, the length of time the process stays in the given stage may take on M different discrete values, each of these having a given probability. We can denote as p_{ijm}^k the probability that the process will go from the ith state to the jth state with the time length of the stage being m if the kth action is taken.

$$\sum_{m=1}^{M} p_{ijm}^k = p_{ij}^k$$

The immediate economic return may also depend on m. We can denote as r_{ijm}^k the immediate economic return when the process goes from state i to state j, with the time length of the stage being m under action k. Similar to before, we will assume that in determining r_{ijm}^k, outlays and incomes have already been discounted to the first day of the stage.

Expected immediate economic return for the ith state and the kth action can now be determined as

$$q_i^k = \sum_{j=1}^{J} \sum_{m=1}^{M} p_{ijm}^k R_{ijm}^k \tag{3.19}$$

We will further denote as β_m the discount factor for discounting an amount from the first day of one stage to the first day of the previous stage when the stage length is m.

If the expected present value of a process in the jth state on the first day of the (n-1)th stage is $f_{j(n-1)}$, then this value discounted to the first day of the previous stage, assuming a stage length m, is $\beta_m f_{j(n-1)}$, and the expected present value of future returns under the kth action is

$$\sum_{j=1}^{J} \sum_{m=1}^{M} p_{ijm}^{k} \beta_m f_{j(n-1)}$$

The total expected present value is

$$f_{i(n)}^{k} = q_i^k + \sum_{j=1}^{J} \sum_{m=1}^{M} p_{ijm}^{k} \beta_m f_{j(n-1)} \tag{3.20}$$

$$(i = 1, 2, \ldots, I)$$

Say that we decide on a given action for each state, so that the set of actions defines a policy s. We can now define a matrix $\underline{G}^{(s)}$, where each element is a sum of products of probabilities and discount factors:

$$\underline{G}^{(s)} = \left\{ \sum_{m=1}^{M} p_{ijm}^{k(s)} \beta_m \right\}$$

where the k(s) is the action for the ith state defined by the sth policy. Equation (3.20) can now be written in matrix notation as

$$\underline{f}_{(n)}^{s} = \underline{q}^{s} + \underline{G}^{(s)} \underline{f}^{s}_{(n-1)} \tag{3.21}$$

If the policy is kept constant over stages, we can expand the right-hand side of equation (3.21) in the same way as was done on page 39 for a Markov chain with constant stage length. Letting n approach infinity, we will arrive at the result:

$$\lim_{n \to \infty} \underline{f}^{s} = \left[\underline{I} - \underline{G}^{(s)} \right]^{-1} \underline{q}^{s} \tag{3.22}$$

Our objective will be to select s so that the right-hand side of equation (3.22) is maximized. The objective can be achieved by any of the methods discussed above.

By the value-iteration method, we will follow an iterative procedure whereby

$$\bar{f}_{i(n)} = \underset{k}{\text{Max}} \left[q_i^k + \sum_{j=1}^{J} \sum_{m=1}^{M} p_{ijm}^k \beta^m \bar{f}_{j(n-1)} \right] \qquad (3.23)$$

This method can be used to find the optimal set of policies (one for each stage) when there is a limited and known number of stages left under the planning horizon. However, when each stage is of variable length, this may be of limited usefulness, since a limited planning horizon usually will be given as some limited time span rather than as a limited number of stages. The method can also be used to find the optimal policy under an infinite planning horizon by repeating the procedure until convergence has taken place. In this case, more iterations may be required to reach convergence than what would be required with a constant stage length. The reason is that until present values through the iterative procedure have come close to the values they will take on under an infinite horizon, there may be a tendency for actions resulting in long stage lengths to be selected if such actions at the same time give high values for q_i^k. To reach convergence sooner, it may help if one can use a priori information to estimate approximate present values under an infinite horizon and use these values as the $\bar{f}_{j(o)}$-values in the procedure. This method was used in the empirical part of this study and in this case proved itself very satisfactory.

When using the policy-iteration method or the linear programming method, the procedure will be the same as under constant stage length, except that the matrix

$$\underline{G}^{(s)} = \left\{ \sum_{m=1}^{M} p_{ijm} \beta_m \right\}$$

will be substituted for the matrix

$$\underline{P}^{(s)} = \left\{ \beta \, p_{ij}^{(s)} \right\}$$

E. Notation Used in Chapter III

The notations most used in this chapter are listed below for reference. More complete definitions are contained in the text.

Small letters denote scalars, underlined small letters denote vectors, and underlined capital letters denote matrices. The maximum value a variable can take on is denoted with $\bar{\ }$ above the letter.

Subscripts i and j ($i = 1, 2, \ldots, I$; $j = 1, 2, \ldots, J$) denote states.

Subscript (h) denotes the stage, counted forward in time.

Subscript (n) denotes the stage, counted from the end of the planning horizon.

Superscript k ($k = 1, 2, \ldots, K_i$) denotes action for a given state.

Superscripts s and (s) denote a policy consisting of a set of actions, one for each state.

Other notations used are:

$\left.\begin{array}{l} p_{ij(h)} \\ p_{ij(n)} \end{array}\right\}$ = transition probabilities in a Markov process where the transition probabilities depend on stage number h or n.

p_{ij} = transition probabilities in a Markov chain where the probabilities do not depend on stage number.

p_{ij}^{k} = transition probabilities under the $\underline{k}$th action.

p_{ijm}^{k} = the probability under action k that a process in state i will go to stage j and the stage length will be m.

$\left.\begin{array}{l} \underline{P}(h) \\ \underline{P}(n) \\ \underline{P} \end{array}\right\}$ = matrices of transition probabilities, the element in the $\underline{i}$th row and $\underline{j}$th column being $p_{ij(h)}$, $p_{ij(n)}$, and p_{ij}, respectively.

$\underline{P}^{(s)}$ = the matrix of transition probabilities under the $\underline{s}$th policy.

$\pi_{i(h)}$ = the probability that the process is in the $\underline{i}$th state during the $\underline{h}$th stage.

$\underline{\Pi}_{(h)}$ = an I-dimensional row vector, the $\underline{i}$th element of which is ${}_{i(h)}$.

r_{ij} = immediate economic return (incomes less outlays) during one stage when the process goes from state i to state j.

r_{ijm} = immediate economic returns during one stage when the process goes from state i to state j and the stage length is m.

$\underline{R}$ = a matrix of immediate economic returns, the element in the $\underline{i}$th row and the $\underline{j}$th column being r_{ij}.

$q_{i(n)}$ = expected immediate economic return for the $\underline{i}$th state and the $\underline{n}$th stage.

$\underline{q}(n)$ = a column vector of expected immediate economic returns, the $\underline{i}$th element of which is $q_{i(n)}$.

$q_{i(n)}^{k}$ = expected immediate economic returns under the $\underline{k}$th action for the $\underline{i}$th state and the $\underline{n}$th stage.

$\underline{q}^{s}$ = a column vector of expected immediate economic returns corresponding to the $\underline{s}$th policy.

β = the discount factor for discounting an amount from the first day of one stage to the first day of the preceding stage when the stage length is constant.

β_m = the discount factor for discounting an amount from the first day of one stage to the first day of the preceding stage when the length of the preceding stage is m.

$f_{i(n)}$ = expected present value of a process in the ith state and the nth stage.

$\underline{f}_{(n)}$ = a column vector of expected present values, the ith element of which is $f_{i(n)}$.

IV. DERIVATION OF TRANSITION PROBABILITIES WITH RESPECT TO PRODUCTION

A. The Problem

As explained in the last chapter, a stochastic process with discrete time variable is a Markov process if, and only if, the probability distribution of outcomes at any stage is completely determined when the outcome at the last preceding stage is given. In order to use the Markovian dynamic programming framework as a basis for a dairy cow replacement model, we must be able to define the variables determining the states in such a way that this condition is satisfied.

In the simple example used until now, we have used only one <u>state variable</u>, namely, production during the last lactation, to describe the "outcome" of a given stage. The outcome may be described by means of two or more state variables, as, for example, when we measure milk production in both pounds and butterfat percentage. Some variables which we may want to include as state variables in the replacement decision model are pounds of milk, butterfat percentage, age, length of calving interval, degree of mastitis, body weight, etc. Which variables to include will be discussed in more detail later; however, it will be assumed here that at each stage at least one variable should be used to express level of milk production. The problem to be discussed here is how this variable or these variables should be defined in order to get a Markov process.

To define states according to production during the last stage alone will not do. If 305 days production during the first, the second, ..., the <u>n</u>th lactation are denoted $x_1, x_2, \ldots, x_n$, respectively, we may denote the conditional probability that x_3 will fall between a lower limit, $\bar{x}_3$, and an upper limit, $\bar{\bar{x}}_3$, for given values of x_1 and x_2 as $P(\bar{x}_3 < x_3 \leqslant \bar{\bar{x}}_3 \mid x_1 = x_1^o, x_2 = x_2^o)$, and the conditional probability that it will fall between the same limits for the same given value of x_2 alone as $P(\bar{x}_3 < x_3 \leqslant \bar{\bar{x}}_3 \mid x_2 = x_2^o)$. In order to have a Markov process, we would have to assume that the two conditional probabilities were the same for all values of $\bar{x}_3$, $\bar{\bar{x}}_3$, x_1^o, and x_2^o. In the same way, we would have to assume that the conditional probability distributions for x_4 for given values of x_1, x_2, and x_3 were the same as the conditional probability distribution of x_4, with only x_3 given, and so on for x_5, x_6, etc. It can be tested whether these assumptions are consistent with empirical observations; however, there does not seem to be any a priori reason to believe that they are.

One possibly way to satisfy the Markov requirement, even if these assumptions are not true, is to use two or more state variables to describe the state of the "outcome." We would then use one state variable for each lactation up to the present. Say, for example, that we want to define a finite Markov process where production for each lactation is classified only as either "high" or "low". After completion of the second lactation, the process could be in any of four alternative states:

State number	Production: First lactation	Production: Second lactation
1	High	High
2	High	Low
3	Low	High
4	Low	Low

There would be $2^3 = 8$ states after completion of the third lactation, $2^4 = 16$ states after completion of the fourth lactation, and so on. Although this way of defining states would completely satisfy the Markov requirement, it would be computationally unfeasible when we consider that a realistic model would require much more than two alternative values for each production variable.

It will be shown below that under certain assumptions about the nature of the probability distributions, it is possible to redefine the variable(s) representing production for each stage in such a way that the Markov requirement is exactly satisfied. In this case, the variable(s) representing production will be defined as linear combinations of 305-day production during previous lactations.

B. A Multivariate Normal Model for Describing Milk Production

A stochastic model which seems to give a good description of production relationships in milk production is based on multivariate theory.

We will define a vector $\underline{x}'_\alpha = [x_{1\alpha}\ x_{2\alpha}\ \dots\ x_{F\alpha}]$ as a vector of yields for the $\underline{\alpha}$th cow belonging to a given population of dairy cows and under given management conditions. $x_{f\alpha}$ ($f = 1, 2, \dots, F$) is production measured as pounds of milk, pounds of butterfat, or pounds of fat-corrected milk, for the first 305 days of the $\underline{f}$th lactation for the $\underline{\alpha}$th cow.

Each lactation yield is influenced by some measurable causes of variation, the effects of which can be expressed through a regression equation. If we look at the $\underline{f}$th lactation independently of the others, the following model can be specified:

$$x_{f\alpha} = \gamma_{fo} + \gamma_{f1} z_{f1\alpha} + \dots + \gamma_{fD} z_{fD\alpha} + u_{f\alpha} \qquad (4.1)$$

where the γ's are population parameters and the $z_{fd\alpha}$'s (d = 1, 2, ..., D) are variables measured for the αth cow. As variables to be included among the z's, may be mentioned length of the preceding and the current calving interval, age of the cow at the beginning of the given lactation, one variable expressing feed level or other environmental conditions during the given lactation, etc.

We will assume that the error terms, $u_{f\alpha}$, for different cows are normally independently distributed with expected values, 0, and variance, σ_f^2:

$$E(u_{f\alpha}) = 0$$

$$E(u_{f\alpha}^{\ 2}) = \sigma_f^2$$

$$E(u_{f\alpha}\, u_{f\beta}) = 0 \quad (\alpha \neq \beta)$$

Independence is also assumed between the error terms, $u_{f\alpha}$, and the explanatory variables, $z_{fd\alpha}$.

Looking at all F lactations simultaneously, we will also be interested in the covariance terms $\sigma_{fg} = E(u_{f\alpha}\, u_{g\alpha})$ or, with words, in the relationship between the deviations from the regression line for one lactation and the deviation from the regression line for another lactation for the same cow.

We may extend the model to the following:

$$\begin{bmatrix} x_{1\alpha} \\ x_{2\alpha} \\ \cdot \\ \cdot \\ \cdot \\ x_{F\alpha} \end{bmatrix} = \begin{bmatrix} \gamma_{10} + \gamma_{11} z_{11\alpha} + \cdots\cdots + \gamma_{1D} z_{1D\alpha} \\ \gamma_{20} + \gamma_{21} z_{21\alpha} + \cdots\cdots + \gamma_{2D} z_{2D\alpha} \\ \cdot \qquad \cdot \qquad\qquad \cdot \\ \cdot \qquad \cdot \qquad\qquad \cdot \\ \cdot \qquad \cdot \qquad\qquad \cdot \\ \gamma_{Fo} \quad \gamma_{F1} z_{F1\alpha} + \cdots\cdots + \gamma_{FD} z_{FD\alpha} \end{bmatrix} + \begin{bmatrix} u_{1\alpha} \\ u_{2\alpha} \\ \cdot \\ \cdot \\ \cdot \\ u_{F\alpha} \end{bmatrix} \qquad (4.2)$$

To arrive at a more condensed notation, define vectors:

$$\underline{\gamma}_f' = [\gamma_{fo} \quad \gamma_{f1} \quad \cdots\cdots \quad \gamma_{fD}] \qquad (f = 1, 2, \ldots, F)$$

$$\underline{z}_{f\alpha}' = [1 \quad z_{f1\alpha} \quad \cdots\cdots \quad z_{fD\alpha}] \qquad (f = 1, 2, \ldots, F)$$

$$\underline{u}_\alpha' = [u_{1\alpha} \quad u_{2\alpha} \quad \cdots\cdots \quad u_{F\alpha}]$$

Model (4.2) can now be written:

$$\underline{x}_\alpha = \begin{bmatrix} x_{1\alpha} \\ x_{2\alpha} \\ \cdot \\ \cdot \\ \cdot \\ x_{F\alpha} \end{bmatrix} = \begin{bmatrix} \underline{\gamma}_1'\underline{z}_{1\alpha} \\ \underline{\gamma}_2'\underline{z}_{2\alpha} \\ \cdot \\ \cdot \\ \cdot \\ \underline{\gamma}_F'\underline{z}_{F\alpha} \end{bmatrix} + \begin{bmatrix} u_{1\alpha} \\ u_{2\alpha} \\ \cdot \\ \cdot \\ \cdot \\ u_{F\alpha} \end{bmatrix} \qquad (4.3)$$

We will assume that the vector $\underline{u}_\alpha$ has a multivariate normal distribution with mean vector $E(\underline{u}_\alpha) = \underline{0}$ and with covariance matrix $E(\underline{u}_\alpha\underline{u}_\alpha') = \Sigma$. Further, we assume independence between the error terms $u_{f\alpha}$ and the explanatory variables $z_{fd\alpha}$.

It may be worth while to discuss this stochastic model in more verbal terms before going on to show how it can be used to define a Markov process.

It is known from practical experience and from empirical research that if we measure 305 days production for a given lactation--for example, the second--for different cows belonging to the same breed and under similar management conditions, we will find a fairly large variation in production between cows. Part of this variation can be explained as due to variation in factors like age, length of the previous calving interval, length of the present calving interval, health conditions, feed input, etc. The effect of variation in such factors can be described by means of a multiple regression equation for the given lactation, as is done by equation (4.1). Even when variation in all such factors are accounted for, however, there usually remains a residual which measures the difference between the observed yield for the given cow and the yield which we would expect according to the given regression equation. The residual for the αth cow and the fth lactation is here denoted $u_{f\alpha}$. The unexplained variance for a given cow population is denoted σ_f^2 or, for convenience, will in the following sometimes be denoted σ_{ff}.

If we compare the residual terms for the same cows for different lactations, we will find in most cases that a cow which has a positive residual for the fth lactation also has a positive residual for the gth lactation, and a cow which has a negative residual for the fth lactation also has a negative residual for the gth lactation. The strength of this tendency in a cow population can be expressed by the covariance term denoted σ_{fg} and defined as:

$$\sigma_{fg} = E(u_{f\alpha}\, u_{g\alpha})$$

Another measure of this relationship is the correlation coefficient between the fth and the gth lactation, denoted r_{fg} and defined as

$$r_{fg} = \sqrt{\frac{\sigma_{fg}^2}{\sigma_{ff}\,\sigma_{gg}}}$$

All covariances can be arranged together with the variances in a variance-covariance matrix--for brevity, often only called a covariance matrix:

$$\Sigma = \{\sigma_{fg}\} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots\cdots & \sigma_{1F} \\ \sigma_{21} & \sigma_{22} & \cdots\cdots & \sigma_{2F} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \sigma_{F1} & \sigma_{F2} & \cdots\cdots & \sigma_{FF} \end{bmatrix}$$

The covariance matrix gives important information about the relationship between production for different lactations. Under the assumption that the distribution is multivariate normal, it can be used to predict the size of the residual for any given lactation when the size of the residuals for the previous lactations are known.

An alternative way of giving the information contained in the covariance matrix is to give the matrix of correlation coefficients and either the variances σ_{ff} $(f = 1, 2, \ldots, F)$ or the standard deviations which are the square roots of the variances. The correlation matrix and the standard deviations contain together exactly the same information as the covariance matrix, and it is easy to derive one when we have the other. Both the covariance matrix and the correlation matrix are symmetric matrices, and the lower part is therefore often not listed. All elements in the diagonal of the correlation matrix are 1 and can therefore also be excluded when the actual parameters are listed.

In the stochastic model described above, all γ-elements and all elements in the covariance matrix are parameters, the values of which usually are not known. The values of these parameters can be estimated based on a sample of observations from a dairy breed. This estimation causes some problems because of the culling which normally takes place in dairy herds. These problems are discussed and a method to arrive at unbiased estimates is developed in Appendix B of this thesis.

In the following, we will proceed as if the estimates at which we have arrived are the true parameter values. This is the same as is usually done when planning under a "subjective risk" situation.[1]

C. Some Properties of a Multivariate Normal Distribution

1. Conditional Distributions

If one or more variables in a multivariate normal distribution are kept fixed, we will get conditional probability distributions of the remaining variables. It can be shown that these conditional probability distributions themselves are multivariate normal distributions, the mean vectors of which depend only linearly on the values of the variables kept fixed and the covariance matrices of which depend only on which variables are kept fixed and not on the value of these fixed variables.[2]

We will consider the case where a random vector $\underline{y}' = [y_1 \; y_2 \; \ldots \; \ldots \; y_F]$ has a multivariate normal distribution with mean vector

$$\underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \mu_F \end{bmatrix}$$

---

[1] See page 11.

[2] T. W. Anderson, An Introduction to Multivariate Statistical Analysis (New York: John Wiley & Sons, 1958), p. 27.

and covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots\cdots & \sigma_{1F} \\ \sigma_{21} & \sigma_{22} & \cdots\cdots & \sigma_{2F} \\ \vdots & \vdots & & \\ \vdots & \vdots & & \\ \sigma_{F1} & {}_{F2} & \cdots\cdots & \sigma_{FF} \end{bmatrix}$$

For our purpose here, we need only be conserned with the conditional distribution of one variable y_f when the values of the variables $y_1, y_2, \ldots y_{f-1}$ are kept fixed. This conditional distribution is normal with a mean which can be denoted $\mu_{f \cdot 1,2,\ldots,f-1}$ and a variance which can be denoted $\sigma_{ff \cdot 1,2,\ldots,f-1}$. It can be shown that:

$$\mu_{f \cdot 1,2,\ldots,f-1} = \mu_f + \sum_{j=1}^{f-1} \beta_{fj}(y_j^o - \mu_j) \tag{4.4}$$

where $\{\beta_{fj}\}$ is the set of elements in the vector

$$[\beta_{f1} \quad \beta_{f2} \cdots\cdots \beta_{f,f-1}] = \tag{4.5}$$

$$= [\sigma_{f1} \quad \sigma_{f2} \cdots\cdots \sigma_{f,f-1}] \cdot \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots\cdots & \sigma_{1,f-1} \\ \sigma_{21} & \sigma_{22} & \cdots\cdots & \sigma_{2,f-1} \\ \vdots & \vdots & & \vdots \\ \sigma_{f-1,1} & \sigma_{f-1,2} & \cdots\cdots & \sigma_{f-1,f-1} \end{bmatrix}^{-1}$$

The β-vector here is the vector of regression coefficients of y_f on

$y_1, y_2, \ldots, y_{f-1}$.[1] It can be shown that of all linear combinations $\sum_{j=1}^{f-1} a_j y_j^o$, the one that minimizes the variance of $(y_f - \sum_{j=1}^{j-1} a_j y_j^o)$ is the linear combination $\sum_{j=1}^{f-1} \beta_{fj} y_j^o$, where the set of β_{fj}'s is defined as above.[2]

The variance of the conditional distribution of y_f is:

$$\sigma_{ff\cdot 1,2,\ldots,f-1} = \tag{4.6}$$

$$= \sigma_{ff} - [\sigma_{f1} \;\; \sigma_{f2} \;..\; \sigma_{f,f-1}] \cdot \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots\cdots & \sigma_{1,f-1} \\ \sigma_{21} & \sigma_{22} & & \sigma_{2,f-2} \\ \vdots & \vdots & & \vdots \\ \sigma_{f-1,1} & \sigma_{f-1,2} & \cdots\cdots & \sigma_{f-1,f-1} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \sigma_{1f} \\ \sigma_{2f} \\ \vdots \\ \sigma_{f-1,f} \end{bmatrix}$$

$$= \sigma_{ff} - [\beta_{f1} \;\; \beta_{f2} \;\cdots\cdots\; \beta_{f,f-1}] \cdot \begin{bmatrix} \sigma_{1f} \\ \sigma_{2f} \\ \vdots \\ \sigma_{f-1,f} \end{bmatrix}$$

2. The Effects of Linear Transformations

Assume that the F-dimensional random vector $\underline{y}$ has a multivariate normal distribution with mean vector $\underline{\mu}$ and covariance matrix Σ. It can be shown that by premultiplying $\underline{y}$ with a GxF-dimensional matrix

---

1/ *Ibid.*, p. 28.

2/ *Ibid.*, p. 32.

$\underline{C}$, the product $\underline{Cy}$, which will be a G-dimensional vector, will also have a multivariate normal distribution where the mean vector will be $\underline{C\mu}$ and the covariance matrix will be $\underline{C}\Sigma\underline{C}'$.[1/]

D. Transformation of a Normal Stochastic Process to a Markov Process

1. The Concept of a Normal Stochastic Process

The concept of a stochastic process was discussed previously. A stochastic process is a normal process if the "outcome" of each succeeding stage is measured as a continuous variable and a vector of outcomes for all stages has a multivariate normal distribution.[2/]

2. The Transformation to a Markov Process

A normal stochastic process can be transformed to a Markov process if we define the outcome of each stage in a certain way. Say that the observed variable for the $\underline{f}$th stage is denoted y_f. We have a series of observed variables: $y_1, y_2, \ldots, y_F$. We can define "outcomes" so that the outcome of the $\underline{f}$th stage is defined as a linear combination of all observed variables up to and including the $\underline{f}$th stage. The new outcomes will be denoted v_f. We have:

1/ *Ibid.*, p. 19.

2/ Parzen, *op. cit.*, pp. 88-94.

$$v_1 = c_{11}y_1$$

$$v_2 = c_{21}y_1 + c_{22}y_2$$

$$v_3 = c_{31}y_1 + c_{32}y_2 + c_{33}y_3$$

$$\vdots$$

$$v_F = c_{F1}y_1 + c_{F2}y_2 + c_{F3}y_2 + \cdots\cdots + c_{FF}y_F$$

The c's are here coefficients in the linear combinations. It is possible to select the c's in such a way that a stochastic process consisting of successive values of v's is a Markov process. In fact, many different sets of c's will satisfy this condition. The rules for determining the c's which will be followed here are the following:

(1) The choice of c_{11} is arbitrary; however for convenience c_{11} will be set equal to the regression coefficient in the regression of y_2 on y_1.

(2) c_{21} and c_{22} will be set equal to the regression coeffients in the multiple regression of y_3 on y_1 and y_2.

(3) c_{31}, c_{32}, and c_{33} will be selected in such a way that the Markov requirement is satisfied and that, subject to this restriction, the conditional variance of y_4 for a given value of v_3 is minimized.

(4) c_{41}, c_{42}, c_{43}, and c_{44} will be selected in such a way that the Markov requirement is satisfied and that, subject to this restriction, the conditional variance of y_5 for a given value of v_4 is minimized.

(5) The same procedure is followed step for step for higher values of f. For each step, we will impose restrictions to secure satisfaction of the Markov requirement and, subject to these restrictions, select the c-values in such a way that the conditional variance of y_{f+1} for a given value of v_f is minimized.

c_{11}, c_{21}, and c_{22} are easily derived. Thus:

$$c_{11} = \sigma_{12} \cdot \sigma_{11}^{-1} \tag{4.7}$$

$$\begin{bmatrix} c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} \sigma_{31} & \sigma_{32} \end{bmatrix} \cdot \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}^{-1} \tag{4.8}$$

The coefficients for higher values of f require a more complicated procedure. We must first derive the restrictions to be imposed in order to secure the Markov requirement, then derive coefficients subject to these restrictions.

To start with v_3, the Markov requirement says that the conditional distribution of v_3 for a given value of v_2 should equal the conditional distribution of v_3 for given values of both v_1 and v_2. Since v_3 is normally distributed, it is enough to make sure that the conditional means and the conditional variances in the two conditional distributions are the same. In symbols, we require

$$E(v_3 | v_2) = E(v_3 | v_1, v_2)$$

$$V(v_3 | v_2) = V(v_3 | v_1, v_2)$$

We can satisfy both these conditions by selecting c_{31}, c_{32}, and c_{33} in such a way that in the regression of v_3 on v_1 and v_2, the regression coefficient for v_1 is zero.[1/] [2/] It is evident that this satisfies the first condition, since if this regression coefficient is zero, it makes no difference for the expected value of v_3 whether v_1 is kept fixed or is allowed to vary. It will also satisfy the second condition, since a zero regression coefficient in a multiple regression means that the corresponding variable has no "explanatory value" in addition to the other variables included in the regression equation. We will therefore not reduce the conditional variance by including this variable.

The point of departure is the covariance matrix Σ. Following the rules for linear transformations given on page 74, we can easily derive the covariance matrix for a random vector $\underline{v}^f$ which consists of the first f v-elements. The elements in this new covariance matrix will be denoted ϕ_{ij}, and the whole matrix will be denoted $\underline{\phi}^{(f)}$. We have:

$$\underline{\phi}(2) = \begin{bmatrix} c_{11} & \\ c_{21} & c_{22} \end{bmatrix} \cdot \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \cdot \begin{bmatrix} c_{11} & c_{21} \\ & c_{22} \end{bmatrix} \qquad (4.9)$$

$$\underline{\phi}^{(3)} = \begin{bmatrix} c_{11} & & \\ c_{21} & c_{22} & \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \cdot \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \cdot \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ & c_{22} & c_{32} \\ & & cc_{33} \end{bmatrix} \qquad (4.10)$$

From the covariance matrix given by (4.10), we can find the regression coefficients in the regression of v_3 on v_1 and v_2. We will require the first of these regression coefficients to be zero, while no

---

1/ This is asserted as true by T. W. Anderson, "Determination of the Order of Dependence in Normally Distributed Time Series," Time Series Analysis, ed. M. Rosenblatt (New York: John Wiley & Sons, 1963), Cht. 26.

2/ A formal proof would be relatively complicated, and only some intuitive arguments are offered here. A numerical example which demonstrates the validity for given parameter values is given below.

restriction is imposed on the other one. Therefore:

$$\begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \phi_{13} \\ \phi_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix} \quad (4.11)$$

where no restriction is imposed on b. The coefficients c_{11}, c_{21}, and c_{22} are known and therefore also the matrix $\Phi^{(2)}$ from (4.9). By inverting this matrix and denoting the elements in the inverse matrix by superscripts, the restriction can be written as:

$$\begin{bmatrix} \phi^{11} & \phi^{12} \end{bmatrix} \cdot \begin{bmatrix} \phi_{13} \\ \phi_{23} \end{bmatrix} = 0 \quad (4.12)$$

The covariance terms ϕ_{13} and ϕ_{23} are not known, but are functions of the unknown coefficients c_{31}, c_{32}, and c_{33}. From (4.10):

$$\begin{bmatrix} \phi_{13} \\ \phi_{23} \end{bmatrix} = \begin{bmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{22} & 0 \end{bmatrix} \cdot \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \cdot \begin{bmatrix} c_{31} \\ c_{32} \\ c_{33} \end{bmatrix} \quad (4.13)$$

Substituting (4.13) into (4.12), we get the following condition:

$$\begin{bmatrix} \phi^{11} & \phi^{12} \end{bmatrix} \cdot \begin{bmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{22} & 0 \end{bmatrix} \cdot \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \cdot \begin{bmatrix} c_{31} \\ c_{32} \\ c_{33} \end{bmatrix} = 0 \quad (4.14)$$

The first three matrices in the expression above are known. By carrying out the multiplication and denoting the resulting vector $\begin{bmatrix} a^3_{11} & a^3_{12} & a^3_{13} \end{bmatrix}$, we can write the restriction as:

$$\begin{bmatrix} a^3_{11} & a^3_{12} & a^3_{13} \end{bmatrix} \cdot \begin{bmatrix} c_{31} \\ c_{32} \\ c_{33} \end{bmatrix} = 0 \quad (4.15)$$

There are three variables to be determined, while one restriction is imposed. The restriction can conveniently be transformed so that one of the variables is expressed as a (homogeneous) function of the two others. We will write:

$$c_{31} = [c_{32} \quad c_{33}] \cdot \begin{bmatrix} d_{11}^{3} \\ d_{21}^{3} \end{bmatrix} \tag{4.16}$$

where the values of d_{11}^{3} and d_{21}^{3} are found from equation (4.15) by solving for c_{31}.

Any set of c_{31}, c_{32}, and c_{33} which satisfies the restriction will make the transformed process a Markov process up to and including the third stage. We want to select such values for c_{31}, c_{32}, and c_{33} that the resulting v_3 gives the best possible basis for predicting y_4, which means that we want to select the values in such a way that the conditional variance $V(y_4 \mid v_3)$ is minimized. Before doing this, we will see how restrictions can be formulated for the c-coefficients for v_4, v_5, etc.

For v_4, we will require that in the regression of v_4 on v_1, v_2, and v_3, the regression coeffisients for v_1 and v_2 must be zero. This gives the following restriction:

$$\begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \phi_{14} \\ \phi_{24} \\ \phi_{34} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} \tag{4.17}$$

where, like in equation (4.11), no restriction is imposed on b. We can assume that, at this step, c_{31}, c_{32}, and c_{33} have been determined, so the matrix $\underline{\phi}^{(3)}$ is known and the inverse of this matrix can be found. As before, we will denote the elements of this matrix with superscripts instead of subscripts.

The elements ϕ_{14}, ϕ_{24}, and ϕ_{34} are yet unknown, but are functions of the unknown coefficients c_{41}, c_{42}, c_{43}, and c_{44}. If we express this functional relationship and substitute it into (4.17), we get the following restriction:

$$\begin{bmatrix} \phi^{11} & \phi^{12} & \phi^{13} \\ \phi^{21} & \phi^{22} & \phi^{23} \end{bmatrix} \cdot \begin{bmatrix} c_{11} & 0 & 0 & 0 \\ c_{21} & c_{22} & 0 & 0 \\ c_{31} & c_{32} & c_{33} & 0 \end{bmatrix} \cdot \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & c_{41} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} & c_{42} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} & c_{43} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} & c_{44} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.18}$$

The first three matrices in the expression above are known. By carrying out the multiplication, we can write the restriction as:

$$\begin{bmatrix} a^4_{11} & a^4_{12} & a^4_{13} & a^4_{14} \\ a^4_{21} & a^4_{22} & a^4_{23} & a^4_{24} \end{bmatrix} \cdot \begin{bmatrix} c_{41} \\ c_{42} \\ c_{43} \\ c_{44} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.19}$$

There are now four variables to be determined, while there are two restrictions imposed. The restrictions can be transformed so that two of the variables are expressed as (homogeneous) functions of the two other. We will write:

$$\begin{bmatrix} c_{41} & c_{42} \end{bmatrix} = \begin{bmatrix} c_{43} & c_{44} \end{bmatrix} \cdot \begin{bmatrix} d^4_{11} & d^4_{12} \\ d^4_{21} & d^4_{22} \end{bmatrix} \tag{4.20}$$

where the coefficients in the last matrix are derived from (4.19) by solving for c_{41} and c_{42}.

Any set of c_{41}, c_{42}, c_{43}, and c_{44} which satisfy the restrictions will make the transformed process a Markov process; however, again we want to select such values that the variance $V(y_5 | v_4)$ is minimized.

I general, we will require that in the regression of v_f on v_1, v_2, ..., v_{f-1}, the regression coefficients for v_1, v_2, ..., v_{f-1} must be zero. This gives f-2 restrictions while there are f coefficients to be determined, so there are always two degrees of freedom available for determining the coefficients in such a way that the conditional variance $V(y_{f+1} | v_f)$ is minimized. We can derive the restrictions in the same way as is shown above and transform them so that they give the f-2 first coefficients as functions of the last two:

$$[c_{f1} \quad c_{f2} \cdots c_{f,f-2}] = [c_{f,f-1} \quad c_{f,f}] \cdot \begin{bmatrix} d^f_{11} & d^f_{12} & \cdots & d^f_{1,f-2} \\ d^f_{21} & d^f_{22} & \cdots & d^f_{2,f-2} \end{bmatrix} \qquad (4.21)$$

We will go on to see how the c-coefficients can be determined, subject to the given restrictions. For simplicity, we will assume

$$\mu_1 = \mu_2 = \cdots\cdots = \mu_F = 0$$

This assumption is unnecessary for the results, but serves to simplify notation. It can be mentioned that the assumption also holds true in the specific application we will make in this thesis. Under the assumption,

$$E(v_1) = E(v_2) = \cdots\cdots = E(v_F) = 0$$

We will examine the conditional mean of y_{f+1} for given value of v_f. Since v_f is a linear combination of y_1, y_2, ..., y_f and the whole distribution is multivariate normal, it is evident that the conditional mean of y_{f+1} is a linear function of v_f. We can write:

$$E(y_{f+1} \mid v_f) = bv_f \qquad (4.22)$$

where b is the regression coefficient in the regression of y_{f+1} on v_f. Substituting the definition of v_f, we get:

$$E(y_{f+1} \mid v_f) = b(c_{f1}y_1 + c_{f2}y_2 + \cdots + c_{ff}y_f) \qquad (4.23)$$

We have from (4.21):

$$c_{fj} = d^f_{1j}c_{f,f-1} + d^f_{2j}c_{f,f} \qquad (4.24)$$
$$(j = 1, 2, \ldots, f-2)$$

Substituting this in (4.23) gives

$$E(y_{f+1} \mid v_f) = bc_{f,f-1}\left(\sum_{j=1}^{f-2} d^f_{1j}y_j + y_{f-1}\right) + bc_{f,f}\left(\sum_{j=1}^{f-2} d^f_{2j}y_j + y_f\right) \qquad (4.25)$$

The terms inside the parentheses are linear combinations of y_1, y_2, ..., y_f, and we may define two new variables:

$$w^f_1 = \sum_{j=1}^{f-2} d^f_{1j}y_j + y_{f-1}$$
$$(4.26)$$
$$w^f_2 = \sum_{j=1}^{f-2} d^f_{2j}y_j + y_f$$

Substituting these definitions in (4.26), we get:

$$E(y_{f+1} \mid v_f) = bc_{f,f-1}w^f_1 + bc_{f,f}w^f_2 \qquad (4.27)$$

We want to select b, $c_{f,f-1}$, and $c_{f,f}$ so that $V(y_{f+1} \mid v_f)$ is minimized. This is equal to seeking the regression coefficients in a regression of y_{f+1} on w^f_1 and w^f_2. We will minimize the variance if $bc_{f,f-1}$ is set equal to the regression coefficient for w^f_1 and $bc_{f,f}$ is set equal to the regression coefficient for w^f_2. We can select any arbitrary value ($\neq 0$) for b, and it is most convenient to set $b = 1$.

$c_{f,f-1}$ and $c_{f,f}$ are then set equal to the regression coefficients. In order to arrive at these regression coefficients, we must derive the covariance matrix for the random vector:

$$\begin{bmatrix} w_1^f \\ w_2^f \\ y_{f+1} \end{bmatrix}$$

This is done by defining the transformation matrix:

$$\underline{D} = \begin{bmatrix} d_{11}^f & d_{21}^f & \cdots\cdots & d_{1,f-2}^f & 1 & 0 & 0 \\ d_{21}^f & d_{22}^f & \cdots\cdots & d_{2,f-2}^f & 0 & 1 & 0 \\ 0 & 0 & \cdots\cdots & 0 & 0 & 0 & 1 \end{bmatrix}$$

By carrying out the matrix multiplication $\underline{D}\underline{\Sigma}\underline{D}'$, we will get the covariance matrix we are seeking and can find the regression coefficients in the usual way. When $c_{f,f-1}$ and $c_{f,f}$ are found, we can find c_{fj} $(j = 1, 2, \ldots, f-2)$ from equation (4.21).

We have developed a method for defining linear combinations v_f $(f = 1, \ldots, F)$ of the observed production variables y_f $(f = 1, \ldots, F)$ in such a way that the sequence $v_1, v_2, \ldots, v_F$ satisfies the condition for being a Markov process and that, subject to this condition, each v_f gives the best possible basis for predicting y_{f+1}. It is evident that if we had ignored the Markov requirement, the best possible way to define v_f would have been to use as coefficients the multiple regression coefficients in the regression of y_{f+1} on $y_1, y_2, \ldots, y_f$. Thus, we may compare two alternative definitions of v_f:

$$v_f^* = \beta_{f1}y_1 + \beta_{f2}y_2 + \cdots\cdots + \beta_{ff}y_f$$

$$v_f = c_{f1}y_1 + c_{f2}y_2 + \cdots\cdots + c_{ff}y_f$$

where β_{fj} $(j = 1, \ldots, f)$ are the multiple regression coefficients referred to above. Since more restrictions are imposed in the determination of the c_{fj}'s than in the determination of the β_{fj}'s, v_f^* is at least as efficient as a basis for predicting y_{f+1} than v_f is and will in most cases be more efficient. In symbols:

$$V(y_{f+1} \mid v_f) \geq V(y_{f+1} \mid v_f^*)$$

The size of the difference will give a measure of the sacrifice in precision of predictions by imposing the Markov requirement. Such measure will be derived in the empirical part of this study.

Until now we have assumed that only one variable, v_f, will be used for each stage to express the outcome. If computational facilities and the nature of the problem allow, we may use two or more variables to express outcome as far as production is concerned. This would allow us to transform a normal process to a Markov process with less loss in precision of predictions, since less restrictions would have to be imposed in the determination of coefficients. If, for example, we can use both v_f and v_{f-1} to express the outcome, then we would have three degrees of freedom available for determining the coefficients c_{fj} and would seek to minimize the conditional variance $V(y_{f+1} \mid v_f, v_{f-1})$, which would be at least as small and more likely smaller than the conditional variance $V(y_{f+1} \mid v_f)$.

3. A Numerical Example

The numerical example below is taken from the empirical part of this study. Only the covariance matrix for the first four elements in the vector (y_1 to y_4) is given, and we will show how we derive the coefficients for the first three linear combinations (v_1, v_2, and v_3).

The data are coded by multiplying the y-elements with 10^{-3}. This means that the variances and covariances given below should be multiplied by 10^6 in order to give the "true" variances and covariances when the y-variables are measured in pounds.

The covariance matrix for the vector $\underline{y}' = [y_1 \quad y_2 \quad y_3 \quad y_4]$ is the following:

$$\begin{bmatrix} 2.290233 & .945019 & .590079 & 1.024213 \\ .945019 & 2.571982 & 1.162668 & 1.174593 \\ .590079 & 1.162668 & 3.044812 & 1.445305 \\ 1.024213 & 1.174593 & 1.445305 & 3.692158 \end{bmatrix}$$

From this, we can derive the coefficients in the first two linear combinations as follows:[1/]

$$c_{11} = .945019 \cdot 2.290233^{-1} = .412630$$

$$[c_{21} \quad c_{22}] = [.590079 \quad 1.162668] \cdot \begin{bmatrix} 2.290233 & .945019 \\ .945019 & 2.571982 \end{bmatrix}^{-1}$$

$$= [\,.083830 \quad .421250\,]$$

1/ Equations (4.7) and (4.8).

We can now derive the covariance matrix for the vector $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ by the matrix multiplication:[1/]

$$\underline{\Phi}^{(2)} = \begin{bmatrix} .412630 & 0 \\ .083830 & .421250 \end{bmatrix} \cdot \begin{bmatrix} 2.290233 & .945019 \\ .945019 & 2.571982 \end{bmatrix} \cdot \begin{bmatrix} .412630 & .083830 \\ 0 & .421250 \end{bmatrix}$$

$$= \begin{bmatrix} .389943 & .243484 \\ .243484 & .539240 \end{bmatrix}$$

We must invert this matrix in order to quantify the restriction. The inverse matrix is:

$$\underline{\Phi}^{(2)-1} = \begin{bmatrix} 3.571400 & -1.612601 \\ -1.612601 & 2.582602 \end{bmatrix}$$

The last row of the inverse matrix is irrelevant, and we can write the restriction as:[2/] :

$$\begin{bmatrix} 3.571400 & -1.612601 \end{bmatrix} \begin{bmatrix} \Phi_{31} \\ \Phi_{32} \end{bmatrix} = 0$$

The covariance elements Φ_{31} and Φ_{32} are functions of the unknown coefficients c_{31}, c_{32}, and c_{33}:

$$\begin{bmatrix} \Phi_{31} \\ \Phi_{32} \end{bmatrix} = \begin{bmatrix} .412630 & 0 & 0 \\ .083830 & .421250 & 0 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} 2.290233 & .945019 & .590079 \\ .945019 & 2.571982 & 1.162668 \\ .590079 & 1.162668 & 3.044812 \end{bmatrix} \cdot \begin{bmatrix} c_{31} \\ c_{32} \\ c_{33} \end{bmatrix}$$

1/ Equation (4.9).

2/ Equation (4.12).

Substituting the last expression in the former and multiplying out, we get the restriction:

$$\begin{bmatrix} 2.423480 & -0.482277 & 0 \end{bmatrix} \cdot \begin{bmatrix} c_{31} \\ c_{32} \\ c_{33} \end{bmatrix} = 0$$

We can express c_{31} as a function of c_{32} and c_{33} and get:1/

$$c_{31} = \begin{bmatrix} c_{32} & c_{33} \end{bmatrix} \cdot \begin{bmatrix} .199002 \\ 0 \end{bmatrix}$$

Now we can form the transformation matrix D:

$$\begin{bmatrix} .199002 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By carrying out the multiplication $\underline{D}\Sigma\underline{D}'$, we get the covariance matrix

$$\begin{bmatrix} 3.038801 & 1.280140 & 1.378413 \\ 1.280140 & 3.044812 & 1.445305 \\ 1.378413 & 1.445305 & 3.692158 \end{bmatrix}$$

From this, we get the coefficients c_{32} and c_{33} as:

$$\begin{bmatrix} 1.378413 & 1.445305 \end{bmatrix} \cdot \begin{bmatrix} 3.038801 & 1.280140 \\ 1.280140 & 3.044812 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} .308231 & .345088 \end{bmatrix}$$

1/ Equation (4.16).

Now we can find the conditional variance of y_4, given v_3:

$$V(y_4 \; v_3) = 3.692158 - [.308231 \quad .345088] \cdot \begin{bmatrix} 1.378413 \\ 1.445305 \end{bmatrix}$$

$$= 3.692158 - .923627$$

$$= 2.768531$$

For comparison, we can list the conditional variance of y_4 if v_3 had been defined without the Markov requirement:

$$V(y_4 \mid v_3^*) = 3.692158 - 1.186586$$

$$= 2.505572$$

c_{31} is found as $.199002 \times .308231 = .061339$. For comparison, v_3, which ensures a process satisfying the Markov condition, is listed in complete form below together with v_3^*:[1/]

$$v_3 = .061339\, y_1 + .308231\, y_2 + .345088\, y_3$$

$$v_3^* = .276080\, y_1 + .199250\, y_2 + .345090\, y_3$$

Finally, we will demonstrate that the regression coefficients of v_3 on v_2 alone and of v_3 on v_1 and v_2 are the same and that the conditional variance of v_3 is the same whether v_2 alone or both v_1 and v_2 are kept constant. For this purpose, we need the covariance matrix $\underline{\Phi}^{(3)}$. We can find this by forming the transformation matrix $\underline{C}$ and performing the matrix multiplication $\underline{C}\underline{\Sigma}\underline{C}'$:

$$\underline{C} = \begin{bmatrix} .412630 & 0 & 0 \\ .083830 & .421250 & 0 \\ .061339 & .308231 & .345088 \end{bmatrix}$$

[1/] Rounding errors are responsible for the difference between the last coefficient in each equation.

$$\underline{\Phi}^{(3)} = \underline{C}\underline{X}\underline{C}' = \begin{bmatrix} .389943 & .243484 & .262185 \\ .243484 & .539240 & .580656 \\ .262185 & .580656 & .923634 \end{bmatrix}$$

The regression coefficient of v_3 on v_2 is:

$$.580656 \cdot .539240^{-1} = 1.076804$$

The regression coefficients of v_3 on v_1 and v_2 are:[1/]

$$\begin{bmatrix} 3.571400 & -1.612601 \\ -1.612601 & 2.582602 \end{bmatrix} \cdot \begin{bmatrix} .262185 \\ .580656 \end{bmatrix} = \begin{bmatrix} .000001 \\ 1.076804 \end{bmatrix}$$

The conditional variances with v_2 alone or with both v_1 and v_2 held fixed are:

$$V(v_3 \mid v_2) = .923634 - 1.076804 \cdot .580656 = .298381$$

$$V(v_3 \mid v_1, v_2) = .923634 - [.262185 \quad .580656] \cdot \begin{bmatrix} .000001 \\ 1.076804 \end{bmatrix} = .298381$$

E. Derivation of Transition Probabilities in a Finite Markov Process

The Markov process we have derived above by a transformation of a normal stochastic process is an infinite Markov process in which all state variables are continuous variables. To get a finite Markov process, the range of each continuous variable v_f must be divided up in some

---

1/ Rounding errors have caused a minor positive value for the first regression coefficient.

number of class intervals. We will denote the lower and upper limits for the ith class of v_f as $\bar{v}_f^i$ and $\bar{\bar{v}}_f^i$, respectively, and the lower and upper limits for the jth class of v_{f+1} as $\bar{v}_{f+1}^j$ and $\bar{\bar{v}}_{f+1}^j$. If numbered from above so that $i = 1$ denotes the highest class, then the lower limit of the ith class will be the upper limit of the i+1th class.

The conditional distribution of v_{f+1} for given value of v_f is independent of the values of $v_1, \ldots, v_{f-1}$, and we can therefore derive the transition probabilities from the bivariate normal distribution of v_f and v_{f+1} alone. These bivariate distributions are determined by five parameters: the expected values of v_f and v_{f+1}, the variances of v_f and v_{f+1}, and the covariance term between the two.

To simplify notation, we will here only examine the derivation of transition probabilities when going from the end of the first lactation $(f = 1)$ to the end of the second lactation; however, the results for the general case of going from the end of the fth lactation to the end of the f+1th lactation are the same—we can simply substitute subscript f for subscript 1 and subscript $f+1$ for subscript 2.

The bivariate normal probability density function of the variables v_1 and v_2 will be denoted $f(v_1,v_2)$. The marginal probability density function of v_1 will be denoted $g(v_1)$:

$$g(v_1) = \int_{-\infty}^{\infty} f(v_1,v_2)dv_2$$

The transition probabilities are given as:

$$p_{ij} = P(\bar{v}_2^j < v_2 \leqslant \bar{\bar{v}}_2^j \mid \bar{v}_1^i < v_1 \leqslant \bar{\bar{v}}_1^i) \tag{4.27}$$

$$= \frac{\displaystyle\int_{\bar{v}_1^i}^{\bar{\bar{v}}_1^i}\int_{\bar{v}_2^i}^{\bar{\bar{v}}_2^j} f(v_1, v_2)\,dv_1\,dv_2}{\displaystyle\int_{\bar{v}_1^i}^{\bar{\bar{v}}_1^i} g(v_1)\,dv_1}$$

The normal density functions, whether univariate or bivariate, are difficult to integrate, and in the empirical part of this study, the transition probabilities to be used were derived in a more approximate way by means of tabulated values of areas under the (univariate) normal curve. In order to reduce error, the transition probabilities were first derived for classes defined according to rather small class intervals, then several of these "small" classes were summed to obtain larger classes. This procedure seems particularly desirable for the classes covering the two "tails" of the distribution.

The procedure followed is the following:

1. Define small class intervals for v_1. Call these small classes $a_1, a_2, \ldots, a_M$.
2. Define small class intervals for v_2. Call these small class intervals, $b_1, b_2, \ldots, b_N$.
3. From a normal table, find the probabilities that v_1 will fall within each of the a-classes. Denote these probabilities p_1 $p_2, \ldots, p_M$.

4. Find the conditional probabilities that v_2 will fall within each of the b-classes _if_ v_1 fell within the _n_th of the a-classes. Denote these conditional probabilities $p_{n|m}$ $(n = 1, \ldots, N; \;\; m = 1, \ldots, M)$.

5. The probability that the process will fall in a_m and b_n is $p_m \cdot p_{n|m}$.

6. Define larger classes A_i and B_j by summing smaller classes. Say A_i is the sum of adjacent smaller classes from (and including) the m_i'th to (and including) the m_{ii}'th and that B_j is the sum of smaller classes from and including the n_j'th to and including the n_{jj}'th.

7. Analogous to (4.27), we have now:

$$p_{ij} = \frac{\sum_{m=m_i}^{m_{ii}} \sum_{n=n_j}^{n_{jj}} p_m \cdot p_{n|m}}{\sum_{m=m_i}^{m_{ii}} p_m} \tag{4.28}$$

The conditional probabilities under point 4 were derived under the simplifying assumption that v_1 had a value like the midpoint of each of the small classes a_m. With this assumption and knowing the regression coefficient of v_2 on v_1, we can find the expected value of v_2 for the given value of v_1. Since we also know the variance in the conditional distribution of v_2 with v_1 fixed, we can use a normal table to find the probabilities that v_2 will fall within given class intervals when v_1 is given. To reduce errors for the extreme classes for which no midpoint is defined, the "small" classes were defined so far out to both sides of the expected values of v_1 that the probability that v_1 would fall within an extreme class was very small $(< .0004)$.

F. Application to the Dairy Case

The general stochastic model assumed to be valid for milk production of dairy cows is given as (4.3). 305 days production for each lactation is assumed to depend on certain variables given for that lactation. Knowing the value of these variables for a given dairy cow but ignoring previous lactation records for the same cow, we may derive the expected production for the given lactation from the "within lactation" regression equation. For each lactation, we will measure the deviation between this expected production and the realized production and denote this deviation for the $\underline{f}$th lactation, u_f. Knowing the size of $u_1, \ldots, u_{f-1}$, we can derive the expected value of u_f from the "between lactation" regression equation which is known when we know the covariance matrix of the vector $\underline{u}$. However, since these complete between-lactation regression equations do not fit into a Markov process model, we have defined new variables v_f which are linear combinations of the u's. The way we defined the v_f's, the expected value of u_{f+1} for a given value of v_f will be exactly equal to v_f.

In order to get a finite Markov process, we had to divide the total range of each v_f up in a number of class intervals. When a cow has completed f lactations, she can be classified in the correct class by taking the deviation between expected and realized 305 days production for each of the lactations up to and including the $\underline{f}$th, deriving the linear combination v_f, and seeing which class interval that value falls within.

In the complete Markovian dynamic programming model which will be discussed in chapter V, cows will be classified not only according to lactation number and "production class" but also according to one or more other relevant variables. Each cow, however, can be classified in one and only one class based on the value of these different variables, and the results from the optimization procedure will show whether it is profitable (in the sense of maximization of expected present value) to replace the given cow.

V. PHYSICAL RELATIONSHIPS IN DAIRY PRODUCTION, AND THE SPECIFICATION OF A DAIRY COW REPLACEMENT MODEL

A. The Problem

The problem is now to decide on a specific model for dairy cow replacement decisions within the general framework of a Markovian dynamic programming model described in chapter III. We have to make the following choices:

1. Specification of state variables, and definition of class intervals for those state variables which are continuous in the real world.
2. Definition of a stage.
3. Specification of alternative actions for each state.

Computational limitations restrict the model to some maximum size. With the computer program available for this study, size of the model was restricted to 300 states if only two alternative actions were considered for each state, and to less if more alternative actions were considered. Restrictions on professional working time must be considered as well. Since no computer program for derivation of model parameters was written in connection with this study, this part of the work had to be done on a desk calculator, and would have been excessively time-consuming if the model had been very large.[1/]

Within such limitations, we want to specify a replacement model which describes reality with a reasonable degree of approximation and can be as helpful as possible to the dairy producer. After the model is

---

[1/] With the size of model used in this study, derivation of parameters for each model required at least three weeks of work with a desk calculator, even after the basic estimation of probability parameters based on empirical data had been completed.

specified, parameters must be estimated based on one source of data or another. Both as a basis for the specification of a realistic and useful model and for estimation of parameters, knowledge of the physical and biological aspects of dairy production is indispensable. This chapter will deal with previous research findings relevant to our problem, and will arrive at a conclusion with respect to the detailed specification of a model.

B. Removal Causes in Dairy Herds

An extensive study of removal causes in dairy herds is reported by Asdell, who compiled data for 2,792.188 cows on DHIA-tests in 17 states from 1932 to 1949, inclusive.[1/] O'Bleness and Van Vleck conducted a mail survey covering 7,362 cows removed from New York DHIA-herds during the period October, 1960 to March, 1961.[2/] Arave examined disposal reasons for 2,596 cows which were removed from 12 Californian Jersey herds from 1930 to 1960.[3/] A summary of some of the results is given in table 5.1.

More extensive lists of alternative removal causes were used in O'Bleness and Van Vleck's and in Arave's studies than in that by Asdell. This may at least partly explain why "low production" weighs more and "other reasons" weigh less in Asdell's results than in those of the other two studies.

1/ S. A. Asdell, "Variations in Amount of Culling from D.H.I.A. Herds," Journal of Dairy Science, XXXIV (June, 1951), pp. 529-35.

2/ G. V. O'Bleness and L. D. Van Vleck, "Reasons for Disposals of Dairy Cows from New York Herds," Journal of Dairy Science, XLV (Sept., 1962), pp. 1087-93.

3/ Arave, loc. cit.

TABLE 5.1. Reasons for Cow Culling[a]

Removal reasons	Asdell		O'Bleness/ Van Vleck	Arave
	Per cent of cows on test	Per cent of cullings[b]	Per cent of cullings[b/c]	Per cent of cullings[b/d]
Removals, total	16.8			
Low production	7.3	44	31.6	27.0
Mastitis, udder trouble	2.5	15	16.8	14.3
Abortion, brucellosis	1.5	9	2.0	1.6
Sterility, breeding trouble	1.8	11	18.8	24.4
Old age	0.6	4	4.2	4.6
Deaths	1.1	7	6.5	} 38.8
Other reasons	1.7	10	18.9	

a/ Cows sold for dairy purposes excluded.

b/ Per cent of total number of cows culled when cows sold for dairy purposes are excluded.

c/ The column sum deviates from 100 due to some inaccuracy in the reported data.

d/ Since cows sold for two different reasons are counted in both groups, the sum of this column exceeds 100.

In Asdell's and O'Bleness and Van Vleck's studies, either only one disposal reason was given for each cow, or if more than one reason were given, only the one listed as most important is counted in the results given in table 5.1. In Arave's study, two reasons were sometimes given for disposal of a cow, therefore the percentage sum of all reasons exceeds 100. Cows sold for dairy purposes are not included in the percentages given in the table. These cows were 23.3 per cent of the total number of cows removed in Asdell's study, 14.2 per cent in O'Bleness and Van Vleck's study and 4.8 per cent in Arave's study.

It is seen that the pattern of disposal reasons was similar in the three studies. "Other reasons" are mostly reasons related to poor health. Low production, breeding trouble, and reasons related to poor health were in all studies in three important groups of removal causes.

There is a fairly strong indication that the pattern of removal reasons has changed over the last 30 years. According to results reported by Asdell, removals due to abortion (brucellosis) decreased from 3.8 per cent of cows on test in 1935 to 0.7 per cent in 1949. In O'Bleness and Van Vleck's study, the importance of this reason was very small. On the other hand, sterility or breeding trouble seems to have increased in importance. Asdell found that removals due to sterility increased from 1.5 per cent of cows on test in 1937 to 2.4 per cent in 1949. In O'Bleness and Van Vleck's study, sterility was even more important. It is difficult to judge whether the increase in the importance of sterility as a disposal reason is due to a real increase in the rate of breeding problems, or just to changes in management practices. It may well be that dairy producers have become less willing to accept the losses in production which result from prolonged calving intervals, and that more cows are culled because of "sterility" og "breeding problems" than before even if the real frequency of such problems has not increased.

While most replacement theory of the kind discussed in chapter II is concerned about replacement of durable assets as related to the age of those assets, age has been found to be rather unimportant as a removal cause in all three studies. We should not infer from this that age can be ignored in a replacement model. First, age is related to production in the sense that expected production increases substantially from the first up to the third lactation. Second, other removal causes may be related to age. Expected production starts to decrease slowly with increasing age as an effect of senescence after some higher age is reached. More important, poor health and breeding trouble may be related to age. In Asdell's study, data on age of removed cows were reported from three of the seventeen states. The results suggest that removals due to udder trouble, sterility, and deaths were more frequent among cows older than 4-5 years than among younger cows. The total frequency of disposals for reasons other than dairy purposes and low production increased with increasing age in Indiana and New York, while in Kansas it increased with increasing age up to an age of 6-7 years, and thereafter decreased.

Based on data covering over 10,000 removals from Pennsylvania DHIA-herds during 1960, Jenkins and Halter found that the "probability of failure" of a dairy cow during a given year was conditional on the lactation number and the milk fat level of the herd.[1/] "Failure" was defined as removal from the herd for reasons other than dairy purposes and low production. According to the Jenkins-Halter results, "probability of failure" increased for each lactation up to the sixth, and almost trebled from the first to the sixth lactation. This is a much stronger effect of age than the effect found by Asdell.

Again, it is difficult to judge whether the reported differences between age groups with respect to frequencies of removals are due to

---

1/ Jenkins and Halter, loc. cit.

real differences in the occurence and severity of sickness, accidents, etc., or to the effect of management practices. It is conceivable that many dairy farmers remove older cows suffering from a given sickness, where they would keep a younger cow suffering from the same cause, and that this explains part of or all the difference found.

C. Measurement and Prediction of Milk Production

1. Unit of Measurement

The value of a given quantity of milk, as well as the quantity of feed required for the production of that output quantity, depends on the chemical composition of milk. Milk production is therefore often expressed in terms of a set of two variables, for example:

Pounds of milk, and fat percentage

Pounds of milk, and pounds of milk fat

Pounds of non-fat milk solids, and pounds of milk fat

The statistical treatment of available data, as well as the computational work connected with making predictions for a given cow, will be made much easier if we can use only one variable to express milk yield. Alternative variables for this use are:

Pounds of milk

Pounds of milk fat

Pounds of fat-corrected milk

Raw data can be converted to data expressing fat-corrected milk by the formula

$$FCM = 0.4\ M + 15\ F \qquad (5.1)$$

where FCM = fat-corrected milk in pounds

M = uncorrected milk in pounds

F = milk fat in pounds

Fat-corrected milk expresses milk production in terms of a product of constant energy value. Milk production expressed as pounds FCM gives the best measure of the physical work the cow has performed, and this measure has for this reason often been preferred by research workers in the biological fields.

For our purpose, however, the choice between the three variables should rather be made on the basis of other criteria. Eventually, we want to predict the differences between milk sales income and feed costs for different cows. Two criteria for choosing between the given variables are: (1) how well they express the difference between milk sales income and feed costs under variations in the fat percentage of the milk, (2) how exactly they can be predicted based on available information.

With the price formulae for class I milk now in use in California, milk fat is valued at 95 cents per pound; non-fat milk solids at 19 cents per pound; and the water in the milk at 0.00223 cents per pound.

Since cows are not tested individually for milk content of non-fat solids, an economic comparison of cows will have to assume the average relationship between fat content and non-fat solids content. E. L. Jack et. al. found this relationship in bulked milk to be

$$NFS = 7.07 + 0.444 F \qquad (5.2)$$

where NFS = percentage content of non-fat solids

F = percentage content of fat[1/]

1/ Quoted from: D. C. Clarke and J. B. Hassler, Pricing Fat and Skim Components of Milk, Calif. Agric. Exp.Sta. Bull. 737 (Berkeley, Calif.: University of California, 1953), p. 5.

Feed input required for the production of a given quantity of milk increases with the fat content of the milk. It can be assumed that feed required for milk production (feed requirements for maintenance and growth excluded) is directly proportional to the energy content of the milk. In table 5.2, it is assumed that feed requirements for milk production correspond to 0.4 pounds of grain for each pound of FCM.[1/] A grain price of 3.1 cents per pound is assumed.

TABLE 5.2. Milk Revenue less Variable Feed Costs for Quantities of Milk Corresponding to 100 Pounds Milk, 1 Pound Milk Fat, and 100 Pounds FCM, for Different Fat Percentages[a/]

Fat percentage	100 pounds milk		1 pound milk fat		100 pounds FCM	
	cents	relative figures	cents	relative figures	cents	relative figures
3.0	339.4		113.1		399.3	
4.0	424.3	83.3	106.1	104.2	424.3	95.8
5.0	509.1	100.0	101.8	100.0	442.7	100.0
6.0	593.9	116.7	99.0	97.2	456.8	103.2

a/ The table is based on certain assumptions given in the text.

The table demonstrates how a given quantity of, for example, milk fat production will give a different economic result depending on the fat percentage of the milk. Of two cows with the same production measured in lbs. of fat, the one with the lowest fat percentage will give the best economic result, so milk fat used as a measure of milk production will

1/ Source: Scandinavian feeding standards.

underevaluate (from an economic point of view) cows with low fat percentage and will over-evaluate cows with high fat content in their milk. Of two cows with the same production measured in FCM, the one with the highest fat percentage will give the highest economic return, so FCM used as a measure of milk production will over-evaluate cows with low fat percentage and will under-evaluate cows with high milk fat tests.

The degree of bias can best be seen from the relative figures. Since the empirical part of this study will be dealing with Jersey cows, relative figures are given only for a range in fat percentage approximating what we can expect for Jerseys. The figures show that lbs. of fat and FCM used as measures of milk production with the present price relationships are about equally biased, but in opposite directions. Pounds milk used as a measure of milk production will greatly over-evaluate cows with low fat percentage.

For most cows, the economic error committed when expressing their production either as lbs. of fat or as FCM will be relatively small, at least as long as only cows of the same breed are being compared. There are reasons to assume that the price relationship will move in the direction of a higher price for non-fat milk solids as compared with milk fat than what is the case today. This will make FCM a better measure from an economic point of view than it is today, and it will make lbs. of milk fat a less satisfactory measure.

We will now turn to the second of the two criteria. The most relevant variable for prediction of future milk production of a given cow is a record of previous milk production of the same cow. If we could show that correlation between yields during different time periods for the same cow was higher when production was expressed in terms of one measure than in terms of another, then this would give reasons for preferring the first measure.

Few results pertaining to this question are available. The author has found no study where results relevant for a comparison between milk fat and FCM are given. Some results seem to indicate that repeatability and heritability[1/] of production expressed in pounds milk are somewhat higher than if production is expressed in pounds of milk fat.[2/] It seems clear, however, that this is not enough to outweigh the other disadvantages "pounds milk" has as a measure of milk production.

In the following, pounds FCM has been used as a measure of milk production. The choice between this measure and pounds of milk fat is rather arbitrary under the present price conditions. We should remember that FCM used as a measure results in a somewhat unjustified preference for cows with low fat percentage.

2. The Relations between Monthly Production, 305 Days Production, and Production for a Full Lactation

Milk production of a dairy cow is cyclic by nature. Daily production normally increases rapidly through the first weeks after freshening, then the production curve flattens out and starts to decline again. A new pregnancy tends to decrease production further after the twentieth

---

1/ These concepts are discussed on page 111 and page 114respectively.

2/ D. E. Madden, J. L. Lush, and L. D. McGilliard, "Relations between Parts of Lactations and Producing Ability of Holstein Cows," Journal of Dairy Science, XXXVIII (Nov., 1955), p. 1267.

L. D. Van Vleck and C. R. Henderson, "Estimates of Genetic Parameters of Some Functions of Part Lactation Milk Records," Journal of Dairy Science, XLIV (June, 1961), p. 1078.

week.[1/] The recommended, and widely followed, management practice is to allow cows a dry period of from six to eight weeks before the start of each succeeding lactation.

Since total production for the lactation varies with the length of the calving interval, it is common to "amputate" or "standardize" the lactation record at 305 days after calving and to use the 305 days yield as a measure of the productivity of the given cow. This practice will be followed in this study, however we will also be interested in production per month both within the 305 days period and after.

With respect to predictions of future production of a given cow, it seems practical to go through two steps: First, we may predict 305 days production based on available information of the kind which will be discussed below. Second, we may predict production for each month within the lactation based on predicted 305 days production and predicted length of the calving interval.

Several authors have examined the relationship between monthly or part lactation records and 305 days production records. For selection purposes, it is often desirable to assess the productive ability of a cow before the 305 days period has expired. All results reviewed indicate that correlations between the sum of monthly records for the first months of a lactation and 305 days production are high.[2/] For example,

---

[1/] Ivar Johansson and Artur Hansson, "Causes of Variation in Milk and Butterfat Yields of Dairy Cows," Kungl. Lantbruksakademiens Tidsskrift Journal of the Royal Swedish Academy, LXXIX (No. 6 1/2, 1940), p. 46.

[2/] The following literature deal with this question:
Madden, Lush, and McGilliard, op. cit., pp. 1264-71.

D. E. Madden, L. D. McGilliard, and N. P. Ralston, "Relations between Test-Day Milk Production of Holstein Cows," Journal of Dairy

Madden, McGilliard, and Ralston found a correlation coefficient of 0.90 between the sum of the first four monthly test-days and the sum of the ten test-days which form the basis for assessing 305 days production. The correlation coefficient between the sum of the first seven test-days and the sum of the ten was 0.97.[1/]

The data which are reported in the reviewed literature are not sufficient for our purpose, which is to arrive at formulaes for expected production for each month of the lactation when 305 days production and length of the calving interval are known. Estimation of relationships of this nature will be done as one part of the empirical part of this study.

3. Variables Related to 305 Days Production

If we consider the variable "305 days FCM" for a given lactation, we will find that a number of other variables are related to this. For some of these variables, the relationship is clearly of a causal nature. For other variables, the causal nature of the relationship is uncertain but a correlation is known to exist. We want to examine such relationships for two reasons. First, if we want to use past 305 days records as a basis for predictions of future 305 days records, we may want to

---

Science, XLII (Feb., 1959), pp. 319-26.

G. R. Fritz, L. D. McGilliard, and D. E. Madden, "Environmental Influences of Regression Factors for Estimating 305-Day Production from Part Lactations," Journal of Dairy Science, XLIII (Aug., 1960), pp. 1108-17.

L. D. Van Vleck and C. R. Henderson, "Regression Factors for Extending Part Lactation Milk Records," Journal of Dairy Science, XLIV (June, 1961), pp. 1085-92.

[1/] Madden, McGilliard, and Ralston, loc. cit. The figures quoted here are for cows 3 years and older. For cows under 3 years, the correlation coefficients were 0.93 and 0.98 respectively.

correct the past records for the influence of certain variables in order to remove disturbing influences. Second, variables which can be used to predict future 305 days records are of interest because we may consider to include them as state variables in our model. Some of the variables related to 305 days production will be discussed in the following.

a. Feeding level

Of environmental factors influencing milk production, feeding level is probably the most important. Previous studies have shown that dairy producers often vary the feeding level over time and that such variations tend to follow fluctuations in the milk/grain price ratio.[1/] Part of observed variation in milk production between lactations for a given cow may, therefore, be due to variations in feeding level.

As will be discussed below, it will be assumed in this study that dairy producers follow a given "feeding system" which may vary over time as a result of price variations, but which, at any given time, is the same for all cows in the herd. The difference in production which we can observe between cows for the same year should then be due to other factors. When we compare different lactations of the same cow, we should, in principle, correct for effects of differences in feeding level. In most cases, reliable information on feed quantities actually given are not available from dairy herds. Another method, to be discussed below, will be used to correct for such variations.

b. Length of the present calving interval

"Amputation" of lactation records 305 days after freshening removes most but not all of the variation which is due to variations in length

---

1/ Arave, loc. cit.

of the calving interval. In this study, past lactation records will be corrected for length of the calving interval before they are used for predictive purposes. The correction factors will be estimated from empirical data.

c. Age of cow, and number of the lactation cycle

On the average, lactation yields increase with increasing age up to maturity, then stay approximately constant over some age range, and finally decrease as an effect of senescence. Lactation yield is also influenced by the number of previous lactations, so that we will generally expect different production records from two cows of the same age if one of them is in the first and the other in the second lactation. This effect is particularly important for the very first lactations.

A large number of studies have sought to quantify the relationship between age and/or lactation number and production.1/ It has usually been assumed that the relationship between age and production follows a pattern which is approximately the same for all cows of a given breed.

1/ Some of these studies are:

Johansson and Hansson, op. cit., pp. 29-40.

J. F. Kendrick, Standardizing Dairy-Herd-Improvement-Association Records in Proving Sires, (U. S. D. A. Agric. Res. Service, Dairy Husbandry Research Branch, ARS-52-1).

Jay L. Lush and Robert R. Shrode, "Changes in Milk Production with Age and Milking Frequency," Journal of Dairy Science, XXXIII (May, 1950), pp. 338-57.

Arave, op. cit., pp. 22-34.

S. R. Searle and C. R. Henderson, "Establishing Age-Correction Factors Related to the Level of Herd Production," Journal of Dairy Science, XLII (May, 1959), pp. 824-35.

In contrast to this view, Arave found different aging patterns between different herds of genetically related Jersey cows.[1]

In the empirical part of this study, estimates of the relationship between age and production will be developed.

d. Body weight

On the average, milk production increases with increasing body weight. Both body weight and milk production will normally increase with age, so that part of the variation in production which is "explainable" by variation in body weight will be taken care of if we include a variable for age in the prediction formulae. However some research findings indicate that it is possible to improve predictions by including a separate variable for body weight in addition to a variable for age. Heady et. al. arrived at a production function estimate where cow weight had additional explanatory value even when both age and previous production were included as explanatory variables in the regression model.[2] Clark and Touchberry found that in multiple regressions of milk yield or milk fat yields on weight and age, weight seemed to be more closely associated with production than age was.[3]

1/ Arave, loc. cit.

2/ Earl O. Heady et. al., "Milk Production Functions Incorporating Variables for Cow Characteristics and Environment," Journal of Farm Economics, XLVI (Febr., 1964), pp. 1-19.

3/ R. David Clark and R. W. Touchberry, "Effect of Body Weight and Age at Calving on Milk Production in Holstein Cattle," Journal of Dairy Science, XLV (Dec., 1962), pp. 1500-1510.

e. Previous production records of the same cow

Previous production records of the same cow may provide the most useful information for prediction of future production. A measure of the value of previous lactation records as a basis for predicting future lactation records is the correlation coefficient between 305 days records for different lactations. Most research workers dealing with these questions have not undertaken to estimate correlation coefficients between individual pairs of lactations, but have estimated the "repeatability" of lactation records. The "repeatability" is the same as the "intra-class correlation coefficient" as defined by R. A. Fisher, and corresponds to the average of the correlation coefficients between pairs of lactations.1/

Repeatability estimates for lactation records are of special interest to geneticists because they give an upper limit to the part of the variation in production which is due to genetic variation. Repeatability is greater in populations where there is a large genetic variation between individuals with respect to production than in populations where this variation is smaller. Repeatability can be expected to increase if variation in environment is decreased, for example by improvements in management practices.2/ We will usually find higher repeatability estimates in a sample of cows from different herds than if repeatability is

---

1/ George W. Snedecor, Statistical Methods, (5th ed., Ames, Iowa: Iowa State College Press, 1956), p. 282.

Henry Scheffé, The Analysis of Variance, (New York: John Wiley & Sons, 1959), p. 223.

2/ R. R. Schrode et. al., "Changes in Repeatability with Changes in Herd Environment," Journal of Dairy Science, XLIII (Sept., 1960), p. 1343.

estimated on an intra-herd basis, because of similarity in feeding and care within a herd which causes a likeness between successive records of the same cow which is non-genetic.[1/]

Estimation of repeatability of lactation records measured as pounds milk or as pounds of fat has given varying results, with most estimates falling between 0.30 and 0.60.[2/] The proportion 0.40 is often taken as an average value.[3/] Castle and Searle and Arave have shown that repeatability estimates may vary widely between different herds, and Arave has shown that the differences in repeatability between herds is greater than what can be explained as due to randomness in small samples.[4/] [5/]

It is a question of interest for prediction purposes whether correlation coefficients between different pairs of lactation records can be assumed to be equal or not. If they can be assumed to be equal, a weighted average (where the weights are proportional to the inverse of the within lactation standard deviation) of previous lactation records would be the most efficient measure for predicting future lactation records. It is often assumed that the coefficient of variation is constant between lactations, or in other words that the within-lactation

---

1/ R. L. Laben and H. A. Herman, Genetic Factors Affecting Milk Production in a Selected Holstein-Friesian Herd, University of Missouri Agric. Exp. Sta. Research Bulletin 459 (Columbia, 1950).

2/ A survey of repeatability estimates is given by Arave, op. cit., p. 16.

3/ Arave, op. cit., p. 36.

4/ Olive M. Castle and S. R. Searle, "Repeatability of Dairy Cow Butterfat Records in New Zealand," Journal of Dairy Science, XL (Oct., 1957), pp. 1277-83.

5/ Arave, op. cit., p. 37.

standard deviation is proportional to the lactation average.[1] If this holds true and the correlation coefficients can be assumed to be equal, then the average of previous age-corrected lactation records would be the most efficient measure for predicting future lactation records.

Johansson and Hansson reported that correlations between adjacent lactations were higher than between more remote lactations.[2] Castle and Searle reported the following correlations between milk fat production of the same cows at different ages:[3]

Age in years	Age in years				
	3	4	5	6	7
2	.67	.61	.59	.53	.43
3		.67	.62	.58	.54
4			.65	.62	.57
5				.67	.60
6					.66

Lower correlation coefficients but the same pattern of decreasing correlation coefficients for lactations more distant in time were reported by Berry.[4] Rendel et al. found the same pattern in correlation coefficients between the first four lactations.[5]

1/ Castle and Searle, op. cit., pp. 1278-79.

2/ Johansson and Hansson, op. cit., pp. 108-109.

3/ Castle and Searle, op. cit., p. 1278.

4/ J. C. Berry, "Reliability of Averages of Different Numbers of Lactation Records for Comparing Dairy Cows," Journal of Dairy Science, XXVIII (May, 1945), pp. 355-66.

5/ J. M. Rendel et. al., "The Inheritance of Milk Production Characteristics," The Journal of Agricultural Science, XLVIII (April, 1957), pp. 426-431.

These results suggest that the assumption of equal correlation coefficients between different pairs of lactations does not hold. If so, a more complicated formula for predicting future lactation records based on previous lactation records will give better predictions. The most relevant alternative is to use the multivariate approach outlined in chapter IV.

f. Production records of relatives

The correlation between a trait measured on one of the parents and on the offspring gives a measure of the degree to which this trait is inherited. "Heritability" of a trait may be estimated by the doubling of this correlation coefficient, if it is assumed that the offspring receives one half of its genetic characteristics for the trait from each of its parents. Most heritability estimates fall somewhat below repeatability estimates based on the same data. While it is often assumed that the repeatability of milk production is close to 0.40, heritability of the same trait may be in the range of 0.20 to 0.30.[1/]

Since the propensity for high or low milk production to some degree is heritable, knowledge about production records of dam, of sire's dam, and of halfsibs may be useful for predicting the level of milk production of a given cow. For heifers without a previous lactation record, a figure for "pedigree promise" which summarizes the information available from known relatives may be a useful measure for predictions. After the cow has attained its own milking record, it is a question whether information on "pedigree promise" will add anything to the precision of predictions based on previous lactation records alone. The author does not know about any study where this question has been investigated.

1/ See page 112.

g. Length of the preceding calving interval

On the average, milk production during a given lactation will increase with increasing length of the preceding calving interval. Johansson and Hansson found that length of the preceding calving interval was a "rather powerful" cause of variation in lactation yields.[1/] Correction of 300 days records for length of the preceding calving interval reduced the total variance by 4.5 per cent.[2/]

h. Length of the preceding dry period

For given cows, lactation yields will normally increase with increasing length of the preceding dry period up to a certain limit. This refers to results from intra-cow comparisons of different lactations, and must not be confused with inter-cow comparisons, where it is usually found that cows with long dry periods have lower lactation yields.

i. Season of calving

It is likely that the effect of season of calving on lactation yields has to do with different environmental factors which may vary between seasons, like temperature, availability and quality of various feeds, etc. Johansson and Hansson found that differences in yields due to season of calving seemed to be closely related to management, and that the effect of season also varied considerably between years for the same herd.[3/] Season of calving seems to be a less important source

---

[1/] Johansson and Hansson, op. cit., p. 41.

[2/] Ibid., p. 103.

[3/] Ibid., p. 57

of variation in yields for California market milk producers than it is for dairy producers in the eastern parts of the United States.[1/]

j. Type ratings

Attempts are sometimes made to estimate the productive ability of cows from their external appearance. Several authors have examined the relationship between different "type ratings" and production. Mitchell, Corley and Tyler calculated sample correlation coefficients between milk and milk fat production and such "type ratings" as "general appearance", "dairy character", "body capacity", "mammary system", and "final ratings".[2/] The type rating of best predictive value proved to be the rating for "dairy character", which showed a correlation with milk fat production of 0.22, 0.25, and 0.24 in low, medium, and high-producing herds respectively. If "type rating" of this predictive value can be made on heifers before their first freshening, it may be of some practical value in lieu of or in addition to "pedigree promise". In the study referred to, the ratings were performed after first calving, however. For animals with a production record, "type ratings" appear to be of small practical value.

[1/] Dr. Robert C. Laben, personal communication.

[2/] R. G. Mitchell, E. L. Corley, and W. J. Tyler, "Heritability, Phenotypic and Genetic Correlations between Type Ratings and Milk and Fat Production in Holstein-Friesian Cattle," Journal of Dairy Science, XLIV (Aug., 1961), pp. 1502-10.

D. The Feed Input - Milk Output Relationship

The replacement decision requires that we are able to make predictions about feed inputs and milk output for given cows. Since feed input and milk output are closely interrelated, we need a framework or a method for dealing with the relationship between them.

The daily milk output for a cow at any given time is determined by a number of factors. Of these factors, feed input is one. Conceptually, it may be useful to think of a functional relationship, where milk output is determined as a function of inputs of different feeds and of a number of other factors which for the purpose of this discussion conveniently may be summarized as "cow ability". "Cow ability" as the term is used here depends on the cow's genetic constitution, its age, number of the lactation, distance in time since last freshening, influences from past and present environmental factors other than feeding, etc.

If we assume that the dairy producer wants to feed at the level which maximizes his profit, and if we know the functional relationship, "cow ability" for the given cow, and prices for feed and milk, then the determination of feed and milk quantities is a matter of routine. Under the assumption of profit-maximizing behaviour, the prediction problem within this framework consists in predicting "cow ability". When this is predicted, both feed quantities and milk quantity follow authomatically.

At the present state of knowledge, the conceptual framework outlined above is difficult to use in an operational replacement model, both because we usually have no satisfactory measure of "cow ability", and because the existing estimates of the functional relationship between feed inputs and milk output are far from satisfactory.

An alternative approach includes the assumption that the dairy producer follows a given "feeding system" which may vary over time but at any given time is the same for all cows in the herd. With a "feeding system" is here meant a policy for allocation of feed, whether this policy consists in giving all cows in the herd the same ration, or in regulating feed rations according to actual production in a given way. Realized milk production as well as feed consumed for any cow will then be a function both of feeding system and of cow ability. Under this assumption, however, we do not have to predict cow ability in order to predict milk production, but can, under a given feeding system, predict milk production directly.

It is evident that under most of the feeding systems which are likely to be used in practice, milk production and feed consumption vary together, so that cows with high production on the average also will consume more feed.

It is known from experiments that there are differences between cows with respect to utilization of feed, so that two cows of the same weight and with the same production may consume different quantities of feed. Even if such differences are large enough to be of economic significance, they are very difficult to measure under practical dairy conditions.

It will be assumed in this study that with the feeding system given, there exists a relationship between milk production and feed input in such a way that we know the expected values for feed consumed during a given time period when we know the age of the cow and the quantity of milk produced during the same time period.

E. Prediction of Accidents and of Sickness

Accidents and sickness are of considerable economic importance in dairy herds. If we could identify variables which were useful for prediction of such mishaps, we might want to include them as state variables in the model.

For some diseases, it seems likely that the probability of future occurence is higher for some cows than for other. Some diseases are chronic. If a cow suffers from a such disease during one production period, there is a high probability that it will suffer from the same disease during future production periods. It is also possible that some relationship exists between the level of milk production and the probabilities of some diseases. No empirical study related to this question on an within-herd basis has been reviewed. An attempt has been made in the empirical part of this study to examine the relationship between production level and the probability of "involuntary replacement", which again is indicative of diseases or accidents.

Different authors have found that susceptibility to mastitis is partly heritable. Lush estimated heritability of mastitis to be 0.38.[1/] This estimate was based on a relatively small sample, however, so that the confidence region associated with the estimate was very wide. Legates and Grinnells estimated heritability to be 0.27 ± 0.10, while Young, Legates, and Lecce found low heritability in another study but a

---

1/ Jay L. Lush, "Inheritance of Susceptibility to Mastitis," Journal of Dairy Science, XXXIII (Feb., 1950), pp. 121-25.

repeatability of mastitis between lactation cycles of 0.31 ± 0.06.[1/2/] These results indicate that mastitis is predictable to a certain degree, based on information on the individual cow.

F. Prediction of Breeding Trouble and of Length of the Calving Interval

1. Nature of the Problem

Because of breeding difficulties, length of the calving interval for dairy cows has a marked skewed distribution. While some cows develop different kinds of reproductive abnormalities early during the lactation, most cows show normal heat in normal time after last freshening. Of these cows, we may expect that approximately 60 per cent conceive at the first service, a fairly large per cent of the remaining conceive at the second service and a fairly large percent of those which still are open conceive at the third service. Some cows which show normal heat can go through a large number of unsuccessful breeding attempts. In a study by Buch, Tyler, and Casida, the most important sources of breeding inefficiency were infertile services (average 21.2 days lost), loss of pregnancy (11.9 days lost), interruption of the estrual cycle (8.0 days lost), and out-of-breeding conditions (5.2 days lost).[3/] These

---

1/ J. E. Legates and C. D. Grinnells, "Genetic Relationships in Resistance to Mastitis in Dairy Cattle," Journal of Dairy Science, XXXV (Oct., 1952), pp. 829-33.

2/ C. W. Young, J. E. Legates, and J. G. Lecce, "Genetic and Phenotypic Relationships between Clinical Mastitis, Laboratory Criteria, and Udder Height," Journal of Dairy Science, XLIII (Jan., 1960), pp. 54-62.

3/ N. C. Buch, W. J. Tyler, and L. E. Casida, "Variations in Some Factors Affecting the Length of the Calving Interval," Journal of Dairy Science, XLII (Feb., 1959), pp. 298-304.

results are from an experimental herd where cows which did not conceive were kept until 233 open days following a 75 days postpartum interval, and in such cases, all 233 days were counted as days lost.

Absolute sterility is of relatively low frequency among dairy cow. However, because of the economic loss incurred if the calving interval gets too long, a dairy producer will seldom keep a cow which has not conceived within a reasonable time after last freshening. For our purpose here, we will want to estimate the probability distribution of the length of the calving interval, as this distribution is related to other factors. It will then be left to the replacement model to find out how long calving intervals should be allowed to be before the cow should be replaced.

2. Variables Which May Influence the Probability Distribution of Calving Intervals

a. Management practice

Management practice may differ between herds with respect to how much time is allowed to expire after the last calving before a cow is bred, and with respect to veterinary treatment of cows with breeding problems. Management practices conducive to discovery of heat may also vary between herds. Information on such practices may therefore improve the precision of predictions. In this study, probability distributions of length of the calving interval will be estimated separately for different herds, in order to take into account the possible differences in management.

b. Production level

Menge et. al. reported that there appeared to be a positive correlation between milk production and frequency of cystic ovaries both in their own study and in other studies reported.[1] There seemed to be a positive correlation also between milk production and interval to first post-partum estrus, while they found no correlation between milk production and conception to first service.

From these results, it seems to be some relationship between level of production and breeding trouble, however the importance of the relationship is difficult to evaluate from the data given.

c. Age

No study has been reviewed where the relationship between age and breeding problems is examined. An attempt will be made to examine such relationships in the empirical part of this study.

d. Information on relatives and on previous breeding problems for the same cow

Most authors agree that both heritability and repeatability of factors related to breeding problems are very low. Johansson and Hansson found a very slight tendency for cows to repeat a short or a long calving interval. They estimated within-herd repeatability of length of the calving interval to be 0.036,[2] and assumed the heritability of length of the calving interval to be in the range 0 - 5 per cent.[3] Trimberger

---

1/ A. C. Menge et. al., "Variation and Association among Postpartum Reproduction and Production Characteristics in Holstein-Friesian Cattle," Journal of Dairy Science, XLV (Feb., 1962), pp. 233-41.

2/ Johansson and Hansson, op. cit., p. 28.

3/ Ibid., p. 118.

and Davis used as criterion for breeding efficiency the number of services per conception. They found it impossible to predict breeding efficiency of a cow from her previous breeding records, but found an indication of differences in breeding efficiency between cow families and between sire groups.[1] Dunbar and Henderson estimated repeatability of non-return to first service in usual herds consisting partly of half-sibs and partly of unrelated cows to be in the range of 0.027 - 0.051, while they estimated heritability of the same trait to be 0.[2] Inskeep, Tyler and Casida used as a measure of fertility whether the first insemination resulted in the birth of a live calf. This is a more restricted measure than the one used by Dunbar and Henderson. They found differences between sire groups in fertility, and estimated heritability of fertility to be 0.09 for heifers and 0.08 for elder cows.[3]

G. Specification of a Dairy Cow Replacement Model

1. The Model

Among many different alternatives considered, the following model was selected:

1/ George W. Trimberger and H. P. Davis, "Predictability of Breeding Efficiency in Dairy Cattle from Their Previous Conception Rate and from Their Heredity," Journal of Dairy Science, XXVIII (Sept., 1945), pp. 659-669.

2/ R. S. Dunbar, Jr. and C. R. Henderson, "Heritability of Fertility in Dairy Cattle," Journal of Dairy Science, XXXVI (Oct., 1953), pp. 1063-71.

3/ E. K. Inskeep, W. J. Tyler, and L. E. Casida, "Hereditary Variation in Conception Rates of Holstein-Friesian Cattle," Journal of Dairy Science, XLIV (Oct., 1961), pp. 1857-62.

a. Definition of a stage

By definition, an old stage ends and a new stage begins at the following points in time: (1) immediately before the inclusion of a new heifer in the herd; (2) except for cows in the sixth lactation, seven months after last freshening.[1/]

b. Specification of state variables

Three state variables are included in the model:

1. Age, measured in lactations
2. One variable for production history, defined differently for different lactation numbers and derived by the method explained in chapter IV
3. One variable for length of the calving interval

Each of these state variables can take on a specific number of different values, as will be discussed below. In addition to the states which can be formed from all possible combinations of the state variables listed above, one state is defined to represent the process immediately before the purchase of a heifer.

c. Specification of alternative actions

For each state, two alternative actions are considered:

1. For cows known to be pregnant: Keep

For cows not yet known to be pregnant: Keep the cow three more

---

1/ Cows in the sixth lactation are assumed kept until ten months after freshening, when they will be replaced with a heifer. Provided no involuntary removal of such cows, the last stage in the life of a given cow will last from seven months after fifth freshening to ten months after sixth freshening, when the cow will be replaced with a heifer. For cows of all ages, the stage may expire before seven months after last freshening if the cow is involuntary removed and a replacement heifer included in the herd.

months, replace her then if she has not conceived in the meantime, but keep otherwise.

2. Replace the cow immediately with a heifer.

d. Some comments

The size of the model depends on the number of alternative values for each state variable. In the present application, cows which are not decided removed during the fifth lactation are assumed kept until the end of the sixth lactation and then replaced, so we need five alternative values for the state variable "age". The variable for production history is allowed seven alternative values. As a simplification, it is assumed that the calving interval will be either 12 months, 15 months, 18 months, or longer. On the first day of the stage, we know whether the calving interval is 12 months, 15 months, or longer than 15 months, so we need three alternative values for the state variable "length of the calving interval". Since one state, which will be referred to as "state 1", represents the process immediately before the purchase of a heifer, the model contains a sum of $1 + 5 \times 7 \times 3 = 106$ states.

For each state and decision, we can estimate transition probabilities for transitions to other states. A great number of these transition probabilities will be zero. If, for example, the state variable for age for a given state has the value "second lactation" and the decision is "keep", the process can only go to states for which the state variable for age has the value "third lactation", or to state 1 if the cow is involuntary removed. If the state variable for age has the value "fifth lactation", the next state will be state 1 anyhow, but the length of the stage will vary depending on whether the cow is involuntarily removed or not before the end of the sixth lactation.

In this model, stage length will vary between zero and 21 months. In any case where the decision is to "replace", the process will go to state 1 and the length of the stage will be zero. If the decision is "keep", the age is anywhere below "fifth lactation"; and the cow is not involuntarily removed before seven months after next freshening, the stage length will be 12, 15, or 18 months depending on the length of the calving interval. If the age is "fifth lactation", then the stage length may be 15, 18, or 21 months respectively. If the decision is to "keep" but the cow is involuntarily removed, the stage can, theoretically, end at any time between the first day of the stage and seven (ten) months after next freshening. For simplicity, it is assumed that involuntary removal will always take place in the middle of a month.

Discounting plays an important role in a model with variable stage length. With zero interest rate, the model would not be useful. A stage length of, say, 18 months will normally yield a higher "immediate expected return" than a stage length of 12 months. With zero interest rate, this would result in an unjustified preference for cows with long calving intervals. Differences in stage length is taken into account by the discounting of future returns.

If the policy is allowed to vary over stages, we will get a finite Markov process with 106 states. If the policy is kept constant over stages, the transition probabilities will also be invariable over stages since they are assumed to depend on the decisions only, and we will get a finite Markov chain. This will be the case under an infinite planning horizon, when the optimal policy is invariable over stages.

2. Justification for the Choice of Model

A model should be able to consider the three important groups of removal causes: low production, breeding trouble, and diseases and accidents. If possible, it should be able to measure each of these factors or groups of factors so that it can be left to the optimization procedure to find out in which cases a cow should be replaced because of one or another reason.

While the level of production and the degree of breeding trouble are relatively easy to measure and express through given state variables, it is very difficult to find any simple measure of "degree of sickness". Nor do we have any possibility for estimating the relationship between such a variable and economic important future variables like production, feed consumption, and death. In the replacement model, therefore, replacement because of sickness or accidents will be taken as given. Probabilities of such "involuntary replacements" will be estimated based on empirical data and these estimated probabilities will be taken as some of the assumptions underlying the model.

One state variable for production and one state variable for length of the calving interval are included in the model since production and breeding trouble are the other two important replacement causes. Age is included as the third state variable because production level as well as probabilities of involuntary replacement can be assumed to be related to age, and breeding problems may be related to age.

In addition to the three state variables specified, we might want to include a number of others. The future profitability of a dairy cow depends first of all on her future production, her future feed consumption, length of future calving intervals, and future sicknesses and accidents. Any variable which may add something to the prediction of these future variables could be considered for inclusion. It would have improved the precision of predictions of future production if we had included more than one variable for production history.[1/] Other variables of interest are age measured in years or in months, body weight, pedigree promise, season of calving, and variables for health history. In addition to these, we might want to include two more variables which the dairy producer may want to consider: milking qualities, and disposition. On the other hand, a variable for past degree of breeding trouble is of little interest, since research quoted above indicate that the repeatability of breeding performance is very low.

The limitations set on the total number of states in the model restrict the number of state variables which can be included. In the present application, the model contains 1 + 105 = 106 states. With one additional state variable with for example five alternative values, there would be 1 +(5 x 105) = 526 states, which is far beyond computational capacity. By restricting the number of alternative values for each state variable, we could have been able to include one more state variable. The actual choice of state variables and of number of alternative values for each state variable is to some degree a matter of subjective judgement. Within the limits set by capacity, we want to make a choice which gives as good representation as possible of the actual replacement problem. Before the choice was made in this case, the previous research findings discussed in this chapter were weighted and the variables which seemed to be the most important ones were selected for inclusion.

[1/] See page 85.

Definition of a stage is another important question. In this case, actual alternatives were to use as stages months, two- or three-month periods, years, or lactation cycles. In order to be able to keep track of the age of the cow, we need one alternative value for the age variable for each stage length addition in age. For example, if maximum time in dairy production for a cow is set equal to six years and stages are defined to be three-month periods, we would need $\frac{6 \times 12}{3} - 1 = 23$ alternative values for the age variable. The number of alternative values which could then be allowed the other variables would be very low. A model with annual stages is not very satisfactory either. It is known that the calving interval varies in length, and the optimal time for replacement within the lactation is somewhere during the later part. In a model with annual stages, we would need one separate state variable to describe the location of calving within the stage, and since the decision has to be made at the same point in time within each stage (for example on the first day), the time for decisionmaking and actual time for replacement would often not fall together.

Lactation cycles as stages were selected as the most satisfactory alternative. With stages defined in this way, stages will not be of constant length, but it has been shown in chapter III that this problem can be handled within the general framework.

Except in the case of heifers just included in the herd, each stage is defined to begin seven months after last freshening, because voluntary replacement usually will take place around this time. Preferably, we should have let the initial day of the stage depend on the production class and on the length of the calving interval, since cows which are replaces because of low production alone often should be replaced before seven months, and cows which are replaced because of breeding trouble often should be replaced somewhat later. It is feasible without increasing the size of the model to let the last day of a stage depend on the actual classification of the cow, however this would have increased the task of deriving parameters for the model tremendeously. It will be shown later that the error committed when assuming all replacements to take place seven months after last freshening is of quite small importance, even if it may change the replacement decision in marginal cases.[1]

One apparent inconsistency should be mentioned here. The state variable for production is defined according to the method explained in chapter IV, whereby production is expressed as a linear combination of all 305 days records up to and including the present. Seven months after last freshening, however, the present 305 days record is still unknown. It has been shown that there is a very high correlation between the record for the first seven months of a lactation and the 305 days record. We will assume that the record for the first seven months is used to predict the 305 days record, and that this predicted record is used to derive the production variable.

It is assumed in the model that all cows which are removed from the herd are replaced with heifers. This is not quite realistic but the deviation from reality is not likely to be of much significance in this case. Even if older cows are sometimes purchased as replacement animals, the prices of such older cows are likely to be such that the difference in economic result between replacement with a heifer and replacement with an older cow is small. Replacement with a heifer can then be taken as representative of the economic result under the alternative "replace".

---

[1] See pages 221-224.

It is assumed in the actual models that a calving interval will be either 12 months, 15 months, 18 months, or longer. If the current calving interval will be 12 or 15 months, then this is known 7 months after last freshening, and we have all information required for making the decisions "keep" or "replace". If the current calving interval will be 18 months or longer, then the actual length is not known 7 months after last freshening, and the decisions we can make are either "replace" or "wait and see". If we decide to "wait and see", then it is assumed that we will wait three more months, replace then if the cow has not conceived in the meantime, but keep the cow if she has conceived.

It is assumed in the model that all cows which have not been replaced before will be replaced at the end of the sixth lactation. Under this assumption, cows will normally not be bred during the sixth lactation, and will therefore maintain a higher production during the later part of the lactation than they would have done otherwise. This is why the last stage is assumed to last until ten instead of seven months after the sixth freshening.

In reality, a few cows are kept longer than to the end of the sixth lactation. The reason it was decided to cut off the model at the end of the sixth lactation is that very few observations exist by use of which parameters for lactations after the sixth can be estimated. Even then, it might have been better to "guess at" parameters for the later lactations, maybe by extrapolating trends found within earlier lactations, and allow lactations beyond the sixth in the model. Few cows will last beyond the sixth lactation anyhow and the method followed here is not likely to bias the decisions rules for the first lactations to any significant degree. It may, however, possibly give some bias in the decision rules for cows in their fourth and especially in their fifth lactation.

It is assumed in the model that any removed cow will be replaced immediately with a new animal.[1/] In reality, a dairy producer may prefer to wait some time before he includes a new cow in the herd. We could, therefore, consider to distinguish between a "removal decision" and a "purchase decision". Computationally, this problem could be handled without much difficulty by introducing an extra state characterized as "no cow present" in the model. With the other simplifying assumptions made in this model, however, this would have no purpose. The reasons why a dairy producer may prefer to wait before he introduces a new animal in the herd are: (1) considerations of seasonal distribution of milk production and of seasonal differences in level of milk production of a cow, (2) considerations of short-run variations in meat prices and replacement prices, (3) that he raises his own heifers and wants to wait until a heifer is ready for introduction in the milking herd rather than to buy a replacement animal at the market. The model used here makes no distinction between seasons, assumes constant livestock prices, and makes no distinction between replacement with a purchased animal and replacement with a self-raised heifer.

Allthough the considerations mentioned in reality may influence the exact time for removal of one animal and the exact time for inclusion of a new animal in the herd, it is believed that under Californian market milk production conditions they will seldom change the decision as to whether or not to remove a cow during a given lactation. It is for this

---

[1/] It will be assumed that in the case of "involuntary removed" cows one month will expire from the cow is removed to a replacement animal is purchased. See page 192.

decision the specified model is thought most helpful, while details in the determination of exact time for removal and replacement can be left to judgement or to more simple methods.[1/]

The present model does not consider effects of genetic changes over time. The different ways in which such considerations could be included have not been considered in detail. Two alternative possibilities should be mentioned:

In a model with constant stage length and optimization under a limited planning horizon, we could introduce a separate state variable for "year of birth" and let the expected immediate economic returns depend on this state variable. This alternative would have resulted in a larger model, or would have required more simplifications with respect to other state variables in order to keep the model within a manageable size.

In a model with variable stage length and optimization under an infinite planning horizon, the problem seems manageable if we are willing to make some additional assumptions. For example, assume that each year's batch of heifers, as a result of rational breeding programs, has an average productive ability which is a given percentage above the average ability of the preceding year's batch. Further, assume that as a result of competition, the over-all profitability of dairy production is constant over time in spite of this increase in productivity. The result would be a vector of expected immediate economic returns which could be assumed to be invariable over stages, but in which the expected

---

[1/] See pages 219 ff.

immediate economic returns for younger cows would be somewhat higher and the same parameters for older cows somewhat lower than what we get in a model where genetic change is not considered. If we are willing to accept such assumptions, this seems a feasible way the handle the problem of genetic change in the same basic type of model as the one defined above.

VI. ESTIMATION OF PARAMETERS

A. Description of Data

Data used in this study were obtained from the University of California Dairy Cattle Breeding project. This project has been in operation from 1925 to the present time, and involves the cooperation with a number of dairy herds which are all enrolled in the Dairy Herd Improvement Association of the state. Each cooperating herd is visited twice each year by a University staff member for the purpose of recording different information concerning the individual cows in the herd into permanent herd books, as well as to handle problems which arise in herd management. The herds consist predominantly of Jersey cows, allthough there are some cows of other breeds in some of the herds.

At the time this study was initiated, much of the information collected in this way had been transferred to IBM-cards, with one card for each lactation for each cow. Among the various data transferred to cards, the following are most relevant to this study: cow identification number and lactation number, age of the cow at calving, month and year of calving, length of the lactation, length of the calving interval, and 305 days production of milk, of milk fat, and of fat-corrected milk. A code number for removal reason was given for the last lactation for each cow which had been removed.

Data for the distribution of production within the calving interval had not been transferred to cards, but for each lactation prior to 1958, monthly production data were available in the herd books.

Only data for Jersey cows were used in this study. At the time the analysis was carried out, more than 12,000 lactation records for about

4,000 cows in 12 cooperating herds were available. The original plans for the study called for performing analysis on all 12 herds. Since the analysis of data proved to be more time-consuming than had been anticipated, the analysis was restricted to the two largest herds, data from which were analyzed separately.

Previously, Arave had used the same data to measure genetic change in the 12 herds.[1/] Some results from Arave's analysis could have been used as part of the basis for the numerical replacement models in this study. Such results as Arave had produced for these herds will normally not be available, however, and it seemed desirable to develop or try out estimation methods which can be used directly on field data in more general cases. Therefore, the analysis of data was performed independent of Arave's results.

Arave's analysis, which was based on another stochastic model, had indicated a considerable difference between population parameters in the two selected herds. One of the purposes of selecting these two herds for analysis was to examine whether such differences would result in considerable differences in replacement policy between them. The other reason for selecting just these herds was that they were the largest and therefore would provide the largest number of observations.

The two herds will be referred to as the ME-herd and the MA-herd. The total number of cows from which records were used in this study was 702 in the ME-herd and 654 in the MA-herd.

1/ Arave, op. cit.

See also: C. W. Arave, R. C. Laben, and S. W. Mead, "Measurement of Genetic Change in Twelve California Dairy Herds," Journal of Dairy Science, XLVII (March, 1964), pp. 278-83.

In order to study the relationship between 305 days production, length of the calving interval, and production for each month during the lactation, a sample of lactations was drawn by a random procedure from the ME-herd. For the selected lactations, the relevant data were copied directly from the herd books and transferred to a new set of IBM-cards for analysis.

B. The Multivariate Analysis of Production Data

1. Specification of a Stochastic Model and Exclusion of Some Data

As dependent variable in this analysis is used 305 days FCM. The method of analysis described in appendix B requires for each cow a complete sequence of lactation records starting with the first lactation. Thus, a cow was included in the sample if there was a complete record for the first lactation alone, for the first and the second lactation, or in general, for the first, the second, ..., the i'th lactation. If the record for one lactation for a cow was missing, all subsequent lactation records had to be excluded from the sample but the lactations prior to the missing one were retained in the sample. For example, if a cow had complete records for the first, the second, and the fourth lactation, but some of the information required was missing for the third lactation, then only the first two lactations were included in the processed data.

The last lactation record in a sequence was considered incomplete and therefore excluded if the lactation length was less than 270 days.[1/]

[1/] In principle, such records should have been excluded only if the cow had had a legitimate reason for the short lactation, such as accidents, sold while still milking, etc. A cow with short last lactation

A lactation record was considered complete even if the lactation length was less than 270 days if the cow had one or more following lactations.

Some lactation records were excluded because some variable other than production was missing. For some lactation records, information on age at calving or on length of the calving interval was not available, and in such cases, this lactation and the following lactations of the same cow had to be excluded.

It seems reasonable to regard the heifers freshening in these herds as a sample frawn from conceptual infinite populations, namely the populations of dairy cows which conceiveably could have been born from the given genetic stock and reached the age of first freshening in these herds.

To avoid bias in the results, it is important that all heifers which freshened for the first time in these herds had an equal chance to enter the sample. Bias would occur if heifers with low production had been removed from the herds before 270 days after freshening, so that their records had been excluded from the sample. The cooperating dairy producers had obliged themselves to keep all heifers at least 270 days unless they dried up before, so this source of bias in unimportant. Some heifers were removed before 270 days because of disease and accidents, but it does not seem likely that such removal causes are correlated with production level in a way which would bias results.

The general model which has served as a basis for this analysis is given as (4.3).[1/] The model assumes that there are some observable

---

may have been a low producer due to lack of persistency, and removal of the record in such cases will give some bias in results.

[1/] See page 69.

variables z_{fd} which are able to explain part of the variation in 305 day FCM between cows within each lactation number. The variables listed below are such as either previous research findings or common sense points out as relevant:

1. Age at calving
2. Length of the previous calving interval
3. Length of the present calving interval
4. Sickness, if such has occurred during the lactation
5. Environmental factors, especially feeding level
6. Average genetic constitution of the population at the time the cow was born

The effect of variation in age at calving is supposedly most important for the first lactation, somewhat less important for the second, and of small importance for later lactations.

Increasing length of the previous calving interval tends to increase 305 days FCM, because the cow is better prepared for a new lactation after a long calving interval.

Increasing length of the present calving interval tends to increase 305 days FCM because the effect of a new pregnancy during the later part of the 305 days period is less important when the present calving interval is longer. A new pregnancy has little effect upon production during the first 20 weeks (140 days).[1/] If the calving interval is 335 days, this means that the cow has become pregnant approximately 335 - 280 = 55 days after last calving, since average gestation length is close to 280 days. The part of the 305 days period within which the effects of pregnancy are noticeable is 305 - (55 + 140) = 110 days. If the calving interval is 440 days, this means that the cow has become pregnant 160

[1/] Johansson and Hansson, op. cit., p. 46.

days after last calving, and the part of the 305 days period within which pregnancy is noticeable is 305 - (160 + 140) = 5 days. Whether the calving interval is 440 days or longer than 440 days will make very little difference for 305 days production.

Since variations in the present calving interval beyond 440 days have little or no effect on 305 days production, the variable "length of present calving interval" was set equal to 440 days whenever the observed length was 440 days or longer.

There may be some reasons to expect variations in length of the previous calving interval beyond some limit to have little effect on 305 days production as well, however the author knows about no study where this question has been examined. In this study, the variable "length of previous calving interval" was also set equal to 440 days if the observed previous calving interval was 440 days or longer. The only justification for setting this limit to 440 was computational convenience. In fact, it would have increased the computational burden considerably if different limits had been used for the previous and the present calving intervals.

A special problem was met when trying to take into account the effects of sickness and of other extraordinary environmental factors. A close inspection of the data from both herds revealed that in the MA-herd, some peculiarly small production data occurred for some lactations for a few of the cows. As an example, 305 days FCM for the first through the sixth lactation for one of the cows were: 9410, 11600, 12460, 4880, 13030, 12340. The underlined figure for the fourth lactation deviates so much from the average that the probability of finding such a large deviation in a normally distributed population is very small. There are quite good reasons to believe that deviations of this size are due either to sickness or to some other observable source of variation.

In the MA-herd, approximately 60 lactation records were found to be "suspiciously low" in the sense that they showed very large negative deviations from the other lactation records of the same cow. Very few such records were found in the ME-herd. Classification of such strange records by years revealed that more than half of these "suspiciously low" records in the MA-herd had occurred for lactations initiated in the years 1947, 1948, 1955, and 1956. Probably some specific environmental factors accounted for these unusual deviations.

Notes on sickness and on other environmental factors were available in the original log of notes on the herds, but had not been transferred to the cards which were used in this analysis. Time did not permit full utilization of these additional data. Under doubt, it was decided to remove from the sample records which were "suspiciously low" during the given four years, while records with large negative deviations were retained in the sample if the deviations had occurred during other years.

It perhaps would have been more satisfactory if information on sickness or other extraordinary causes of deviations had been used so that either these causes could have been included as "independent variables" in the model, or rejection of data could have been based on this information rather than on production records. It is recognized that rejection of extreme data based on no other information than the data themselves may bias the results. However in view of the numbers involved and the difficulties of coordinating and interpreting subjective notes on environment, the bias introduced by this decision was largely unavoidable and probably not great in magnitude.

We may also look at this from the point of view of the purpose of this study. A dairy producer will normally use his knowledge about previous production records of a cow to predict the expected future productivity of the same cow. However if he knows that a cow has been seriously sick during a given lactation or have had a low production record for some other specific reason related to environment, he will most likely ignore that lactation record when predicting about the future. If the purpose of estimating correlations or covariances between lactation numbers is to obtain prediction formulae, then it is better to obtain estimates of parameters which are valid for the cases where no such extraordinary causes of variation have been in effect.

Of the six possible indenpendent variables listed on page 139, it now remains to discuss number five and six. Inclusion of these variables or of some substitute for them is desirable because data for the analysis have been collected over a long span of time, within which considerable changes may have taken place both in environmental factors and in the average genetic constitution of the herd.1/ A relatively large part of the total variation in data may be due to such changes. If we do not correct for them, we may arrive at estimates of between lactation correlation coefficients which are too high, because different lactation records of the same cow have a likeness which is due to a closeness in time rather than to a true tendency for high and low production records to repeat themselves.

1/ For the given herds, such changes had been measured by Arave -- see Arave, op. cit. Even if, in this case, Arave's estimates could have been used to correct the individual observations for such changes, it was decided to use a method which is more generally applicable to analysis of herd data.

It can be argued that the herd average, after corrections for variation due to variation in age composition of the herd, is a measure which in a somewhat crude way summarizes the "genetic change effect" and the "environmental effect". Another argument for the inclusion of an "age-corrected herd average" as a variable in the model is the following: As a basis for a replacement decision, we are first of all interested in a comparison of the productive ability of a given cow with the expected productive ability of a possible replacement animal. If the replacement animal will be drawn from the same genetic stock as the one from which the cows in the herd have been recruited, then the herd average may give a reasonable good estimate of the expected productive ability of a replacement animal under the same environmental conditions as the given cow had. The deviation between a cow's record and the herd average is therefore a more relevant measure than the absolute level of the record is.

As a result of these considerations, it was decided to include "age-corrected herd average" as an independent variable in the model. As an alternative, however, parameters were also estimated for a model where this variable was excluded.

The "age-corrected herd averages" presented in table A.1 in appendix A were calculated by multiplying each individual 305 days production record up to and including the fifth lactation with an age-correction factor given by the lactation number, and then calculating the arithmetic mean of all such corrected lactation records which were initiated in the given calendar year.

The age-correction factors used were the following:

Lactation number	ME-herd	MA-herd
1	1.19	1.19
2	1.08	1.06
3	1.00	1.00
4	1.00	1.00
5	1.00	0.99

These factors were developed from the data by comparing the unweighted average over all years from 1936 through 1962 of average (for the year) first-lactation records, second-lactation records, ..., fifth-lactation records.

This method of obtaining age-correction factors is a crude one. Many authors are also of the opinion that age-correction factors based on age in months give a better age-correction than age-correction based on lactation number. It should be remembered that the purpose of age-correction in this case only is to improve the measure "herd average" as a measure of combined environmental and average genetic effects, not to evaluate the productive ability of individual cows. Therefore, a very exact method of age-correction is not required.[1/]

One of the possible ways in which a variable for herd average can be introduced in the stochastic model is to write each line in the model of type (4.2) as:

$$(x_{f\alpha} - k_f A_{f\alpha}) = \gamma_{fo} + \gamma_{f1} z_{f1\alpha} + \cdots\cdots + \gamma_{1D} z_{fD\alpha} + u_{f\alpha} \qquad (6.1)$$

1/ Previous studies have shown that even rather crude age-correction factors will remove up to 90 per cent of the variance due to age. For example, see Arave, op. cit.

where $A_{f\alpha}$ is age-corrected herd average for the year in which the lactation was initiated, and k_f is a predetermined coefficient which may vary between lactation numbers. Since, for example, we have found that average 305 days production for the first lactation is approximately 84 per cent $(\frac{1}{1.19} \times 100)$ of mature lactation records, it would be reasonable to use 0.84 as the value of k_1.

An alternative way is to include A among the independent variables:

$$x_{f\alpha} = \gamma_{fo} + \gamma_{f1} z_{f1\alpha} + \cdots\cdots + \gamma_{1D} z_{fD\alpha} + \gamma_{fA} A_{f\alpha} + u_{f\alpha} \qquad (6.2)$$

The regression coefficient γ_{fA} in (6.2) corresponds to the predetermined coefficient k_f in (6.1), but in model (6.2), the coefficient will be estimated as part of the estimation procedure. (6.2) is a somewhat more flexible model, since it allows for the possibility that changes in environmental conditions may have a stronger influence on some lactation numbers than on other. Model (6.2) can be criticized on the grounds that regression theory assumes independence between the independent variables and the error term $u_{f\alpha}$, while in this model there will be some degree of correlation between the residual $u_{f\alpha}$ and the age-corrected herd average $A_{f\alpha}$. The bias introduced by this correlation will be small if the number of observations from which $A_{f\alpha}$ is formed is large.

Model (6.2) was used in this study in spite of the objections mentioned above.

2. Results

The estimation procedure is described in appendix B. A stepwise procedure was used. First, coefficients were estimated in a multiple regression equation where the dependent variable was 305 days FCM for the first lactation, and the independent variables were the following:

1. Age in months at first calving
2. Age-corrected herd average for the given year
3. Length of the first calving interval

Second, coefficients were estimated in a multiple regression equation where the dependent variable was 305 days FCM for the second lactation, and the independent variables were:

1. Age in months at second calving
2. Age-corrected herd average for the given year
3. Length of the previous (first) calving interval
4. Length of the present (second) calving interval
5. The deviation u_1 between expected and observed 305 days FCM for the first lactation, where expected 305 days FCM was derived using the regression coefficients which were determined for the first lactation

An alternative set of regression equations were estimated for the case where age-corrected herd average is excluded from the regression equation.

The results for the first two lactations for both herds showed very small effects of the age variable. The estimated regression coefficients for age were far from significant, and in many cases had the opposite sign of what we would expect. The stochastic model which is used here allows for consideration of age both by estimating separate regression equations for each lactation number, and by including the age variable in the within lactation number regression equations. It appears that most of the total age effect has been covered by the difference between lactation numbers, and that the age effect which may exist within lactation numbers has been overshadowed by other changes which have taken place in the herds over the years from which the data were collected.

The effects of age are supposedly most important for the first two lactations, so it is highly unlikely that we would find any age effect for the later lactations when we did not find any for these first two. Therefore, it was decided to remove the age variable from the model, and new regression equations of the same type as before but without the age variable were estimated.

After deletion of the age variable, the general form of the regression equation for the r'th lactation contains the following independent variables:

1. Age-corrected herd average for the given year
2. Length of the r-1'th calving interval
3. Length of the r'th calving interval
4. The deviation u_1 between expected and observed 305 days FCM for the first lactation

•

•

r+2. The deviation u_{r-1} between expected and observed 305 days FCM for the r-1'th lactation

The first three variables represent the z-variables in model (4.2). The regression coefficients for the u-variables describe the relationships between deviations from "within lactation number regression lines" for different lactations. We need to estimate these regression coefficients in order to arrive at estimates of the variance-covariance matrix in the multivariate distribution, as is explained in appendix B.

The estimated regression coefficients and the standard deviations of the estimates are given in appendix A, tables A.2 - A.5. Estimates were derived for each lactation up to and including the sixth. The number of observations for higher lactation numbers was so small that no purpose was seen in trying to estimate parameters for these lactation numbers. This could have been done if data for several herds had been pooled, however since it was believed that important differences exist between parameter values for different herds, this possibility was not regarded as relevant.

Some comments to the results will be given here:

The regression coefficients for the "age-corrected herd average" were expected to be in the range 0.80 - 0.85 for the first lactation, in the range 0.90 - 0.95 for the second lactation, and close to 1.00 for higher lactation numbers. It is seen that all estimates are close to these values and that the actual deviations can be explained as due to random error. Nothing, therefore, would have been lost if we had used a model of the type (6.1) for including the herd average variable rather than model (6.2). Model (6.1) would have had the advantage that we would have avoided the correlation between independent variables and error term.

For all independent variables, there is a tendency for standard deviations of estimates to increase with higher lactation numbers. This reflects smaller sample sizes for the higher lactation numbers, and has no other significance.

The regression coefficients for length of the previous and the present calving intervals are significantly different from zero except for the high lactation numbers where sample sizes are small. The estimated coefficients vary relatively little between lactation numbers in the ME-herd, while this variation is considerable in the ME-herd. When we look at the standard errors of estimates, however, it seems likely that the difference between lactation numbers may be due to random errors, and we would loose little in precision by assuming the same regression coefficients for length of the calving intervals for all lactation numbers.

Estimates of the conditional covariance matrix were derived by the method explained in appendix B, based on the results given in tables A.2 - A.5. The results are given in appendix A, tables A.6 - A.7.

Since it gives a clearer picture of the results, the covariance matrices have been transformed to matrices of correlation coefficients and corresponding standard deviations. The covariance matrices can easily be generated from the results given in tables A.6 and A.7. For example, the lower part of table A.6 gives the results for the ME-herd when herd average is included as an explanatory variable. It is seen that the estimated standard deviation of u_1 is 1,513.35 and of u_2 1,603.74. The estimated correlation coefficient between u_1 and u_2 is 0.3894. Thus, the estimated variance of u_1 is $1,513.35^2$ = 2,290,228, and the estimated covariance between u_1 and u_2 is 0.3894 x 1,513.35 x 1,603.74 = 945,082.

Comparison of the different tables shows that when herd average is not included as an explanatory variable, correlation coefficient estimates are substantially higher in the ME-herd than they are in the ME-herd. This corresponds to the results given by Arave, who found that estimated "repeatability" was much higher in the ME-herd than in the MA-herd.[1/]

Introduction of herd average as an explanatory variable tends to decrease the correlation coefficient estimates. This is what we would expect. After introduction of the extra variable, there is no longer any over-all tendency to higher correlation coefficients in the ME-herd than in the MA-herd. There are quite large differences between the two herds for some of the correlation coefficient estimates, however the number of cases where the estimate for the ME-herd is larger than the estimate for the MA-herd is about equal to the number of cases where the opposite is true. It seems possible that the differences in estimates between the two herds can be explained as due to randomness in relatively small samples.

Table A.1 shows that there has been a stronger increase in herd averages over time in the ME-herd than in the MA-herd. This fact may explain why between lactation correlation coefficient estimates and repeatability estimates have been higher in this herd when no correction has been made for herd average.

The standard deviation estimates given in tables A.6 and A.7 are also of interest. These standard deviations express the variation in 305 days FCM between cows of same lactation number within a herd, after corrections have been made for variation due to variation in calving intervals, and in the lower part of the tables, also for variation due to genetic trend and to environmental changes.

[1/] Arave, op. cit.

Table A.8 in appendix A gives expected 305 days FCM for the first six lactations for the two herds, when the independent variables have been given values which approximately correspond to the observed herd averages. These are the averages we would expect in a dairy herd if no cows were culled because of low production. The same table gives estimated coefficients of variation, based on these expected values and the estimates of standard deviations given in the lower part of tables A.6 and A.7. The coefficient of variation estimates are approximately constant both between herds and between lactation numbers within the herds. This result is consistent with the usual assumption about a constant coefficient of variation with respect to milk production.[1/]

Separate analysis for the two herds was decided upon because of an assumed difference in parameter values between the two herds, especially with respect to the covariance matrices. Our results show that after "age-corrected herd average" is included as an explanatory variable, there is little difference between the two herds with respect to correlation coefficients, while the herds still appear to be different with respect to "between cows within lactation" standard deviations. This last difference seems to be explainable as due to a difference in production level between the two herds. If the observed data had been corrected for "herd average" by use of a multiplicative factor instead of an additive factor, then it is likely that this difference in standard deviations would have nearly disappeared.

1/ See pages 112-113.

C. Definition of New Variables and Derivation of Transition Probabilities

1. Derivation of Coefficients c_{ij} in Linear Combinations of the u-variables

We have assumed that the random vector $\underline{u}$ consisting of deviations between observed 305 days FCM and the regression lines for each lactation number has a multivariate normal distribution with mean vector $\underline{0}$ and covariance matrix Σ.[1/] We have arrived at estimates of the covariance matrix separately for the ME-herd and for the MA-herd. Only the estimated covariance matrices for the cases where herd averages are included as explanatory variables will be used in the remaining part of this study.

As explained in chapter IV part D, if we know the covariance matrix in a normal stochastic process we can define linear combinations v_f $(f = 1, \ldots, F)$ of the observed u-variables in such a way that a stochastic process consisting of successive observations of $v_1, v_2, \ldots, v_F$ is a Markov process. The linear combinations presented in appendix A, table A.9, are derived in this way using the estimated covariance matrices as if they were the true covariance matrices.

The linear combinations v_f derived in this way are compared with linear combinations v_f^* based on least squares regression coefficients, which we might consider to use if we had not had to worry about the Markov requirement.[2/] It is seen that the two alternatives are close in some cases but are quite much different in other cases. The coefficients for the last u-variable are the same for the "Markov process variables"

1/ See page 69.

2/ See page 85.

v_f and the "least squares variables" v_f^*. This is a result which has not been proved in the theoretical part of this thesis but which appear from the empirical results presented in table A.9.

We get a measure of the loss in precision due to the imposition of the Markov requirement by comparing the conditional variances $V(u_f|v_{f-1})$ with the unconditional variances $V(u_f)$ and with the conditional variances $V(u_f|v_{f-1}^*)$. These variances are given in appendix A, table A.10.

It is seen from table A.10 that the $V(u_f|v_{f-1}^*)$'s in most cases are not substantially smaller than the $V(u_f|v_{f-1})$'s. We may say that the loss in precision incurred by classifying a cow according to its "Markov process variable" is small as compared with classifying according to its "least squares variable". This is demonstrated for the case where we are interested in predictions of next stage's production only. We should, in principle, be concerned also about the relative value of the two methods of classification with respect to predictions more than one stage ahead. Thus, we could compare $V(u_f|v_{f-2})$ with $V(u_f|v_{f-2}^*)$, $V(u_f|v_{f-3})$ with $V(u_f|v_{f-3}^*)$, and so on. It should be remembered, however, that neither v_{f-2} nor v_{f-2}^* as defined are the most efficient predictors of u_f. The most efficient predictor of u_f when $u_1, \ldots, u_{f-2}$ are known is a variable based on the least squares coefficients of u_f regressed on $u_1, \ldots, u_{f-2}$.

Our problem is that we are forced by the nature of the replacement model to condense all information contained in the knowledge of u_1, $u_2, \ldots, u_r$ into one single variable, and it is not possible to define this single variable in such a way that it is most efficient both for the prediction of u_{r+1}, for the prediction of u_{r+2}, and for the prediction of u-variables further out in the sequence. When in addition we have to define this single variable in such a way that a sequence of such variables forms a Markov process, we are forced to sacrifice even some more of the precision.

2. Definition of Class Intervals and Derivation of Transition Probabilities

For each value of f ($f = 1, \ldots, 5$) and for each herd, the total range of the defined production variable v_f was divided in seven class intervals symmetric around the mean. These class intervals are defined in appendix A, table A.11.

Both the number of class intervals and the limits for each class interval are arbitrarily selected. In this study, the limits for the different class intervals were selected for computational convenience, however it is seen that for each lactation number, all but the two extreme classes have the same width. The actual limits are not identical for the two herds. Since the definition of the v_f-variables differ between the herds anyhow, it would have no purpose to use the same class limits for both herds. However it was attempted to define class intervals in such a way that the percentage of cows (in an unculled population) which would fall within a given class would not vary much between lactation numbers and between herds. These percentages, together with the class means, are also given in table A.11.

For example, we see from table A.11 that for cows in the ME-herd and the second lactation, production class No. 1 (the highest) is defined to encompass all cows for which the v_2-variable as defined in table A.9 exceeds 1248.9 pounds. The relative frequency of cows in an unculled population which would fall within this production class is estimated to 0.0445, and the expected value of the v_2-variable for these cows is 1,556.2 pounds.

After production classes are defined, we can derive transition probabilities by the method described in chapter IV, part E.[1/] The results for the ME-herd and for the MA-herd are given in appendix A, tables A.12 and A.13 respectively. When comparing the two tables, it should be remembered that differences may be due to differences both in the covariance matrix estimates for the two herds, and to differences in the definition of class intervals.

D. The Relationship between 305 Days FCM and Monthly Production

1. Method of Analysis

The replacement model developed in this study requires some method for determining expected production for each separate month of the lactation when 305 days production is known. For this purpose, a separate analysis was performed to estimate the relationship between 305 days FCM and monthly production of FCM for separate months. It was assumed that this relationship is independent of herd average and of length of the preceding calving interval.[2/]

[1/] See pages 90 - 93.

[2/] The truth of this assumption is not selfevident but a statistical test of a corresponding hypothesis would have required a number of observations far beyond what was available. It seems like moderate deviations from the assumption will not change conclusions from the replacement study to any significant degree.

When formulating models to describe the relationship, we should consider the following factors which have been found in earlier studies:

1. The shape of the lactation curve depends on the level of 305 days production. Cows with a high total production tend to produce a larger proportion of this production during the later part of the lactation.
2. The typical shape of the lactation curve tends to differ between the first lactation and the later lactations, while there are small differences between lactations after the first.
3. A new pregnancy tends to decrease monthly production after the twentieth week of the pregnancy, but has little effect before that time.

The model selected for this study assumes that there is a simple linear relationship between expected FCM for the i'th month of a lactation and 305 days FCM for the same lactation after 305 days FCM has been corrected for length of the present calving interval. The parameters in this relationship are different for the first lactation and later lactations, they are different for different values of i, and they are different for the following cases:

When the cow during the i'th month is less than 20 weeks in a new pregnancy.

When the cow during the i'th month is 21 - 24 weeks in a new pregnancy.

When the cow during the i'th month is 25 - 28 weeks in a new pregnancy.

etc.

To describe the assumed relations, simple linear regressions were estimated by least squares method. In these regressions, FCM for single months were regarded as "dependent variables", while "305 days FCM corrected for length of the present calving interval" was regarded as the independent variable. The correction for length of the present calving interval was performed by adding or subtracting 5 lb FCM for each day the calving interval fell short of or exceeded 365 days, however so that the maximum negative correction corresponded to a calving interval of 440 days. A correction factor of 5 lb per day is close to the average regression coefficient for 305 days FCM on length of calving interval found in the regression analysis of the ME-herd.[1/]

The estimation of these relationships was based on a sample drawn from the ME-herd only. The results were used in the replacement models both for the ME-herd and for the MA-herd.

Based on the information available on the original IBM-cards, all lactations in the ME-herd for which information on monthly production were available were stratified according to lactation number and length of calving interval.[2/] A random sample of records was drawn within each stratum. For the selected records, monthly production transformed to FCM was recorded from the herd books and transferred to a new set of IBM-cards for processing purposes. Data for different strata were pooled when the month in question came before the 20th week of a new pregnancy, however data for the first lactation were kept separate from the later lactations. Monthly data in the herd books had been recorded by calendar months. The first month of a lactation could therefore consist of anything between 1 and 31 days, while the average length of this

---

[1/] See appendix A, table A.3.

[2/] Monthly production data were recorded in the herd books for years prior to 1958.

month in the sample was close to 15 days. Regression coefficients for this first, incomplete, month were estimated separately. The next three complete months were added, while separate regression coefficients were estimated for all following months.

2. Results

Some of the results are given in appendix A, tables A.14 - A.16.

Table A.14 gives results for months which are not influenced by a new pregnancy.[1/] The results arrived at by the regression analysis showed that for all months up to and including the thirteenth for the first lactation and the tenth for later lactations, there was a tendency for monthly production to increase with one tenth of the increase in 305 days production. The regression coefficients were in no case anything near significantly different from 0.1. For later months, the increase in monthly production with increasing 305 days production was less than one tenth, and had a decreasing tendency with increasing distance from the last freshening.

The signs of the constant terms showed that the proportion of total 305 days production which falls in the first part of the lactation will decrease with increasing 305 days production, and the proportion which falls in the later part will increase with increasing production. This is consistent with earlier findings.

[1/] Some of the regression equation estimates which should have been in table A.14 were lost when the author moved from California to Norway. Since all uses which should be made of the estimates were completed at that time, and since this part of the analysis was considered a less important part of the total study, it was decided not to calculate the estimates over again.

If the regression equation estimates were used to derive expected monthly production for levels of 305 days FCM close to the average in the data, the results showed that production from the fifth month and on decreased over time in a way which was very close to linear, and with a rate of decline which was much steeper for the later lactations than for the first. This last result is also consistent with earlier findings, which have demonstrated a much higher persistency of production during the first lactation than during the later. For the later months of the lactation, expected production was actually higher for first-lactation cows than for cows in the second or later lactations.

Tables A.15 and A.16 give results for months which are influenced by a new pregnancy. In these tables, only the regression coefficients and not the constant terms are given. Each of the estimates is based on a rather small sample, in most cases between 15 and 20 observations. There is a lot of variation between estimates which may be explained as due to this fact. However by comparing regression coefficient estimates for a given month in tables A.15 and A.16 with the estimates for the same month in table A.14, we will see that on the average the estimates in tables A.15 and A.16 are not much different from those arrived at for the cases where production is not influenced by a new pregnancy.

On the other hand, the constant terms, which are not presented in tables A.15 and A.16, would show that the total level of production is lower for months when production is influenced by a new pregnancy. It appears, therefore, that after 20 weeks of a pregnancy this has the effect of decreasing monthly production, while it does not change the regression coefficients for monthly production regressed on 305 days production to any considerable extent.

For use in the replacement models, some definite assumptions must be made about the relationship between 305 days FCM and monthly FCM. With the replacement models specified in this study, these assumptions must be specified for calving intervals of 12 months, 15 months, and 18 months. One possibility is to use the estimated relationship for these calving intervals directly, however many of the estimates are based on rather small sample sizes and may therefore possibly be far off the true values. It is also an inconvenience that months in the estimated relationships are based on calendar months and have no fixed time distance from the day of freshening.

It was decided to modify the estimated relations by utilizing some of the findings discussed above. Tables A.17 and A.18 present the average relationships which are assumed in the replacement models. Here, all regression coefficients for months up to the thirteenth or the tenth respectively are set equal to 0.1, and the constant terms are adjusted correspondingly so that the new regression lines will still go through the observed averages. Production for the first four months of a lactation is summed. All regression coefficients for the same month after calving are set equal, and the constant terms in the regression equations at the "tail" of each lactation are modified after comparing observed averages for cows with 12 months calving intervals with those for cows with 11 and 13 months calving intervals, observed averages for cows with 15 months calving intervals with those for cows with 14 and 16 months calving intervals, and observed averages for cows with 18 months calving intervals with those for cows with 17 and 19 months calving intervals. Thus, some element of subjective judgement is used when determining the assumed relationships at the "tails" but the relations are based on more observations than would have been possible if estimated relations should have been used directly.

E. The Probability Distributions of Lengths of Calving Intervals

1. Assumptions about the Nature of the Probability Distributions

For use in the replacement models, we will have to assume given probability distributions of lengths of the calving intervals. The parameters of such distributions were estimated based on data on calving intervals in the two herds.

The probability distribution of the calving interval is evidently very skewed, with the long tail extending to the right. Since we have no a priori reason to assume any particular type of probability distribution, the problem of description and estimation was handled by dividing the total range of possible calving intervals in class intervals and estimating the probability that a given calving interval for a given cow would fall within a given class. The following definition of classes was used:

Interval, days	Class midpoint, days	Class midpoint, months
- 319		
320 - 350	335	11
351 - 380	365	12
381 - 411	396	13
412 - 441	426	14
442 - 472	457	15
473 - 502	487	16
503 - 533	518	17
534 - 563	548	18

Interval, days	Class midpoint, days	Class midpoint, months
564 - 594	579	19
595 - 624	609	20
625 -		

The parameters of the probability distributions may possibly depend on the value of such variables as herd (or managerial practices), breed, lactation number (or age), production level, season of freshening, lengths of previous calving intervals, and time period. In this study, the parameters were assumed to be independent of production level, season of freshening, previous calving intervals, and time period. Previous studies have not shown any clear relationship between production level and breeding performance,[1/] and such studies have shown that the repeatability of length of calving interval is very small.[2/] Season of calving may have a significant effect on breeding performance in some countries or regions, however except for some breeding problems during the warmest summer season, the effect of season on breeding performance is believed to be small under Californian conditions.[3/] The question as to whether the parameters might differ between different time periods in the history of the herds was not examined concisely, however inspection of the data did not suggest any significant differences in distribution of calving intervals between the early and the late part of the recorded history of these herds.

1/ See page 122.

2/ See page 122.

3/ Dr. Robert C. Laben, personal communication.

Data were analyzed separately for the two herds and for each lactation. Data for different lactations were pooled later if differences between lactations appeared to be small.

For the purpose of this study, it was necessary to attempt to estimate the parameters of the probability distributions as these parameters would have been if all cows had been allowed to stay in the herds for a long time (at least 625 days) whether they conceived and freshened within reasonable time or not. We want to use the replacement models to decide whether cows with long calving intervals should be allowed to remain in the herds or not. Therefore, we need to know the probabilities of long calving intervals if no cows are replaced because of breeding trouble.

In the observed data, however, the calving interval was known only for those cows which had been allowed to stay in the herd until next freshening. There are reasons to expect that cows which had been removed before next freshening on the average would have had longer calving intervals than cows which had been allowed to stay. In order to avoid a bias due to this, it was necessary to make some additional assumptions about the hypothetical length of the calving interval for cows which had been removed, and to use an estimation procedure based on these assumptions.

The observed lactations could be classified into the following groups:

1. Cows which had been allowed to stay in the herd until next freshening, and for which the exact length of the calving interval was known.
2. Cows which had been removed before next freshening.
    a. Information in the herd books showed that the cow had been successfully bred before it was removed.

b. Information in the herd books made it possible to say that the calving interval would have been at least of a given length.

c. No information available which made it possible to say anything about the length of the calving interval.

Very few observations fell in group 2 a.[1/] For these observations, it was assumed that the calving interval, if the cow had been allowed to stay in the herd, would have been the number of days from last freshening until successful breeding + 282 days.

A substantial number of observations fell in group 2 b. Information was available in the herd books about all breeding attempts. Thus, if a cow had been bred last time for example 100 days after last freshening and had been culled some time thereafter, it was assumed that her calving interval, if she had been kept in the herd, would have been at least 100 + 282 = 382 days, and possibly longer. Likewise, if a cow had been culled before any breeding attempts had been made but there were notes in the herd books indicating that breeding trouble was one of the reasons for removal, it was assumed that the cow, if kept in the herd, would have had a calving interval at least equal to the number of days from freshening to removal + 282 days.

For all observations in group 2 b, it was assumed that the cows, if they had been kept in the herd, would have had a distribution of calving intervals equal to the distribution of calving intervals for those cows which were not culled and which had a calving interval at least of the same length. It will be explained later how this assumption was utilized in the estimation procedure.

[1/] 2 lactations in the ME-herd and 3 lactations in the MA-herd.

The lactations in group 2 c belonged to cows which had been removed during the given lactation before any breeding attempts had been made, and for which no notes were available relating the removal to breeding problems. Probably, most of these cows had not been bred because the dairy producer had already decided to replace them for some other reason. Thus, nothing could be said about the hypothetical length of the calving interval for these cows. It was assumed that they would have had the same distribution of calving intervals as the other cows if they had been kept and had been bred like the other cows. These observations were removed from the sample since they could add no useful information and exclusion from the sample should not bias the estimates.

In the ME-herd, records were missing for some of the culled cows. These cows with missing records were added to the different subgroups of culled cows with records in proportion to the number of culled cows falling in various subgroups.

2. Results

Table A.19 in appendix A presents the observed frequency distributions of calving intervals for lactations for which the exact length was known. Lactations in group 2 a are included in this table.

Table A.20 presents the observed frequency distributions of hypothetical calving intervals for lactations which had not been followed by a new freshening. These frequencies must be interpreted in another way than the frequencies in table A.19. For example, for the first lactation in the ME-herd, there are 8 lactations about which it can be said that the calving interval would have been at least 351 days but about which it can not be said that the calving interval would have been at least 381 days, even if it is possible that the lactation would have been that long or longer.

Table A.20 gives also the number of lactations for which information was missing, and the number of lactations for which evidently no attempts had been made to breed the cow again.

The lactations in tables A.19 and A.20 may be thought of as representing samples from conceptually infinite populations of lactations with given probability distributions of calving intervals. The different populations are defined by herd and by lactation number.[1/] The observation of the relevant variable in the samples has been disturbed by the fact that some cows have been removed from the herds before the calving interval could be observed, so that information on the relevant variable is incomplete for some of the individuals in the sample.

We want to examine whether it can be said that the probability distributions of calving intervals differ between the populations. A test of the null-hypothesis by use of the chi-square statistic is difficult, however, because of the incomplete information available on some of the individuals.

Even without formal tests, the observed frequencies suggest a considerable difference in probability distributions between the two herds. The differences in observed relative frequencies are so large and so consistent for all lactation numbers that formal tests for significance should hardly be necessary.[2/] The greatest differences appear to be

---

[1/] In the results presented in tables A.19 and A.20, observations on the three highest lactation numbers have been added. All lactations were treated separately in the original counting of observations.

[2/] Based on the observed frequencies for the first lactation and for cows with known calving intervals only, a chi-square test for independence was carried out. The hypothesis of no interaction herd - calving interval resulted in a chi-square statistic of 51.31. The corresponding $\chi_{.99}(9) = 21.67$.

with respect to probabilities of calving intervals shorter than 12 months, and may reflect a difference between herds as to how soon cows are bred after freshening. There appear to be a difference also with respect to probabilities of very long calving intervals. While this difference may be much more important from an economic point of view, it can not be explained as due to the same cause.

The results are not equally clear if we attempt to make a comparison between different lactation numbers within one and the same herd. Some tests were made based on observations on lactations with known calving intervals only. When cows with calving intervals exceeding 563 days were placed in one group and all six lactations were treated separately, the hypothesis of no interaction lactation number - calving interval resulted in a chi-square statistic of 44.55 for the MA-herd.[1/] When also cows with calving intervals shorter than 351 days were placed in one group and observations for the fourth, the fifth, and the sixth lactation were pooled, the same hypothesis of no interaction resulted in values of the chi-square statistic of 32.79 and 33.36 for the ME-herd and the MA-herd respectively.[2/]

Thus, the hypothesis of the same probability distribution of calving intervals for different lactation numbers is not rejected at usual levels of significance, however such tests may very well be thought of as invalid because a number of observations in the sample are omitted. Higher percentages of observations are omitted from the later lactations than from the first. If the true probabilities of long calving intervals are

---

1/ $\chi_{.95}(45) = 61.65$.

2/ $\chi_{.95}(24) = 36.42$.

higher for the later lactation numbers but at the same time higher percentages of the cows which would have gotten long calving intervals have been culled, the result may be data which conceal an existing underlying difference in probability distributions.

An attempt was made to arrive at unbiased probability estimates separately for each lactation by the method described below. Then the probability estimates for different lactations were compared. As compared with the relative frequencies we can arrive at by observing lactations with known calving intervals only, the correction for culling implicit in the estimation procedure actually decreased the differences between lactations in the MA-herd, while it increased the differences between lactations in the ME-herd. For the ME-herd, the probability estimates for the first two lactations were close, the probability estimates for the four last lactations were also close, while there appeared to be a difference between the two groups of lactation numbers.

For use in the replacement models, it was decided to assume the same probability distribution for all lactation numbers for the MA-herd. Parameters in this distribution were estimated based on pooled data for all six lactation numbers. For the ME-herd, it was decided to assume one probability distribution for the first two lactation numbers, and another probability distribution for the next four lactation numbers. Corresponding probability estimates were derived from data pooled for the first two lactation numbers and for the next four lactation numbers. Resulting estimates for both herds are presented in appendix A, table A.21.

For use in the replacement models, it was necessary to make further simplifications. As explained earlier, it was assumed in these models that the calving interval would be either 12 months, 15 months, 18 months, or longer than 18 months, and that in the last case the cow would be replaced regardless of other criteria. For use in the replacement models, probabilities of 12 months calving intervals were arrived at from table A.26 by adding probabilities of calving intervals up to 411 days, probabilities of 15 months calving intervals by adding probabilities of calving intervals from 412 days up to 502 days, probabilities of 18 months calving intervals by adding probabilities of calving intervals from 503 days up to 594 days, and probabilities of longer calving intervals by adding probabilities of all longer calving intervals. A calving interval of "15 months" in the replacement model, therefore, really represents calving intervals in the range between 13 1/2 months and 16 1/2 months, and correspondingly for the other calving intervals assumed in the replacement models.

3. The Estimation Method

The estimation procedure will be explained by use of a numerical example from the MA-herd, the case where observations from all six lactations are pooled. If all cows with calving intervals of 412 days and more are placed in the same group, the observed cows with known calving intervals have the following distribution:

Cälving interval	≤ 350 days	243 lactations
– " –	351 - 380 "	519 "
– " –	381 - 411 "	326 "
– " –	≥ 412 "	681 "
	Sum	1,769 lactations

The culled cows about which it can be said that the calving interval would have been at least of a given length have the following distribution:

Calving interval at least	351	days	59	lactations
- " -	381	"	61	"
- " -	412	"	348	"
	Sum		468	lactations

Denote the population probabilities of a cow falling in the four different classes of calving intervals as p_1, p_2, p_3, and p_4 respectively. The total number of observations in the sample is 1769 + 468 = 2,237. p_1 can be estimated directly as:

$$\hat{p}_1 = \frac{243}{2237} = 0.108628$$

According to the assumption stated above, the 59 cows with calving intervals at least 351 days have a probability distribution of calving intervals equal to the probability distribution for cows with known calving intervals and with calving intervals at least 351 days. Thus, the expected number of cows falling in the second class is $\frac{59\ p_2}{1 - p_1}$, the expected number falling in the third class is $\frac{59\ p_3}{1 - p_1}$, and so on.

We can estimate p_2 from the equation:

$$\hat{p}_2 = \frac{519 + \frac{59\ \hat{p}_2}{1 - \hat{p}_1}}{2237}$$

Since $\hat{p}_1$ already is known, we can solve for $\hat{p}_2$ and get:

$$\hat{p}_2 = 0.239081$$

Of the 61 cows with calving interval at least 381 days, the expected number falling in the third class is $\frac{61\ p_3}{1-p_1-p_2}$, and the expected number falling in the fourth class is $\frac{61\ p_4}{1-p_1-p_2}$.

We can estimate p_3 from the equation:

$$\hat{p}_3 = \frac{326 + \frac{59\hat{p}_3}{1-\hat{p}_1} + \frac{61\hat{p}_3}{1-\hat{p}_1-\hat{p}_2}}{2237}$$

$\hat{p}_2$ and $\hat{p}_2$ are known, and we get

$$\hat{p}_3 = 0.156935$$

The procedure can be extended to all classes of longer calving intervals. The sum of the probabilities must sum to 1, and this provides a numerical check of the calculations.

As compared with estimates based on cows with known calving intervals only, the estimation procedure explained here gives higher estimates for the probabilities of long calving intervals. It takes into account that replaced cows, it they had been kept in the herd, would have had a frequency distribution of calving intervals with more lactations falling in the long calving interval groups than cows which have been kept in the herd. This is quite certainly a realistic assumption, however since some of the replaced cows may have had more serious reproductive deficiencies, it is possible that the hypothetical frequency distribution of calving intervals for the replaced cows is even more slanted towards long calving intervals than has been assumed here. At least it can be argued that the estimation method used here gives estimates which are likely to be closer to the true population values than estimates only based on observations of known calving intervals.

F. Probabilities of Involuntary Replacement

1. Classification of Removal Reasons

For all cows which had been removed from the herds, the reason for removal as given by the dairy producer had been recorded in the herd books. The reasons were later classified into one of 20 alternative classes and a code number referring to the class was punched on the card for the last lactation of each cow. The classification did not specifically distinguish between "voluntary" and "involuntary" removals. For the purpose of this study, therefore, each class again was classified into one of two alternative groups. The classes where sickness seemed to be the most important removal reason were placed in the group of "involuntary removals", even if the farmer actually may have had a choice when he decided to remove the given cow. One the other hand, since breeding trouble will be used as a criterion in the replacement models, all classes where breeding trouble alone or in combination with low production seemed to be the most important removal reason were placed in the group of "voluntary removals".

The following classes were grouped as "voluntary removals":

Code	Characterization
1	Low production (only notation)
2	Low production and breeding trouble
3	Low production and mastitis
4	Low production and other reasons, lost quarter
5	Breeding trouble and other reasons -- sterility; not with calf; open; Nymphomania; mummified fetus
8	Dairy purposes - sold
9	Old age (only notation)

Code	Characterization
10	Old age and breeding trouble
11	Old age and other reasons -- age; breeding trouble; low production; age and low production; crippled
19	Miscellaneous: missing not in herd; sold; culled - cause unknown; pink nose; mean animal; holds milk; unthrifty; lost calf; family cow; loaned out

The following classes were grouped as "involuntary removals":

Code	Characterization
6	Mastitis - Gangrenous
7	Mastitis - chronic or acute other than 6 or 3 - and other reasons (garget)
12	Foreign body - wire; hardware disease; traumatic pericarditis
13	Accidents - sold: broken bones; injured in fall; hump-backed; crippled, lame; injured teat; etc.
14	Accidents - dead: drowned; electrocuted; strangled; bled; overeating; crippled; prolapse uterus; Dystocia; died in calving; calving complications; bloated
15	Abortion - Bang's reactor; Brucellosis
16	Abortion - only notation; abortion due to injury
17	Infection or abcesses, diseases except for mastitis or TB
18	TB reactor
20	Poor type; pendulous udder; broken down udder; udder trouble; calved with blind quarter

There may be some doubts as to where some of the classes should be placed, and some of the classes may contain both cases which should have been classified as "voluntary removals" and cases which should have been classified as "involuntary removals".

2. Analysis of Dependences

There are reasons to expect that the probabilities of involuntary removal differ between herds. Within a herd, there are reasons to expect that both age and level of production may influence the probability of involuntary removal. An examination of the data revealed that the relative frequencies of involuntary removals also had varied considerably between time periods over the history of the herds.

For each of the two herds, all lactations on which information was available from the IBM-cards were classified according to the following criteria:

1. Lactation number (seven groups)
2. Level of production (three groups)
3. Time period in five-year groups (seven groups)
4. Whether the cow was involuntary replaced before next freshening or not (two groups)

Since most cows which had been removed during one lactation did not have a complete 305 days record for that lactation, it would have been difficult to classify the cows according to the level of production during the lactation of removal. Instead, data were classified by level of production during the previous lactation into three production groups: a low, a medium, and a high production group. The groups were defined in such a way that approximately one third of the cows in a herd fell in each group.

The result of the classification was a four-way contingency table for each of the two herds. A difference in probability of involuntary removal between herds was apparent already by inspection of the tables. Further examination of the results could concentrate on the question as to whether the probability of involuntary removal depends on level of production, on lactation number, on time period, or on any interaction between two of these factors or on an interaction between all three.

A summary of the contingency tables, classified by one of these three factors at a time, is given in appendix A, tables A.22 - A.24.

Table A.22 shows that there appear to be no consistent pattern of relationship between level of production as defined here and probability of involuntary removal. The hypothesis "the probability of involuntary removal is independent of production level" gave as a result values of the chi-square statistic of 4.36 and 0.47 for the ME-herd and the MA-herd respectively.[1/] The hypothesis can be accepted at usual levels of significance.

This is a result of particular interest, and we should examine whether it can be due to either the method of analysis or the specific nature of the data used. Even if a relationship between level of production and probability of involuntary removal really exists, it could have been concealed by an interaction between production level and either time period or lactation number or both. We would get a more reliable test by examining the four-way contingency table and testing the hypothesis: "The probability of involuntary removal depends on time period x lactation number but is independent of production level". However in many of the cells in this complete table the number of observations was too small to satisfy the requirements for use of the chi-square test.

As a substitute for the complete test, a number of tests were performed based on three-way classification and addition of some of the adjacent classes in order to give enough observations in each cell. Thus, tests of the following hypotheses were performed: (1) "The probability of involuntary removal depends on time period but is independent

---

1/ $\chi_{.95}(2) = 5.99$.

of level of production", (2) "The probability of involuntary removal depends on lactation number but is independent of level of production". All results were consistent with the hypotheses.

Even if these results support the hypothesis of independence between level of production and probability of involuntary removal, we should be aware that the problem has more aspects. First, the results may be due to the way "level of production" is defined. If level of production had been defined either according to production level during the current lactation, or according to a linear combination of lactation records up to and including the present, it is possible that the results would have been different. Second and more important, there is a possibility that the apparent independence found here is the result of two opposite effects which have cancelled each other out: Higher levels of production may have resulted in more sickness but the dairy producer may have been more reluctant to remove the highest producing cows.

Even if there may be some doubts about the results arrived at, it was assumed for use in the replacement models that the probability of involuntary removal really is independent of the level of production.

Table A.23 shows that the actual proportion of cows "involuntary removed" has varied considerably between time periods. The hypothesis "The probability of involuntary removal is independent of time period" resulted in values of the chi-square statistic of 27.01 and 18.05 for the ME-herd and the MA-herd respectively.[1/] More detailed tests based on three-way classifications gave similar results.

The difference between time periods which is found here may very well be real. It could, however, completely or partly be due to a difference

---

1/ $\chi_{.95}(6) = 12.59$.

in culling policies between time periods, so that the dairy producers have culled more intensively for sickness during some time periods. It could also be due to a difference between time periods in description of identical removal causes.

In spite of these objections, there are good reasons to accept the hypothesis that the probability of involuntary replacement may vary between time periods. With regard to the replacement models, this is an unpleasant conclusion. First, such changes are impossible to predict for the future. Second, optimization under an infinite planning horizon requires the assumption of probabilities which are constant over time.

Table A.24 shows that the actual proportion of cows "involuntary removed" tends to increase with increasing lactation number. This effect is particularly strong in the MA-herd. The hypothesis "The probability of involuntary removal is independent of lactation number" resulted in values of the chi-square statistic of 9.52 and 75.51 for the ME-herd and the MA-herd respectively.[1/] The hypothesis is not rejected for the ME-herd, however a more detailed test based on a three-way classification according to lactation number, time period, and removal type gave significant rejection of the hypothesis "The probability of involuntary removal depends on time period but is independent of lactation number".[2/]

3. Estimation of Probabilities of Involuntary Removal for Use in the Replacement Models

With support in the results referred to above, it was assumed that the probabilities of involuntary removal depend on time period and on

---

1/ $\chi_{.95}(6) = 12.59$.

2/ $\chi = 34.20$; $\chi_{.95}(18) = 28.87$.

lactation number but are independent of production level.

Table A.24 suggests that the probabilities of involuntary removal increase with increasing lactation number, however there are irregularities in the table which may be due to the low number of observations in each cell. Further analysis was based on the assumption that the probabilities of involuntary removal increase linearily with increasing lactation number. Thus, the following model was assumed:

$$p_{ij} = \alpha_j + \beta L_i \qquad (6.3)$$

where

i ($i = 1,....,7$) denotes lactation number

j ($j = 1,......$) denotes time period in five-year periods. $j = 1$ represents the time period 1931-35

p_{ij} is the probability during the j'th time period that a cow which has initiated the i'th lactation will be involuntary removed before next freshening

L_i is lactation number

α_j and β are parameters

The parameters α_j and β were estimated by analysis of covariance. The "dependent variable" was set equal to 1 for each observation where the cow was involuntary removed during the given lactation, and was set equal to 0 otherwise.[1/] The results are presented in appendix A, table A.25.

We see from the table that the estimated probabilities for the MA-herd start out lower but increase more rapidly with increasing lactation

---

[1/] To use least squares method for an estimation problem of this kind (with a dichotomous dependent variable) appears to be uncommon but is sometimes used. An attempt was made to develop maximum likelyhood estimators, however such estimators turned out to be very complicated and were non-linear functions of the observations.

number than they do in the ME-herd. This is consistent with the results in table A.24. The "time period" parameters show a similar kind of variation as is shown by the data in table A.23.

For use in the replacement models, the "time period parameter" was set equal to the unweighted average of the estimates for the last three time period. Thus, the assumed equation describing the probability of involuntary removal for the ME-herd is:

$$p_j = 0.0660 + 0.0067L_i \qquad (6.4)$$

For the MA-herd, it is:

$$p_i = 0.0247 + 0.0266L_i \qquad (6.5)$$

So far, neither the distribution of involuntary removals over the calving interval nor the effects of variations in calving interval on the probabilities of involuntary removal have been considered.

A separate study of time length from last freshening to the time of an "involuntary removal" suggested that involuntary removals can be assumed to be equally distributed over the whole calving interval. Thus, it seems justified to assume that for each month of the calving interval there is some constant probability of involuntary removal. This assumption was made in the replacement models.

The monthly probabilities were derived by dividing the total probabilities as derived from (6.4) and (6.5) with an estimated average length of the calving interval.[1/] The results, which were used in the replacement models, are given in appendix A, table A.26.

[1/] These averages were set equal to 12.0 months for the ME-herd and to 12.5 months for the MA-herd. These are the average lengths of time a cow has spent per calving interval, when calving intervals are counted as time from freshening to freshening for cows which have not been removed, and as time from freshening to removal for cows which have been removed.

We have assumed that for each herd and lactation number, there is some stated and constant probability that a cow which has freshened will be involuntarily removed during the first month of that calving interval, during the second month of that calving interval, and so on until the cow freshens the next time. Therefore, the probability that a cow will be involuntarily removed during a given calving interval is assumed to be proportional to the length of the calving interval. When the cow freshens a new time, the monthly probability of involuntary removal is assumed to shift to a new level.

Of course this is a simplified description of what really happens. The assumption is convenient because estimates of the probabilities of involuntary removals had to be based on the number of cows freshening the first time, the second time, etc. It does not seem likely that any other and possibly more realistic assumption about the distribution of involuntary removals over time would have changed the results of the replacement study to any noticeable extent.

In the replacement models, stages are defined to run from seven months after one freshening to seven months after next freshening. A stage is defined to end, however, at the day a replacement animal has been purchased after a cow has been involuntarily removed. Two examples will illustrate how the data in table A.26 were used to derive the probabilities that a cow will survive during a given stage.

Take the ME-herd and the stage running from the second to the third lactation. According to the stated assumptions, the probability that a cow which has freshened the second time will have survived until seven months after freshening is

$$1.000000 - 7 \times 0.006622 = 0.953646$$

Since a probability of 0.006622 is based on the number of cows starting on the second lactation, the monthly probabilities of removal <u>for cows starting on the new stage</u> are:

$$\frac{0.006622}{0.953646} = 0.006944$$

First, take the case where the calving interval between the second and the third freshening is assumed known at the beginning of the stage and is 15 months. The third freshening will take place eight months after the beginning of the stage, so the probability that the cow will survive until freshening is:

$$1.000000 - 8 \times 0.006944 = 0.944448$$

The monthly probability of involuntary removal after the third freshening is therefore:

$$0.944448 \times 0.007183 = 0.006784$$

The probability that a cow which started on the given stage will survive until seven months after next freshening is

$$0.944448 - 7 \times 0.006784 = \underline{0.896960}$$

This is the probability of survival for the given stage. For later use, we need to know not only the probability of survival but also the probabilities for involuntary removal for each separate month of the stage. These probabilities are given above as 0.006944 for the first eigth months of the stage and 0.006784 for the last seven months of the stage.

Second, take the case where the cow has not yet conceived on the first day of the stage. The decision may be: "Keep the cow three more months, replace it then if it is still not known to be pregnant, but keep it otherwise". We will derive the probability that the cow will survive until seven months after next freshening.

In the ME-herd, the probabilities of 12 months, 15 months, 18 months, and longer than 18 months calving intervals for the second lactation are assumed to be 0.6114, 0.2328, 0.0873, and 0.0685 respectively.[1/] Thus, the probability that a cow which is not pregnant on the first day of the stage will conceive within the next three months is

$$\frac{0.0873}{0.0873 + 0.0685} = 0.560334$$

The probability that the cow will survive during the first three months of the stage is

$$1.000000 - 3 \times 0.006944 = 0.979168$$

The probability that the cow will survive during the first three months and conceive within that time is

$$0.979168 \times 0.560334 = 0.548661$$

The monthly probability of involuntary removal after the first three months and until next freshening is

$$0.006944 \times 0.560334 = 0.003891$$

The probability of survival until next freshening is therefore

$$0.548661 - 8 \times 0.003891 = 0.517533$$

The monthly probability or involuntary removal after freshening is

$$0.517533 \times 0.007183 = 0.003717$$

Finally, the probability of survival until seven months after freshening is

$$0.517533 - 7 \times 0.003717 = \underline{0.491514}$$

1/ See appendix A, table A.21.

VII. NUMERICAL REPLACEMENT MODELS

A. Purpose

The purposes of constructing numerical replacement models and of deriving optimal replacement policies for these models as part of this study have been: (1) to demonstrate the feasibility of the kind of replacement models which have been described above; (2) to gain experience with practical problems which have to be faced when constructing and seeking solutions to this kind of models; (3) to study the sensitivity of optimal replacement policies to variations in prices and in other parameter values; (4) to examine the economic loss to dairy producers who follow replacement policies which deviate more or less from the optimal; (5) if possible and depending on the answers to the questions above, derive policies which can be used as basis for recommendations to groups of dairy producers.

Since at the present too little is known both about variations in parameter values between herds in general and about the sensitivity of optimal replacement policies to variations in such parameter values, only very limited emphasis should be put on the last point.

B. Some Assumptions about Parameter Values

In a replacement model which is going to serve as a basis for practical decisions in a real herd or a real group of herds, we should use such parameter values as we expect will be valid for the given herds in the future. The basis for estimation of such parameters, however, are observed data from the past. We must decide whether we want to take these parameter estimates directly as they result from the estimation procedures, or change them according to expected changes in true parameter values.

With respect to probabilities of involuntary removals, probability distributions of calving intervals, and transition probabilities for the "production variables", it seems most reasonable to assume that the same values will be valid in the future as has been estimated from observed data. In this study, these parameters have been taken directly as they resulted from the estimation procedures described above.

On the other hand, it does not seem reasonable to use observed herd averages for past years directly as assumptions of production levels for the future. As table A.1 shows, there has been a marked tendency to increases in "age-corrected herd averages" over time. In most cases, it seems reasonable to expect that the future level of production will be at least as high in the years ahead as it has been over the very last years.[1/]

The actual choice of such assumptions, however, should depend on the purpose of the specific numerical replacement model. If the purpose is to serve as a basis for practical decision-making, assumptions about age-corrected herd averages in the future should be made after careful considerations, in which both the historical records for the respective herds and an evaluation of the present quality of herd management should play a part. On the other hand, if the purpose is to compare optimal replacement policies derived through such models with replacement rules

---

[1/] In the type of model defined in this study, we are forced to assume a constant "age-corrected herd average" for the future. A revised model which would enable us to take into account continued increases in herd averages as a result of genetic change is suggested on page 133.

which have actually been followed in given herds, then it seems reasonable to use an "age-corrected herd average" which is close to the observed average over the years under examination.

In this study, however, other purposes listed above were seen as more important. Since it seemed particularly desirable to examine whether differences in parameter values between herds would result in significantly different replacement policies, it was decided to make assumptions about future age-corrected herd averages which would contrast "more efficient" dairy herds with "less efficient" herds.

Historically, the ME-herd has been "more efficient" than the MA-herd in the sense that frequencies of involuntary removals as well as frequencies of long calving intervals have been smaller. On the other hand, the general level of production has been lower in the ME-herd than in the MA-herd during most years. It was decided to use other parameter estimates from the ME-herd as the basis for a conceptual "more efficient herd" and to use other parameter estimates from the MA-herd as the basis for a conceptual "less efficient herd", but to make assumptions about age-corrected herd averages which would make the ME-herd more efficient also with respect to the general level of production. Accordingly, the assumed age-corrected herd average was set to 12,000 pounds 305 days FCM for the ME-herd, and to 11,000 pounds 305 days FCM for the MA-herd.[1/] For convenience, the two conceptual herds for which replacement models

---

[1/] This assumption is made for experimental purposes and does not express any judgement by the author about possible future differences between the two real herds. It is seen from table A.1 that records for the last three years show a similar difference in favor of the ME-herd, however the MA-herd has had higher herd averages over most years in the past and may very well be ahead again in the future.

have been developed will still be referred to as the "ME-herd" and the "MA-herd". Since one of the assumptions is not based on actual judgement of the most likely outcome of the respectively real herds, however, the resulting "optimal policies" does not represent actual recommendations for the two real herds.

C. Parameters Obtained from Other Sources

Some parameters required for the numerical replacement models were taken from sources other than the actual herd records.

1. Assumed Milk - Feed Relationship

No data on feed consumption were available from the herds which furnished other data for the study. Some theoretical problems of determining the relationship between milk output and feed input have been discussed above.[1/] It was assumed that the dairy producers follow a given "feeding system" when allocating feed to his cows, and that the "system" used in this case implies a feeding of cows according to Morrison's feeding standards.[2/] It was assumed that only two feed types are used: legume hay of good quality, and "grain" or a concentrate mix. Hay was assumed given according to appetite, while grain was assumed rationed according to the level of milk production.

Appendix A, table A.27 gives the assumed relationship between monthly production of FCM and monthly consumption of hay and grain. As is

---

[1/] See page 117.

[2/] Frank B. Morrison, Feeds and Feeding (22nd ed.; Ithaca, New York: The Morrison Publishing Co., 1956), p. 1134.

seen from the table, hay consumption is assumed to be independent of level of production. Grain consumption is assumed to vary with milk production when milk production is beyond a certain level, and to be independent of milk production below this limit.

2. The Probability that a Calving Results in the Birth of a Live Calf

Income from the sale of calves is one of the sources of income from dairy production, and may vary somewhat with different replacement policies. The probability that a calving results in the birth of a live calf was assumed to be 0.969. This figure is taken from a report by Frick and Henry.[1/]

3. Prices and Interest Rates

Appendix A, table A.28 gives the different price sets for which optimal solutions were derived. Prices of milk and feedstuff were selected based on a report by Shultis, Forker, and Appleman.[2/] Livestock prices and prices for removed animals were based on market reports in the Western Dairy Journal over the last years.[3/] The interest rate was set equal to 6 per cent per year for all alternatives. Continuous discounting was used.[4/]

[1/] G. E. Frick and W. F. Henry, Production Efficiency on New England Dairy Farms. V. Adjustments in Obtaining Dairy Herd Replacements, University of New Hampshire, Agricultural Experiment Station (Durham, Aug., 1956).

[2/] Arthur Shultis, Olan D. Forker, and Robert D. Appleman, California Dairy Farm Management, University of California, California Agric. Exsp. Sta. Circular 417 (revised; Berjeley, 1963).

[3/] Western Dairy Journal, Febr., 1961 - Sept., 1964.

[4/] See page .

It is not claimed that the price sets used represent "average" price situations for the respective areas nor that they represent expected prices for the future. Optimal solutions based on these price sets can first of all be regarded as illustrations of the results which can be reached by use of such models. Different price situations were used in order to examine to what degree moderate changes in prices will change the optimal replacement policy.

Price alternatives 1, 2, and 3 in table A.28 are intended to represent fairly "typical" Central Valley price conditions, however in alternative 3 the acquisition cost of a heifer is reduced by $ 50.- as compared with what is assumed to be a typical price. The milk price in alternative 1 and 3 is a fairly typical "blend price", while the corresponding price in alternative 2 is based on the price of manufacturing milk. Alternative 4 is intended to represent a "Los Angelos area" price situation when milk price is set equal to the blend price. Grain price is set lower and hay price higher than in the Central Valley.

To determine realistic price assumptions for replaced animals is difficult without access to detailed and extensive records from dairy herds. Such records were not available for this study, so the prices used were based on average cow weights and market prices for dairy beef. Normally, a cow increases in weight over the first years in milk production, however the percentage of replaced animals which receives prices in a quality class better than the lowest one ("canner, cutter") will decrease with increasing age. It is assumed here that the combined effect of higher weight and lower average price is an expected price per head which stays constant over the first years and then starts to decline.

Selection of relevant milk prices for use in the replacement models is complicated by the system of price regulations in operation in California. Milk used for fluid milk, cream, and related fluid products receives a "Class I" price, while the other milk sold receives the much lower manufacturing milk prices ("Class II" and "Class III" milk prices) which depend on the actual uses and on market prices of butter, cheese, etc. Market milk producers hold contracts with dairy firms, under which they are obliged to deliver certain quantities of market milk per month and are entitled to class I price for this quantity. The quantity of milk delivered beyond this contractual quantity is entitled only to the manufacturing milk price, however part of the surplus may receive class I price if it is actually used for such purposes. The "blend price" which is the weighted average price received by a dairy producer depends both on his contract and on the quantities of milk delivered above what is determined in his contract.

If a dairy producer holds a given milk delivery contract and the total quantity of milk produced by the herd is so high that his contract is fulfilled anyhow, then changes in milk production due to changes in replacement policy can be evaluated at manufacturing milk prices. Since this study is concerned about replacement policies under the assumption of a constant herd size, it can be argued that in most cases the manufacturing milk price is the relevant milk price for use in the replacement models. With this line of reasoning, the difference between manufacturing milk price and class I price is received for a fixed quantity of milk and results in something which from our point of view can be regarded as a fixed income for the firm.

If we consider the possibilities of future changes in milk delivery contracts, however, the situation may very well be different. Little is known about the way contractual quantities are determined, however there may be reasons to believe that when new contracts are negotiated, the historical surplus of milk from the herd is considered, so that a producer who has produced much more milk than his contractual quantity will have a greater change of getting his contract adjusted upwards than another producer who has only produced slightly more than his contract. To increase milk production, for example by more intensive culling, might thus increase the contractual milk quantity in the future. One possible hypothesis is that in the long run, a dairy producer can expect to receive class I price for a given percentage of his total milk production, however in such a way that there is a time lag between a change in milk quantity produced and the adjustment in the contractual quantity of class I milk. If this is true, a milk price close to the blend price would be the most relevant price.

D. Derivation of the Elements $\sum_{m=1}^{M} p_{ijm}^{k}\beta_m$

It is shown in chapter III part D that the Markovian dynamic programming model can be modified to handle the case where the stage length is a stochastic variable. We have assumed that when the process goes from state i to state j, the stage length may take on M different values, and that for each different value of m $(m = 1,\ldots,M)$, there exists a probability p_{ijm}^{k} which is the probability that the process will go from the i'th state to the j'th with the time length of the stage being that denoted by m if the k'th action is taken.

To arrive at numerical solutions, we need the numerical values of the elements $\sum_{m=1}^{M} p^k_{ijm}\beta_m$, where β_m is the discount factor for discounting an amount from the first day of one stage to the first day of the preceding stage when the stage length is m. The numerical values of the factors β_m are given by the interest rate and the stage length m, and the derivation of the actual values of β_m raises no problems. We will be concerned here with the derivation of the elements p^k_{ijm} and with the assumptions underlying this determination. When these elements are known, the derivation of the sum of products of discount factors and probabilities is a matter of routine.

The basic data for the derivation of these probabilities are given in appendix A by tables A.12, A.13, A.21, and A.26.

The replacement models defined in this study contain 106 different states. State 1 is defined as the situation after a cow has been removed, voluntarily or involuntarily, and immediately before the purchase of a heifer. The other 105 states are charactericed by age measured in lactations (5 different values), a production variable (7 different values), and a variable for length of the current calving interval (3 different values).

The derivation of transition probabilities was based on certain assumptions about independence:

1. The probability of involuntary replacement, which can be derived from table A.26, depends on age but is independent of level of production. It depends on the length of the calving interval in the sense that it is defined as a given probability for each month of the lactation, so that the total probability of involuntary replacement increases with increasing calving interval.

2. The probability of a given transition with respect to production variables, given by tables A.12 and A.13, depends on age but is independent of the length of the calving interval.

3. The probability of a given calving interval, which can be derived from table A.21, may depend on age but is independent of the previous lengths of the calving intervals and is independent both of the production variable and of transition with respect to production variables.

Further, it was assumed that if the decision is to replace, the cow will be sold and replaced with a heifer immediately. Since the process is defined to consist of one cow and all successive replacements, this new heifer will be part of the same process. In all cases where a new heifer enters the herd, a time lag of one month was assumed from the day the heifer enters the herd and until it freshens the first time.[1/] If the decision is not to replace, the cow may still be "involuntarily removed" some time before seven months after next freshening, and in this case, it was assumed that one month will expire before a replacement heifer is included in the herd.[2/]

Based on the assumptions above and if the decision is to keep the cow, we can derive the transition probability for a given transition as a product of three different probabilities: (1) the probability of

---

[1/] A dairy producer will usually not be able to find a replacement animal which is exactly "timed" to freshen the day he sells an old animal. Feed costs for the month before freshening are not counted, since they are considered part of the heifer price.

[2/] An extra lag is assumed in the case of involuntary removal, since in such cases the dairy producer has less opportunity to prepare the replacement in advance.

survival, (2) the probability of a given transition with respect to production variables, (3) the probability of a given calving interval during the lactation when the stage expires.

For example, we have derived the probability of survival for a cow in the ME-herd with 15 months calving interval for the second lactation and the stage running from the second to the third lactation as 0.896960.[1/] The probability that the third calving interval will be 12 months long is 0.5832, and the probability of transition from the highest production class in the second lactation to the next highest production class in the third lactation is 0.2246. The probability of the given transition is

$$0.896960 \times 0.5832 \times 0.2246 = 0.11749$$

When the cow survives, the stage length has only one value, in this case 15 months. The discount factor for a stage of 15 months duration is 0.927744, so the combined element is $0.11749 \times 0.927744 = 0.10900$.

If the decision is "keep" but the cow is involuntarily removed at some time before seven months after next freshening, the process will go to state 1 but the length of the stage before this new state is reached has a stochastic distribution. Probabilities p_{i1m} of involuntary removal for each month can be derived based on the data in table A.26. We have derived these probabilities for the same example as above to be 0.006944 for the first 8 months of the stage and 0.006784 for the last 7 months of the stage.[2/] It was assumed in the numerical replacement models that involuntary removal will always take place in the middle of each month. Since a time lag of one month is assumed from the removal

---

1/ See page 181.

2/ See page 181.

of one cow until a heifer is purchased, the discount factors are those for 1 1/2 months, 2 1/2 months, and so on. In the example, the actual figures are:

$$
\begin{array}{rl}
 & \beta_m \qquad\quad p_{i1m} \\
 & 0.992528 \times 0.006944 \\
+ & 0.987578 \times 0.006944 \\
 & \cdot \\
 & \cdot \\
+ & \underline{0.925427 \times 0.006784} \\
= & 0.0907
\end{array}
$$

In the cases where the calving interval exceeds 15 months and therefore is unknown at the first day of the stage, the alternative decisions are "replace at once" and "wait three more months, keep if the cow has conceived within that time but replace otherwise". If the last alternative is chosen, there is a quite high probability that the cow will be replaced after three months, the stage will be of length three months and the new state will be state 1. The method for computation of the combined element is the same as demonstrated above.

In any case where the decision is "replace", the process will go to state 1 with probability 1.000000. The discount factor is also 1 since the stage length is 0, so the combined element is 1.

E. Derivation of Expected Immediate Economic Returns

1. The Procedure

The Markovian dynamic programming model outlined in chapter III assumed that for all states (denoted i) and each possible action (denoted k) for every state, we know the expected immediate economic return q_i^k. Since actually each stage is of a certain length and incomes and outlays are distributed over time within the stage, expected incomes and outlays were discounted within the stage to the first day of the stage.

Prices were assumed given. To arrive at expected incomes and outlays, it was necessary to derive expected physical quantities of products and of inputs. In order to facilitate the derivation of alternative elements q_i^k for alternative price sets, the expected physical quantities were discounted to the first day of each stage. The result was a matrix of discounted expected physical quantities, which was multiplied by a price vector to give the vector of expected immediate economic returns.

2. Derivation of Discounted Expected Physical Quantities

a. Milk

Since stages are defined to go from seven months after one calving to seven months after the next calving, milk production during one stage consists of two parts: one part produced during the latter part of the present lactation, and one part produced during the first seven months of the next lactation.

Derivation of discounted expected milk production for a given state under the decision "keep" went through four steps:

1. Derivation of expected 305 days FCM corrected to 365 days calving interval, for the present and the next lactation under the assumption that the cow is not involuntary removed.
2. Use of the quantities derived under (1) to derive expected milk production for each month of the stage under the assumption that the

cow is not involuntary removed before or during that month.

3. Multiplication of the quantities derived under (2) by the probabilities that the cow will survive at least until and during that month.

4. Multiplication of the expected monthly quantities derived under (3) by the propoer discount factors for the respective months and summation over all months within the stage to arrive at discounted expected quantities for the stage.

The quantities under point 1 were derived by first assessing expected herd averages is an unculled population for all six lactation numbers and for given calving intervals, then adding or subtracting expected class deviations from these herd averages.[1/] The expected herd averages were derived from tables A.3 and A.5. This requires the assumption of a given "age-corrected herd average" as this variable is defined here.[2/] As has been explained above, it was decided to assume an age-corrected herd average of 12,000 pounds for the ME-herd, and of 11,000 pounds for the MA-herd.[3/]

For example, if all calving intervals are set equal to 365 days and the age-corrected herd averages are set equal to the values given above, tables A.3 and A.5 give the following values for expected 305 days FCM for all cows in an unculled population:[4/]

1/ The calving interval for the same lactation number was set equal to 365 days and the preceding calving interval was given different values, as will be explained later.

2/ See page 143.

3/ See page 185.

4/ It is seen that these expected values, even for the third and higher lactation numbers, are lower than the assumed "age-corrected herd

	ME-herd	MA-herd
1st lactation	9705	9113
2nd "	10466	9703
3rd "	11371	10511
4th "	11274	10741
5th "	10953	10803
6th "	10935	10364

The information which can be used to derive the quantities under point 1 is limited to the information which is contained in the state variables, that is, to knowledge about lactation number, production class, and length of the present calving interval. Tables A.3 and A.5 give expected herd averages as a function of both the preceding and the same calving interval. On the first day of one stage, we know the length of the present calving interval and can use this knowledge when deriving expected 305 days FCM for the next lactation, however we do not know the length of the preceding calving interval and must base the derivation of expected 305 days FCM for the present lactation on the herd distribution of calving intervals.[1/]

---

averages". The reason is partly that calving intervals on the average are longer than 365 days, and partly that the "age-corrected herd average", as this variable is defined here, is based on observed values in a culled population.

1/ The herd distribution of calving intervals again depends on the culling policy, which is unknown at the time the replacement model is formulated. In this study, a given culling policy was assumed in order to derive the quantities wanted. Allthough theoretically unpleasant, the practical importance of this problem is small since variations in the preceding calving interval explains only a very minor part of the variation in milk production during the latter part of a given lactation.

For each lactation number and each production class, we need two expected class deviations from the unculled herd averages: the expected deviation of 305 days FCM for the present lactation from the unculled herd average for the same lactation number, and the expected deviation of 305 days FCM for the next lactation from the unculled herd average for that lactation number. Both these quantities can be derived from the covariance matrices represented in tables A.6 and A.7, the definitions of production variables in table A.9, and the definition of classes in table A.11.

Expected 305 days FCM corrected to 365 days calving intervals, for all lactation numbers and production classes and derived in the way explained above, are presented in appendix A, tables A.29 and A.30. From the quantities given in these tables, expected monthly production for each month of the stage were derived by use of the assumed relationships given in tables A.17 and A.18.

Points 3 and 4 in the procedure described on page 196 were now a matter of routine.

b. Feed quantities

The derivation of discounted expected quantities of grain and hay went through three steps:

1. Derivation of expected consumption of grain and hay for each month of the stage, under the assumption that the cow is not involuntarily removed before or during that month.
2. Multiplication of the quantities derived under (1) by the probabilities that the cow will survive at least until and during that month.

3. Multiplication of the expected quantities derived under (2) by the proper discount factors for the respective months and summation over all months within the stage to arrive at discounted expected quantities for the stage.

The quantities under (1) were derived from the corresponding quantities of expected milk production for each month of the stage and the relations between monthly milk production and feed consumption given in table A.27. This procedure is not entirely satisfactory with respect to grain consumption. Since grain consumption is not a continuous linear function of milk production, we would really need the probability distribution of milk production for each month, not only the expected values, in order to derive correct expected values for grain consumption. This would have greatly increased the computational work required, however, and it was believed that the differences in results would not be so important as to warrant such a complicated procedure.

c. Calves and replaced animals.

The probability that a cow which has started on a given stage will survive until calving can be derived from the probabilities of involuntary removal given in table A.26 by the procedure described earlier.[1] The probability that a calving will result in the birth of a live calf is assumed to be 0.969.[2] The expected number of live calves for the stage

---

1/ See page 180 ff.

2/ See page 187.

is thus given, and can be multiplied by the proper discount factor to give the discounted expected number on the first day of the stage.[1/]

For every state under the decision "keep", there are also some probabilities that the cow will be involuntary removed and thus give income from sale of the cow. For each month of the stage, the probability of involuntary removal is derived by the procedure described above.[2/] The probability that an involuntarily removed cow will give income from sale of an animal is assumed to be 0.9.[3/] This probability is multiplied by the probabilities of involuntary removal to give the expected number of sold cows for each month of the stage. Finally, these expected numbers are multiplied by the proper discount factors for the respective months and the products summed over months within the stage to give the discounted expected number of sold cows for the stage.

In the case where a cow is replaced 10 months after last freshening or 3 months after the beginning of the stage because it has not become pregnant, the probability that it will give income from sale of an animal is assumed to be 1.0.

1/ This procedure gives the discounted expected number of live calves when this number is observed at the day of calving. The author has not been aware that high young calfhood mortality has been a serious problem in the given herds. Because of such mortality, discounted expected number of calves which reach the age of, say, six weeks will be lower. In economic calculations, this can be compensated for by setting the "calf value" (see table A.28) lower. Income from calves is only a small proportion of the "economic rewards" as defined here, and variations in this income due to variations in culling policy can be assumed to be very small. This lack of realism in assumptions, therefore, can be assumed to have almost negligible effects on the results from the optimization procedure.

2/ See page 180 ff.

3/ Ten per cent of all involuntary replacements are assumed to be death losses where no income is derived from the lost animal. This estimate is taken from Halter and Jenkins, op. cit.

Also whenever the cow is voluntarily replaced, either on the first day of a stage or at the end of the sixth lactation, the probability that it will give income from sale of an animal is assumed to be 1.0.

In all cases where the decision is "replace", replacement is assumed to take place at once. The expected number of replaced animals sold is 1.0 and the discount factor is also 1.0. Therefore, the decision "replace" gives an expected discounted income of 1.0 times the salvage value of the replaced animal.

d. Replacement animals

State 1 in these models is defined to represent the situation immediately before the purchase of a new heifer. Therefore, for this state the discounted expected number of purchased heifers is 1.0. The corresponding number for all other states is zero.

This definition of the first state gives much simpler computations and results which are identical to those which would have been arrived at if the purchase of replacement animals had been defined to take place during the same state as when the removed animal was sold.

3. Example

The complete matrices of discounted expected physical quantities are quite extensive and will not be reproduced in this paper. For illustrative purposes, an excerpt of one of the matrices is given in table 7.1.

The fifth lactation states give high values for "discounted expected quantities of removed cows" even under the decision "keep", since it is assumed that all cows will be replaced at the end of the tenth month of the sixth lactation if they have not been removed before.

The quantities given in table 7.1 are sold quantities for "milk", "calf", and "removed cow", while they are purchased quantities for "grain", "hay", and "new heifer". The sold quantities are multiplied by positive prices, the purchased quantities are multiplied by negative figures, and the products summed for each state and decision to give the "immediate expected economic return" for the given state and decision.

TABLE 7.1. Selected rows in the matrix of discounted expected physical quantities for the ME-herd

State	Decision	Discounted expected quantities					
		Milk lb	Grain lb	Hay lb	Calf	Remov. cow	New heifer
State 1	Keep	7092	2133	4891	0.964	0.0374	1.000
1st lactation, 1st proc.class, 15 months C.I.	Keep	14987	4677	10730	0.884	0.0821	0.000
	Replace	0	0	0	0	1.0000	0.000
1st lactation, 7th prod.class, 15 months C.I.	Keep	9838	2621	10730	0.884	0.0821	0.000
	Replace	0	0	0	0	1.0000	0.000
5th lactation, 1st prod.class, 12 months C.I.	Keep	15106	4434	11360	0.903	0.9200	0.000
	Replace	0	0	0	0	1.0000	0.000

VIII. REPLACEMENT POLICIES AND IMPUTED VALUES

A. Optimization Procedure

Since the replacement models are defined in such a way that the stage length is allowed to vary, only solutions to the optimization problem under an infinite planning horizon are relevant.1/ In spite of this, the method given by formulae (3.23) was used to attain numerical solutions.2/ This is an iterative method which gives optimal solutions under a finite number of stages, however since it is known that the policies which are optimal for each given stage in a sequence of stages converge to a constant policy when the number of stages gets very large, the method can be used to derive optimal policies under an infinite planning horizon as well.

This method was selected since a computer program for the method was available. A number of 50 iterations were performed for each problem. In order to reach convergence with a smaller number of iterations, the initial values $f_{j(o)}$ of each state were set equal to what was guessed to be approximate present values under an infinite planning horizon.3/ This procedure proved to be very satisfactory in this case. In order to check for convergence, the optimal policy and the corresponding $\bar{f}_{j(n)}$

1/ It could be desirable to get solutions to problems where the planning horizon is specified as a given and finite number of years. Since there is no mathematical relationship between number of years and number of stages when stage length is allowed to vary, the methods described in this paper are not useful for solving this problem except possibly to give approximate results.

2/ See page 61.

3/ See page 61.

-values were printed out for the first three and the last ten of the fifty stages. In the five optimization problems which were run, only minor changes had occurred in policies after the third iteration. In one single case, a change in policy had occurred as late as at the 47th iteration. In the other four cases, no changes in policy had occurred during the last ten iterations.

It is theoretically possible that changes in policy may occur even after a much higher number of iterations than 50. The procedure used gives no absolute guarantee that the optimal policy under an infinite planning horizon has been reached. While this may be regarded as a theoretical objection to the method, the practical importance is small. When changes occur in policy after a large number of iterations, the difference in economic result between the policy rejected and the policy adopted is likely to be very small. From a realistic point of view, both the loss in precision due to imperfect estimates of parameter values and the loss in precision due to the fact that we have to work with a limited number of discrete classes seem to be much more important than the possible minor deviations between the theoretical optimal policy and the policy arrived at after fifty iterations.

In addition to the optimal or near-optimal policy arrived at, the present values of the different states under this policy are of considerable interest. Even if the optimal policy under an infinite planning horizon may have been reached, the present values $\bar{f}_{j(50)}$ are certainly not identical to the present values under an infinite planning horizon, but depend both on the number of iterations, in this case 50, and on the initial values $f_{j(o)}$ chosen. The difference, however, is likely to be much more in the absolute values than in the differences between the present values for different states.

By studying the change in present values of one and the same state over the last ten iterations, an attempt was made to estimate the present value of that state under an infinite planning horizon. The procedure was based on the assumption that after a large number of iterations has been performed, the sequence of present values for the same state can be approximately described as a sequence of sums of a convergent geometric progression as the number of chains in the progression increases.

For example, the present values of state 1 were for the last ten iterations of the replacement model under price alternative 1:[1/]

4284.6973, 4286.1647, 4287.5620, 4288.8926, 4290.1595, 4291.3659, 4292.5144, 4293.6081, 4294.6501, 4295.6430.

The differences between each present value and the preceding one are:

1.4674, 1.3973, 1.3306, 1.2669, 1.2064, 1.1485, 1.0937, 1.0420, 0.9929.

It is seen that the present values approach a constant value which is the present value of the given state under an infinite planning horizon. As an approximate description, we may think of the sequence as being generated by adding a constant periodic amount A to the discounted value of the preceding stage.[2/] If the present value of the given state after n iterations is denoted V_n, then we may write as an approximation:

[1/] The values given here are in dollars.

[2/] Since stages are counted backwards in time, the $n-1$'th stage is the stage which follows the n'th stage in time.

$$V_n = A + \beta V_{n-1} \tag{7.1}$$

A and β can be derived from the sequence. For example, by using the first two values and the last two values in the sequence given above, we can write:

$$4286.1647 = A + \beta \cdot 4284.6973 \tag{7.2}$$

$$4295.6430 = A + \beta \cdot 4294.6501 \tag{7.3}$$

From the two equations, we can solve for A and β:

$$A = 205.74077$$

$$\beta = 0.952325$$

As n goes to infinity, V_n approaches a constant value V which can be derived from the equation:

$$V = A + \beta V \tag{7.4}$$

$$\text{or} \quad V = \frac{A}{1 - \beta} \tag{7.5}$$

In the numerical example, we have:

$$V = \frac{205.74077}{1 - 0.952325} = 4315.48$$

The derivation of present values in this way is based on an approximation which is not even proved here. It is felt, however, that the absolute present values derived in this way give better estimates of present values under an infinite planning horizon than we can get by taking the derived values of $\bar{f}_{j(50)}$ as they are. Theoretically, it would have been much more satisfactory if the linear programming procedure described in chapter III had been used.[1/] If rounding errors due

[1/] Theoretically, one could also have used the method described by Burt to verify optimality of the derived policy and to compute expected present values under the given policy under an infinite planning horizon. In the present application, this would have required the solution of a system of linear equations of dimension 106. See the discussion on page 48 and the following reference: Oscar R. Burt, "The Economics of Conjunctive Use of Ground and Surface Water," Hilgardia, XXXVI (Des., 1964), p.41.

Note:

The original page has been deleted.

to small determinant values were of no importance, this procedure would have given the optimal policy as well as the present values under an infinite planning horizon with absolute certainty. When the possibility of rounding errors is taken into account; it may well be that the procedure described here gives results equally close to the true values. From a practical point of view, it is not likely that the difference in results due to a theoretically less satisfactory procedure is of any importance.

B. Results

Results for the two herds and for different price situations specified in table A.28 are given in tables A.31 - A.35. The tables give both the optimal replacement policies arrived at and the corresponding present values of all states.

The present value of state 1 is derived by the method explained on pages 205-206, however in no case is the difference between the value arrived at by this method and the present value after 50 iterations more than $ 20.00. The differences are so small because the $f_{j(o)}$ -values had been set by judgement to something which proved to be quite close to the true values under an infinite planning horizon.

Present values for other states than state 1 are presented as differences (after 50 iterations) between the present value of state 1 and the present values of the other states. These differences can be said to represent imputed values to the dairy producer of animals with the given characteristics. For all states where the optimal policy prescribes replacement, this value is equal to the salvage value of the animal. For other states, it is higher.

Since complete convergence with respect to present values has not been reached, the relative present values given in the tables may deviate somewhat from the true present values under an infinite planning horizon. An examination of changes in relative present values over the last ten iterations revealed that the changes had been up to $ 0.85, and it may be surmised that further changes could go up to one and a half or two dollars before complete convergence was reached.

When "expected immediate economic returns" were derived, only outlays for feed and for purchase of replacement animals were deducted from income. Therefore, the absolute present values arrived at do not give any indication as to whether milk production under given price conditions is profitable or not. To know this, we would have to know the present value under an infinite planning horizon of all outlays except those which have already been deducted.

When examining the results, we may ask such questions as: What is the optimal replacement policy? How much, and in what directions, will the optimal policy change under changes in prices and in technical efficiency? What are the differences in values to the dairy producer of animals with different characteristics? Tables A.31 - A.35 give part of the answers to these questions.

As expected, all tables show that more cows should be replaced if the calving interval is expected to be 15 months instead of 12 months, and even more cows should be replaced if the calving interval is expected to exceed 15 months. The effects of differences in age are not so easily ascertained. The replacement rules for different lactation numbers can not be compared directly because the "production variables" is defined differently for each lactation number and the class intervals are also defined differently. The tables give the general impression, however, that a higher production relative to the herd-mates is required to allow an older cow to remain in the herd.

Tables A.31 and A.32 represent the same price situation, however in table A.31 milk has been valued at a blend price while in table A.32 it has been valued at the much lower manufacturing milk price.[1/] It is evident that the culling policy should be much less intensive when changes in total milk production is evaluated at manufacturing milk prices. Table A.34 represents a "Los Angelos area" price situation, with higher blend price of milk, cheaper concentrates and higher hay prices. The effect is a somewhat more intensive culling policy than under Central Valley price conditions.

Table A.33 represents the same feed and milk prices as table A.31, but the price of a replacement heifer has been set at an artificially low level in order to examine the effects of changes in replacement prices or in dairy beef prices. A fall in replacement prices will have very nearly the same effects as a raise in dairy beef prices, since the important consideration with respect to the replacement decision is the difference in price between a replacement animal and the replaced animal, not the absolute prices.

A comparison between tables A.31 and A.33 shows that as expected, the optimal culling policy calls for more intensive culling when replacement prices are lower or when dairy beef prices are higher. The change in replacement policy for such a substantial change in prices is rather small, however. At first sight, this may seem surprising. It is common

---

1/ See the discussion on p. 189.

experience that dairy producers cull much more intensively when meat prices go up. This, however, may be the result of adjustments to short-run fluctations in prices which are not taken care of by this replacement model. The explanation may be that when beef prices go up, the dairy producers will hurry to dispose of some of their marginal animals because they expect prices to fall again after a few months, but most of the disposed animals are such as would have been removed before their next freshening anyhow.

The relative present values give imputed values to the dairy producer of animals with different characteristics. For cows in first lactation, the imputed values are actually higher for cows with 15 months calving intervals than for cows with 12 months calving intervals. This finding is consistent with recommendations often given to dairy producers, and is due to the higher persistency of milk production during the first lactation.[1/] For elder cows, 15 months calving intervals give lower imputed values, but the differences in values between 12 and 15 months calving intervals are mostly less than the differences in imputed values due to one class difference in the production variable. On the other hand, cows with calving intervals 18 months or longer have substantially lower imputed values than those with 15 months calving intervals.

Tables A.31 and A.35 give the optimal replacement policies under the same price conditions for the ME-herd and the MA-herd respectively. Assumptions have been stated so that the model for the ME-herd represents a "more efficient" herd and the model for the MA-herd represents a some-

---

1/ It is often recommended to breed first-lactation cows so late that they get a calving interval of 14 - 16 months rather than 12 months.

what "less efficient" herd. The resulting replacement policies appear to be fairly similar, and a comparison of the results do not suggest any significant differences in optimal replacement policies for such differences in parameter values.

We should be careful not to draw far-reaching conclusions from this result, however. First, the policies for the two herds are not exactly comparable since the variables for production history as well as the class intervals with respect to these variables are not identically defined. Second, the two models differ both in transition probabilities with respect to production variables, in probability distributions of calving intervals, in probabilities of involuntary replacements, and in assumed general level of production. It is possible that each of these differences alone would influence the optimal replacement policy in some direction but that together the effects more or less cancel each other out.

Further studies should try to isolate the effects of each of these differences at a time and possibly of two and two in combinations. One might do this by working with experimental models in which the assumed covariance matrix, and therefore also the definition of variables for production history, class intervals and transition probabilities with respect to production, could be kept constant. In such models, changes in level of production, in probabilities of involuntary replacements, and in probability distributions of calving intervals could be introduced one at a time, and effects on optimal replacement policies as well as on imputed values could be isolated. Such experimenting was contemplated but considered too time-consuming and expensive to be undertaken as part of this study.

As it is now, the most notable difference between the two herds is in the present values of state 1. This difference is almost 600 dollars, which corresponds to an annual difference in profit of almost 37 dollars per cow. The difference can be explained as due to the higher degree of technical efficiency assumed for the ME-herd.[1/]

C. Derivation of Present Values for Sub-Optimal Replacement Policies

1. The Procedure

Present values under some pre-selected replacement policies were derived by the same method as was used for the optimization procedure, simply by assigning artificially low values for "expected immediate economic return" to those decisions which should be excluded from the selected policy. Again, 50 iterations were performed. In order to make the results directly comparable to those arrived at by the optimization procedure, the same initial values $f_{j(0)}$ were used.

The present values of state 1 under an infinite planning horizon were derived in the same way as for the optimization models. It makes very little difference whether we compare the present values derived in this way or the present values after 50 iterations, since the differences are nearly the same. As expected, the differences in the former present values are slightly larger than the differences in the later.

[1/] These results have been reached on the assumption that both herds are fed according to Morrison's feeding standards and that other costs (for example for labor, management, etc.) are not higher in the ME-herd. If the higher-producing herd is fed at a higher level as compared with the feeding standards than the lower-produding herd, as may be the case, then the difference in economic results would be smaller.

All present values for sub-optimal policies were derived for price alternative 1, so the results can be compared with the optimization results in tables A.31 and A.35.

2. Results

The results are given in appendix A, tables A.36 - A.39.

Tables A.36 and A.37 represent results of very extensive and of very intensive replacement policies respectively. Table A.36 is based on no voluntary replacement at all, while table A.37 is based on a replacement policy where all cows in production classes below the average are culled, and in addition cows in the medium production class are culled if they are in the second or later lactations and in the longest calving interval group.

In both cases, the effect of following an extreme sub-optimal culling policy has been to reduce the present value of state 1 by about 100 dollars,[1/] corresponding to a difference in annual profit per cow of about six dollars. While this difference is large enough to be of significance, it is small as compared with the difference in present values between the ME-herd and the MA-herd.

Table A.38 represents the results which would follow if the manager of the ME-herd followed the replacement policy which is optimal for the MA-herd, and table A.39 gives the results if the manager of the MA-herd followed the policy which is optimal for the ME-herd. In both cases,

---

[1/] As compared with the optimal policy given by table A.31, the losses in present value of state 1 are:

For the policy in table A.36: \$ 4,315.48 - \$ 4,224.09 = \$ 91.39

For the policy in table A.37: \$ 4,315.48 - \$ 4,203.47 = \$ 112.01

the losses in present values as compared with the policy which is optimal for the given herd is around ten dollars,[1/] corresponding to an annual loss in profit of about 60 cents per cow.

These results give some indication about the economic importance of the replacement problem. Moderate deviations from the optimal replacement policy seem to give only minor losses, while as large deviations as those represented by tables A.36 and A.37 give losses of practical importance, however much smaller than the losses which can be incurred because of the combined effects of low production, more than normal breeding troubles, and high rates of involuntary removals.

D. Some Implications of Given Replacement Policies

Table A.40 in appendix A presents some implications of different replacement policies for the two herds. The data in this table are derived from the replacement policies specified in tables A.31, A.32, A.35, A.36 and A.37, and from the same probabilities as form the basis for the replacement models.

The upper part of the table gives the probabilities that a heifer which freshens first time in the herd will freshen a second time, a third time, and so on. The sum of these probabilities, including the first, gives the expected number of calvings over the lifetime of a cow, if we assume that all cows are replaced at the end of the sixth lactation if not before. For example, by following the optimal replacement policy

---

[1/] As compared with the optimal policy given by tables A.31 and A.35 respectively, the losses in present value of state 1 are:

For the ME-herd:	$ 4,315.48	-	$ 4,306.61	=	$ 8.87
For the MA-herd:	$ 3,721.32	-	$ 3,709.08	=	$ 12.24

for the ME-herd specified by table A.31, we will expect the average number of freshenings per cow to be 3.239. In reality, a few cows will be kept over the seventh, the eighth, and even over higher lactation numbers, so the total expected number of freshenings per cow can be assumed to be 0.2 to 0.3 higher than this figure indicates.

Frick and Henry estimated the number of potentially fertile living female calves to 0.462 per freshening.[1/] This means that if we follow the replacement policy specified in table A.31 up to the sixth lactation but allow about the same proportion of cows to live beyond the sixth lactation as can be observed in these herds, then the average number of potentially fertile female calves which will be born over the lifetime of each cow can be expected to be about 1.6. Some percentage of these calves will die, not conceive, or be removed for some other reason before their first freshening. Still, this replacement policy, which under the given price conditions is optimal from an economic point of view, also seems to be consistent with the desire to maintain the number of cows of a given breed or population in the long run. There is even room for some culling of calves in order to improve the breed through selection among females, however such culling must be very moderate. From a genetic point of view, the selection among female calves which can be practiced within the given replacement policy is of minor importance.

On the other hand, the intensive replacement represented by table A.37 is evidently too strong to be practiced within a larger part of a cow population without decreasing the number of cows within that population over time.

1/ Frick and Henry, loc. cit.

The lower part of table A.40 gives the probability distribution of removal causes under the given replacement policies. Footnotes to the table give the definitions of removal causes which are used. For example, assume that cows in a given lactation number with 15 months calving intervals and in the next lowest production class are replaced. This is counted as removal due to "low production" if also cows with 12 months calving intervals in the same production class are replaced, while it is counted as removal due to "low production + breeding trouble" if cows with 12 months calving intervals in the same production class are kept.

It would have been interesting to compare optimal replacement policies derived through use of models of this kind with replacement rules actually practiced in real herds. The basis for such comparisons is not satisfactory in this case because we have made assumptions about general production levels which deviate from the observed production levels over past years. In addition, this study has not undertaken to examine the characteristics of the cows which really have been removed from the observed herds. If, tentatively, we surmise that general level of production does not influence the optimal replacement policies to any great extent, we may try to compare results presented in table A.40 with what is known about culling in the observed herds.

Price alternative 1, represented by tables A.31 and A.35, is probably most representative for the price situation under which the given herds have operated. The pribability distributions of replacement causes for these table numbers in table A.40 may be compared with the relative frequencies actually reported from the 12 herds and presented in table 5.1.[1/] As compared with the actual replacement reasons declared

---

1/ See page 98.

by the dairy producers, fewer cows should be replaced because of low production alone, and more cows should be replaced because of either "sterility" or a combination of low production and breeding trouble. However the differences may possibly be due to differences in classification rather than to differences in actual replacement policies.

Table A.40 gives the expected number of cows starting on the second, ..., the sixth lactation relative to the number of cows starting on the first lactation, under the optimal replacement policies for the two herds. Tables A.2 and A.4 give some information on the relative number of completed lactation records for the first through the sixth lactation. The figures in these tables are not quite satisfactory for a comparison with the figures in table A.40, both because some records were excluded because of missing information and because the first lactations are over-represented because many of the cows still living are represented with their first lactation records but not with their later. With these reservations, there does not appear to be any considerable difference between the age distribution of cows resulting from the optimal replacement policies and the age distribution actually observed.

With the limited basis for comparison available, it is not possible to say that the policies found to be optimal deviate much from what has actually been practiced. On the contrary, the differences appear to be rather small, but the examination here is not sufficient to support this assertion with any great strength.

IX. OPTIMAL REPLACEMENT TIME WITHIN A LACTATION

A. Method

In the replacement models developed above, it was assumed that voluntary replacement of a cow will take place seven months after last freshening, or ten months after last freshening in the case of cows which have reached the end of the sixth lactation. This simplifying assumption made it possible to save a substantial amount of work required to construct numerical replacement models. We shall examine now whether the results arrived at would be substantially changes if other, and economically better, points in time had been selected for replacement within the lactation.

We may examine this by looking at the present values of the process at other points in time during the lactation and under different alternatives for replacement time. If the present value at a given point in time before the replacement can be increased by changing the replacement time, then it will pay to do so. If the present value will change much as a result of this, then the replacement policy which we have arrived at may in fact be non-optimal and may even deviate much from the optimal policy. If the present value will change only little, then the replacement decisions may change in marginal cases but such changes are of minor practical significance.

The choice of seven months after last freshening as the point in time at which to replace animals was selected as a compromise between conflicting considerations. For cows in the very lowest production classes, we may expect that the replacement time should be earlier. For cows in higher production classes which are replaced partly because of breeding trouble, we may expect that the replacement time should have been later. We will examine here a few such cases, based on the ME-herd and price alternative 1. The same relationship will be assumed between 305 days FCM and monthly FCM as was derived before and has been assumed in the replacement models.[1/] For simplicity, we will disregard the probabilities of involuntary removal within the relatively short time spans we are considering here.[2/]

Select some point in time during the lactation as a reference point, with the only requirement that it should be before the optimal replacement time. Monthly time intervals will be counted from this reference point. Denote incomes less variable costs per month as a_1, a_2,..... or in general as a_i $(i = 1, \ldots\ldots)$. Denote discount factors for discounting these monthly amounts to the reference point in time as β_i'. It will be assumed in the numerical examples which follow that the receipt of a_i is centered in the i'th month, so that a_1 will be discounted corresponding to one half month, a_2 will be discounted corresponding to one month and a half, and so on.

Assume that replacement takes place after <u>n</u> whole months. When the cow is replaced, the dairy producer receives the salvage value of the removed animal, denoted S, and the process goes to state 1, the present value of which is V at the time replacement takes place. To give present values at the reference point in time, both S and V must be discounted by use of the discount factor β_n, corresponding to n whole months.

1/ See table A.18.

2/ Consideration of involuntary removal will change the results slightly. See page 224.

We will select n so that we maximise the present value of the process at our reference point in time:

$$\underset{n}{\text{Maximize}}\ P = \sum_{i=1}^{n} \beta_i' a_i + \beta_n (S + V) \qquad (7.6)$$

B. Examples

1. Example 1

Consider second lactation, cows with 12 months calving intervals and in production class 7 (the lowest). Select three months after last freshening as the reference point in time. Under price alternative 1, milk sales less feed costs for the first, the second,.... month after this point are in dollars: 30.38, 22.01, 19.31, 16.64, 11.83, 5.06. The salvage value S is \$ 115.– and the present value of state 1, which is taken from table A.31, is \$ 4,315.48. With 6 per cent interest rate and continuous discounting, the same as has been used in the models, we have the following relations:

n	$\sum_{i=1}^{n} \beta_i' a_i$	$\beta_n (S + V)$	P
0	0	4,430.48	4,430.48
1	30.30	4,408.39	<u>4,438.69</u>
2	52.15	4,386.40	4,438.55
3	71.22	4,364.52	4,435.74
4	87.57	4,342.75	4,430.32
5	99.14	4,321.09	4,420.23
6	104.06	4,299.54	4,403.60

The optimal value of n is 1 and the optimal replacement time is four months after last freshening.[1] The difference in present values between optimal replacement time and the replacement time which has been assumed in the models is $ 8.37, measured at a point in time three months after last freshening.

If we make the same type of calculation for the same lactation and calving interval but for the next lowest production class, we will find that the optimal replacement time within the lactation is six months after last freshening, and the difference in present values between optimal replacement time and replacement time assumed in the models is only $ 1.33. For this state, the policy specified in table 31 prescribes to keep the cow. If the case had been marginal, that is, if the present value of the state under the decision "keep" had been only slightly higher than the present value under the decision "replace", then the increase in present value caused by the specification of a more optimal replacement time within the lactation could have been enough to reverse the decision.

2. Example 2

Consider the state "second lactation, calving interval unknown, production class 3". For this state, the optimal policy prescribes "keep" but it is possible that the alternative "replace" is more favorable if a better time is specified for replacement within the lactation.

---

[1] At this time, less information is available for classification of cows with respect to production than what is available seven months after last freshening, however with the high correlation known to exist between part lactation records and 305 days records, this is not a major problem. See pages 106 - 107.

Optimal replacement time is likely to be later than seven months after last freshening in this case, so we can select seven months after last freshening as the point of reference. Milk sales less feed costs for each month after this time are in dollars: 28.00, 25.30, 23.10, 19.65, 15.72, 11.45. If the cow is replaced after n months, we have the following relations:

n	$\sum_{i=1}^{n} \beta_i' a_i$	$\beta_n(S + V)$	P
0	0	4,430.48	4,430.48
1	27.93	4,408.39	4,436.32
2	53.04	4,386.40	4,439.44
3	75.85	4,364.52	4,440.37
4	95.16	4,342.75	4,437.91
5	110.53	4,321.09	4,431.62
6	121.64	4,299.54	4,421.18

The optimal value of n is 3, optimal replacement time within the lactation is ten months after last freshening, and the difference in present values between optimal replacement time and time assumed in the models is $ 9.89. If the case had been marginal, the difference could have made the decision "replace" preferable to the decision "keep". Table A.31 shows that the present value of this state under the decision "keep" actually is more than 30 dollars better than the alternative "replace" when replacement time is set to seven months after last freshening.[1/] In this case, the change in replacement time within the lactation

[1/] The "relative present value" under the alternative "replace" is $ 115.00 for all cows of this age, so the difference in present value of the state between the alternatives "keep" and "replace" is

$ 147.35 - $ 115.00 = $ 32.35

is not enough to change the optimal decision.

If we make the same type of calculation for the same lactation and calving interval but for production class 4, we will find the optimal replacement time to be eight months after last freshening and the difference between present values to be $ 2.55.

In the foregoing we have disregarded the effect which the probabilities of involuntary removal have on the selection of the optimal replacement time. In general, the effect of this risk is to make earlier replacement somewhat more profitable, but the effect over the small spans of time with which we are working in this case is rather small. In example 1, replacement four months after last freshening would still be most profitable, but the difference in present values would increase from $ 8.37 to $ 9.19.[1/] In example 2, the optimal replacement time would also be the same, but the difference in present values would decrease from $ 9.89 to $ 8.88.

C. Repercussion Effects of Change in Replacement Time within the Lactation

The two examples described represent extreme cases where the optimal replacement time within the lactation, for states for which replacement should be considered at all, is either much before seven months or much after seven months. It appears from this that the change in present values by selecting a better replacement time within the lactation than

---

[1/] If we had considered replacement for every day instead of as here for every month, the effect would have been to push the replacement time a few days forward.

seven months seldom exceeds ten dollars, and for most states is less. While this difference is small as compared with the difference in present values between production classes, it is enough to reverse the replacement decision in marginal cases. The selection of a better replacement time within the lactation will always improve the profitability of the alternative "replace", and can therefore be expected to lead to a slightly more intensive culling than is found as a result of the replacement models used in this study.

Since the specification of a better time for replacement within the lactation will increase the present values of most of the states where the decision is "replace", the final effect, through repercussions, will be to increase the present values of all states in the model; in particular, the present value of state 1. This, again, would influence the calculations presented above in favor of a somewhat earlier replacement time within the lactation. This is one of the many cases in economics where we attempt, either be approximation or by some iterative procedure, to solve a complex problem in separate parts where it preferably should have been solved simultaneously. It would, no doubt, have been theoretically more satisfactory if we had been able to determine the optimal time for replacement within the lactation and whether or not to replace a cow during a given lactation by a simultaneous procedure. This would have required a model far exceeding our computational capacity.

In our case, the problem does not seem to be of much practical significance. First, our calculations here suggest that the specification of a better time for replacement within the lactation will influence the replacement decision with respect to whether or not to replace the animal within a given lactation only for a few states for which the decision is marginal. Second, when a dairy producer knows the approximate present value of state 1, he can use very simple methods by which he himself can derive the optimal replacement time within the lactation for animals which he is going to replace. On the other hand, to find out whether it is profitable to replace a given animal during a given lactation or wait until later is a much more complicated problem, and it is here that a replacement model of the type developed in this study can be of help.

D. A Rule for Finding Optimal Replacement Time within a Lactation

We will turn to the derivation of a simple decision rule which a dairy producer can use to determine the optimal replacement time within a lactation, provided he has decided to replace the animal during that lactation. If we start with the formula for present value at a given reference point in time, it will pay to keep cow at least one more month from n to $n+1$ if:[1/]

$$\sum_{i=1}^{n+1} \beta_i' a_i + \beta_{n+1}(S + V) > \sum_{i=1}^{n} \beta_i' a_i + \beta_n(S + V) \qquad (7.7)$$

The condition can be transformed to:

$$\beta_{n+1}' a_{n+1} > (\beta_n - \beta_{n+1})(S + V) \qquad (7.8)$$

Remember that:

$$\beta_i' = e^{\frac{-r(i-1/2)}{12}} \qquad (7.9)$$

$$\beta_i = e^{\frac{-ri}{12}} \qquad (7.10)$$

1/ See formula (7.6), page 221.

where r is the annual interest rate. If we insert (7.9) and (7.10) in (7.8) and multiply both sides by $e^{\frac{(n+1)r}{12}}$, the condition becomes:

$$e^{\frac{1/2r}{12}} \cdot a_{n+1} > \left[e^{\frac{r}{12}} - 1\right](S + V) \qquad (7.11)$$

To multiply a_{n+1} on the left-hand side with the given factor is the same as adding interest for one half month. The factor is close to one, and since a_{n+1} is a very small amount as compared with $(S + V)$, we may set the factor equal to one without any great error. To multiply $(S + V)$ by the factor $\left[e^{\frac{r}{12}} - 1\right]$ is essentially the same as taking one month's interest on the amount $(S + V)$. The decision rule becomes:

If it is decided to replace a cow during the current lactation, it will pay to keep the cow as long as the monthly milk income less feed costs exceeds the monthly interest on (salvage value + present value of state 1.[1/]

The interest on $(S + V)$ represents the opportunity cost of keeping a cow in the herd. In order to defend its place in the herd, milk income less feed costs for a cow which is going to be replaced during its current lactation must be as high or higher than this opportunity cost.[2/]

1/ The rule requires knowledge of the present value of state 1, which again is derived through the replacement model developed in this study. In a situation where the replacement problem is not solved in the way described here, we may get an approximate estimate of the right-hand side om (7.11) by taking the average per month (milk sales + dairy beef sales - feed costs - replacement costs - interest on salvage value) per cow for the herd.

2/ We have assumed here that a removed cow immediately will be replaced by a new cow, and we have disregarded short-run fluctations in meat prices and in replacement prices. In practice, such considerations

In the examples above, this opportunity cost is $ 22.21 per month. We could have arrived at the optimal replacement time simply be seeing when monthly income less feed costs fell below that amount.

---

will cause some modifications in the optimal decision rules. If, for some reason, the dairy producer is going to wait some time from the removal of the cow to the inclusion of a replacement animal in the herd, but at the same time he does not expect changes in livestock prices, then it will pay to keep the old cow as long as monthly income from milk sales exceeds monthly outlay for feed costs.

X. SUMMARY AND CONCLUSIONS

A. Description of the Model

The biological nature of dairy production involves a number of features which, if possible, should be considered in a decision model for replacement of dairy cows. Among such features can be mentioned the cyclic nature of milk production of a cow, stochastic variation in level of milk production both among cows within a herd and among different lactations for the same cow, stochastic variation in length of the calving intervals, relatively high probabilities of "involuntary removals" of dairy cows for such reasons as sickness and accidents, and genetic progress over time in a population of dairy cows.

A replacement policy for dairy cows is defined as a rule which for given characteristics of a cow tells whether or not to replace it. An optimal policy is defined as a policy which maximizes the present value of the difference between the expected future income stream and the expected future outlay stream over a given planning horizon. A general model which provides a framework for dealing with the dairy cow replacement problem is "dynamic programming with Markov processes". Within this framework, the dairy cow replacement problem can be formulated in the following way:

A process is defined to represent a dairy cow and all successive replacements. Over time, this process goes through a number of consecutive stages, where a new stage is defined to begin either immediately before a new replacement animal is included in the dairy herd or when seven months have passed since the last freshening of the existing cow. On the first day of each stage, the process is classified into one of a number of alternative states. State 1 is characterized as the situation immediately before the inclusion of a new heifer. The other states are defined according to given characteristics of the existing animal. Such characteristics are called state variables, and can only take on discrete values. Allthough the general framework allows for the choice among many possible state variables and each state variable can be allowed more or less alternative values, the decision model which is used in this study is based on the following choice:

1. One state variable for age measured in lactations, with five alternative values.
2. One state variable for production history, with seven alternative values.
3. One state variable for expected length of the present calving interval, with three alternative values: 12 months, 15 months, and unknown.

With this choice, the model contains $1 + (5 \times 7 \times 3) = 106$ states. On the first day of each stage and for the 105 states where a cow is present, the dairy producer can choose among the alternative actions "keep" and "replace". For each state and each decision, there are given transition probabilities that, on the first day of the next stage, the process will be in either the same state as before or in any of the other 105 states. Under the decision "replace", the process will go to state 1 with probability 1. Under the decision "keep", the process may either go to state 1 because of involuntary removal of the existing cow, or to a number of other states, all characterized by the state variable for age taking on one unit higher value. However all cows are assumed to be removed at the end of the sixth lactation if they have not been removed before, so when the process is in a state characterized as "5th lactation", the next state will always be state 1. For the states where the expected length of the calving interval is unknown, the cow is assumed to be replaced after three months if it has not become pregnant within that time.

For each state and decision, there are given "expected immediate economic returns" which are defined as the expected differences between variable incomes and variable outlays over the present stage, discounted to the first day of the stage. If we known the transition probabilities and the expected immediate returns for all states and decisions, we can use one of several existing methods to derive the policy, or the set of policies, which maximizes the objective criterion. Under a limited planning horizon, the optimal policy may vary between stages, while under an infinite planning horizon, the optimal policy or policies are invariable over stages.

Optimization methods are described in existing literature and are reviewed in this paper. The "value-iteration method" applied to cases where there are a finite and given number of stages left under the planning horizon, but can also be used to derive optimal or near-optimal policies under an infinite planning horizon. The "policy-iteration method" applies to maximization under an infinite planning horizon. Also the "linear programming method" applies to maximization under an infinite planning horizon. It can be shown that in this case, the problem can be transformed to a linear programming problem whereby one of the existing algorithms for linear programming can be used to obtain an optimal solution. This transformation is shown in more detail in the paper, since it is less generally known.

Most published literature on dynamic programming with Markov processes assumes that stages are of constant length. With the formulation of the dairy cow replacement problem described above, stage length is a stochastic variable with distribution depending on state and transition. It is shown in this paper that the general model easily can be modified to take care of this, whereby any of the existing methods for deriving a solution can be modified to obtain solutions for models with variable stage length. If stage length is variable, only optimization under an infinite planning horizon seems to be of practical interest, since under a limited planning horizon there will usually be the time span rather than the number of stages which is limited. Accordingly, numerical solutions in this study are limited to cases of infinite planning horizon.

In order to apply the general model to the dairy cow replacement case, we must be able to define states or state variables in such a way that the resulting stochastic process is a Markov process. This implies that the probability p_{ij} of transition from the ith state to the jth state under a given decision must depend on the state i and the decision only, and be independent of states which the process has been in during earlier stages.

This requirement is difficult to satisfy in the dairy cow case. It is shown in this paper, however, that the problem can be analyzed and solved within the framework of normal stochastic processes. This method is based on the assumption that 305 days production for consecutive lactations of a cow, after correction for the effects of given explanatory variables, have a multivariate normal distribution. Under this assumption, a set of variables for production history can be defined as linear combinations of 305 days production for all lactations up to the present, in such a way that the resulting sequence of variables gives a Markov process. Coefficients in these linear combinations can be defined and transition probabilities under arbitrarily defined class intervals of the linear combinations can be derived if we know the parameters in the multivariate normal distributions.

The other two state variables fit easily into the Markov process framework, and a stochastic process where states are defined according to age, expected length of the present calving interval, and a variable for production defined as explained above, can be assumed to be a Markov process.

B. Parameter Estimation

For estimation purposes, samples of dairy cows were obtained from herds which had been subject to culling. Estimation methods have been designed to correct for bias due to this. The method used to obtain multivariate parameter estimates is described in appendix B. In this study, production was measured as pounds fat-corrected milk. The stochastic model and the estimation method used can be applied to cases where production is measured as pounds milk fat or as pounds milk as well, if the same assumptions are made about the nature of the distribution of these variables.

Numerical replacement models were constructed for two different dairy herds. Parameters for these models were estimated based on records of about 700 and 650 cows in two real herds. Some parameters were taken from other sources, most important of which are parameters for the feed – milk relationship and price parameters. Estimation results and assumed model parameters derived from these results are given in appendix A, tables A.2 - A.25.

Estimates of parameters in the multivariate distributions of lactation yields differed between herds, however the results suggest that if individual lactation records had been corrected for herd average for the given year by use of multiplicative correction factors, the resulting variance and covariance estimates would differ too little between herds to support the hypothesis of true herd differences.

On the other hand, the estimation results showed clearly that probability distributions of calving intervals and probabilities of involuntary removals did differ considerably between the two herds. Probabilities of involuntary removals increased in both herds with age of the cow. It was less clear whether probability distributions of calving intervals differed between lactation numbers.

As a basis for replacement models, we need to know the relationship between 305 days production and production during separate months of the lactation. Parameters in such relationships were estimated based on a subsample of lactations from one of the herds, and the results applied to both herds. Other parameters were estimated and used separately for each herd.

Expected future "age-corrected herd averages" were arbitrarily set to 12,000 and 11,000 pounds 305 days FCM, in order to contrast a somewhat "more efficient herd" with a somewhat "less efficient herd".

C. Replacement Policies and Present Values

The value-iteration method was used to derive optimal or near-optimal solutions under an infinite planning horizon, since a computer program for this method was the only available. Such solutions were derived under four different price situations for one of the herds, and under one price situation for the other herd. The results are given in appendix A, tables A.31 - A.35. In addition to the optimal solutions, present values of all states in the processes were derived for cases where suboptimal policies were imposed, in order to study the economic loss from deviations from the optimal policy. The results are given in appendix A, tables A.36 - A.39.

Results of the optimization procedure show, as expected, that higher milk prices should be followed by more intensive culling. Also lower replacement prices or higher dairy beef prices resulted in replacement policies with more intensive culling. The change in optimal replacement policy for a substantial change in price difference between replacement heifers and replaced cows was rather small.

The differences in optimal replacement policies between the conceptual "more efficient" dairy herd and the "less efficient" dairy herd were relatively small. On the other hand, the differences in efficiency led to a considerable difference in economic profitability. The effects of the difference in production level, of the difference in probabilities of involuntary removals, and of the difference in probability distributions of calving intervals taken one at a time have not been examined.

The effect of changes in replacement policy on the economic result of the dairy business as a whole can be seen best from the present value of state 1. Moderate deviations from the optimal culling policy had only minor depressing effects, even if the present values of some of the states were depressed with larger amounts. Thus, when the culling policy which had been found to be optimal for one herd was imposed on the other herd, the present value of state 1 for this herd decreased by approximately ten dollars. Very large deviations from the optimal culling policy led to a decrease in present value of state 1 of about 100 dollars. This is approximately one sixth of the difference in present values of state 1 between the "more efficient" and the "less efficient" herd.

The culling policies found to be optimal for these replacement models have not been compared in an exact way with the culling policies actually practiced in the two herds, however available data do not suggest that the differences are very large. Thus, it is not likely that the economic loss from a less-than-optimal culling policy has been of any great importance in the herds examined.

A comparison of present values of different states with the present value of state 1 gives information on the value to the dairy producer of cows with different characteristics. Variations in the production variable as this is defined in this study gave considerable differences in imputed values between cows. In most cases, the difference between each of the seven classes and the next lower or higher one was in the range of 30 - 50 dollars, except in the cases where the optimal replacement policy prescribed replacement. Lower milk prices gave smaller class differences in imputed values. For cows in first lactation and of medium or better productive ability, 15 months calving intervals gave higher imputed values than 12 months calving intervals. For elder cows, the difference was in favor of 12 months calving intervals but the difference was only modest. Calving intervals beyond 15 months gave considerably lower imputed values.

The replacement models described here are useful for determining whether or not to replace a given cow during a given lactation, but give no direct information about the optimal replacement time within the lactation. To make it possible to solve the first problem, it was necessary to assume that voluntary replacement will always take place seven months after last freshening. It is shown in the last part of this paper that the optimal replacement time within the lactation easily can be determined when we know the present value of state 1. While a shift in replacement time within the lactation in principle violates the assumptions underlying these decision models, it is shown that the effect is small and is likely to change the replacement decision only in marginal cases.

D. Validity of Results

The optimization method used has produced replacement policies which are optimal or near-optimal under the assumptions and within the limitations imposed by the model. Some of these assumptions and limitations will be discussed here.

First, the models do not include consideration of genetic improvement over time in a population of dairy cows. While consideration of genetic improvement should result in more intensive culling, the size of this effect is not known. If it is assumed that the rate of genetic improvement corresponds to an increase in herd average milk production of about one per cent per year, five years genetic improvement will correspond to approximately one production class difference as production classes are defined here. It can be surmised that the effect for cows in the highest age groups should be to move the "replacement margin" almost one production class upwards, while the effect is smaller in the younger age groups.

Second, the Markovian dynamic programming framework used in this study is known to give optimal solutions to a decision problem if the underlying assumptions are satisfied. One of these assumptions is that the true transition probabilities for all states and decisions are known, and also that the true expected immediate returns are given. In most empirical studies, these values are not known, and existing estimates may be more or less different from the true values, partly because they are estimated from samples of limited sizes, and partly because the stochastic models used for estimation purposes may contain specification errors. It should also be noted that most parameters are estimated from historical data while replacement models for practical use in principle should be based on parameters which will be realized in the future. The derived policies are not the true optimal policies, but the policies which would be optimal <u>if</u> the estimated parameter values had been the true ones. The planning situation can be characterized as one of subjective risk, where the true probability distributions are unknown but it is decided to act <u>as if</u> the estimated parameters are the true ones.

The problem is conceptually difficult. Intuitively, there seems to be room for improvements in the theoretical apparatus applied. One possible line of development might be to integrate the estimation procedure and the optimizing procedure. Thus, not only the parameter estimates but also the degree of reliability of the estimates should be taken into account when seeking the policy which is optimal under the less than perfect information available. However such methods are not available today when it comes to handling such a complicated decision problem as is dealt with in this study.

A third limitation is related to the simplifications made necessary by computational limitations and by the discrete nature of the general model. It would have been desirable to include more state variables in the model than the three which have been used, and to define smaller class intervals for the state variables for production and for calving interval than have been used. Even if the estimated parameters had been the true ones, the derived policies are optimal only in the sense of being the best ones which can be derived for a dairy producer who, with all the information on a cow which is available to him, uses only the limited information which is available from knowledge of the present state of the process.

For example, a dairy producer who is a member of a dairy herd improvement association knows at the time of decisionmaking 305 days production of milk and milk fat for all previous lactations of a cow, as well as production for each separate month of the present lactation up to the moment of decision-making. The replacement model developed here assumes, however, that among all this information he uses only the information which is contained in knowledge of a discrete production class. The production class is defined according to a production variable which again is defined so as to condense all available information in as useful a way as possible. It is evident, however, that some loss of information will take place. This may be especially important with respect to predictions of production for the remaining part of the current lactation. For cows with long calving intervals, a quite large part of expected production for the current stage will be produced in the remaining part of the present lactation. This part of expected production can be predicted with much higher precision if we know more details about production during the same lactation. The replacement models does not permit use of this information.

A fourth limitation is due to the restrictions imposed on the production variables in order to satisfy the Markov requirement.

The Markov requirement raises another problem which conceptually is very difficult. It was shown in chapter IV that the given method for defining production variables gives a sequence of variables which does satisfy the conditions of a Markov process. Dr. Oscar R. Burt has brought my attention to another problem: Even if Markov dependence in the production variables has been reached, the same is not true for the expected immediate economic returns. Thus, the expected immediate economic return for a given stage does not depend on the present state only, but on which previous states the process has been in as well. The author has not been able to reach safe conclusions about what effect this may have on the validity of the optimization results. Strictly speaking, Markovian dependence in expected immediate economic returns in necessary for validity of the dynamic programming functional equation. With the definition of production variables used in this study, dependence between immediate economic returns and production variables prior to the last one is low. Therefore, we can probably say that we have a good approximate decision rule based on the information used, even if the strict mathematical requirements are not exactly satisfied.

E. Practical Usefulness of the Model

It is pointed out above that the replacement policies derived can be said to be "optimal" only in a more limited sense. From a practical point of view, we may ask the question: Are other available methods for deriving replacement policies likely to give better results?

Since this is the only known study which in an explicit way attempts to deal with the replacement problem without ignoring the many important considerations which have been included in the model, the only practical alternative at the present seems to be to rely solely on an intuitive thought process.

Even if a dairy producer uses more intuitive methods to derive a replacement policy, the limitations imposed by less than perfect knowledge of parameter values would still be present. On the other hand, he would not have to restrict his informations to less than what is really available, and his results would not suffer from the fact that expected return estimates do not meet the Markov specifications.

A decision model of the type developed in this study gives the optimal solution to the stated problem with mathematical certainty. In order to reach this degree of mathematical precision and unbiasedness, however, we have had to sacrifice available information and to formulate the decision problem in such a way that the solution is optimal in a more limited sense. An intuitive thought process leading to a replacement policy would not be restricted by the same simplifications, but on the other hand is limited by the imperfect abilities of the human mind to perform extremely complicated logical deductions.

If we are interested in practical useful results, can we assume that the first method generally leads to better results than the last? The question is relevant not only to this study but to many of the problems for which solution is attempted by operational research methods. We may attempt to answer the question either through extensive empirical explorations or through sheer judgement and intuitive "feeling", and the answer may vary with the type of problem faced.

In the given case, it appears to the author that the problem faced is so complicated that an exact model of the type developed here, even with its limitations, is likely to give better results than what can be arrived at by judgement. On the other hand, other, and yet not developed, models may be able to overcome some of the imperfections of the present model.

When the costs of deriving replacement policies in the way described here are considered, the optimal choice of method may be another. The results suggest that moderate deviations from the optimal culling policies give only modest decreases in present values of state 1, which again corresponds to very minor decreases in net economic return per cow per year. It may well be that a dairy producer, using intuitive methods, is able to develop replacement policies which are so close to the optimal ones that it will not pay to improve them further by use of more sophisticated methods like the one described here. It is also possible that instead of constructing separate models for individual herds, it may be helpful to dairy producers if a model is constructed and optimal policies developed for only one herd. This herd may be an imaginary one and defined so that it is representative for a given breed and a given dairy district. Possibly, dairy producers may be able to derive enough useful information from such results that they can make the necessary adjustments for their own herds by intuitive methods.

Another practical objection to the method should be mentioned: In order to define the process so as to get a Markov process, variables for production were defined in a rather complicated way. A dairy producer is likely to want a simpler method for classification of cows according to level of production than the method which is used here. For practical use, it should be considered how simplifications can be made without decreasing the precision of the model too much.

F. Need for Further Research

If the discussion is limited to dairy cow replacement models which can be formulated within the Markovian dynamic programming framework, the following questions should get more attention:

1. Choice of State Variables

Will the introduction of other state variables improve the precision of replacement decisions substantially? Variables which possibly may be important are: a variable for production during the current lactation; a variable for body weight; a variable for degree of mastitis; a variable for pedigree promise; a variable for season of calving. It should be noted, however, that introduction of some of these variables possibly may introduce difficult problems of parameter estimation and of satisfying the Markov requirement.

2. Herd Differences in Parameter Values

How large herd differences can we expect with respect to various parameter values? Results from the two herds examined in this study give only some indications, and more herds should be examined to give more complete answers.

3. Consequencies of Herd Differences on Optimal Policies and Present Values

Will differences in parameter values between herds lead to important differences in optimal replacement policies? Will a policy which is optimal for one set of parameter values lead to important losses when this policy is imposed on a herd with other parameter values? These questions can be examined in an experimental way by introducing changes in one variable or in the probability distribution of one variable at a time in an existing replacement model.

4. Importance of Genetic Improvement over Time

Can the models be modified to include consideration of genetic improvement within the limitations set by computer capacity? Will consideration of genetic improvement lead to substantial changes in the optimal replacement policies?

LITERATURE CITED

Allen, R. G. D. Mathematical Analysis for Economists. London: Macmillan & Co., 1960.

Anderson, T. W. An Introduction to Multivariate Statistical Analysis. New York: John Wiley & Sons, 1958.

________. "Determination of the Order of Dependence in Normally Distributed Time Series," Time Series Analysis, edited by M. Rosenblatt. New York: John Wiley & Sons, 1963, pp. 425-46.

Arave, Clive W. "A Study of Genetic Change in Twelve California Dairy Herds," Unpublished Ph.D. dissertation in Genetics, Graduate Division, University of California, Davis, 1962.

Arave, C. W., Laben, R. C., and Mead, S. W. "Measurement of Genetic Change in Twelve California Dairy Herds," Journal of Dairy Science, XLVII (March, 1964), pp. 278-83.

Asdell, S. A. "Variations in Amount of Culling from D.H.I.A. Herds," Journal of Dairy Science, XXXIV (June, 1951), pp. 529-35.

Bellman, Richard. Dynamic Programming. Princeton: Princeton University Press, 1959.

Berry, J. C. "Reliability of Averages of Different Numbers of Lactation Records for Comparing Dairy Cows," Journal of Dairy Science, XXVIII (May, 1945), pp. 355-66.

Buch, N. C., Tyler, W. J., and Casida, L.E. "Variations in Some Factors Affecting the Length of the Calving Interval," Journal of Dairy Science, XLII (February, 1959), pp. 298-304.

Burt, Oscar R. "The Economics of Conjunctive Use of Ground and Surface Water," Hilgardia, XXXVI (December, 1964), pp. 31-109.

________. "Optimal Replacement Under Risk," Journal of Farm Economics, XLVII (May, 1965), pp. 324-46.

Burt, Oscar R., and Allison, John R. "Farm Management Decisions with Dynamic Programming," Journal of Farm Economics, XLV (February, 1963), pp. 121-36.

Castle, Olive M., and Searle, S. R. "Repeatability of Dairy Cow Butterfat Records in New Zealand," Journal of Dairy Science, XL (October, 1957), pp. 1277-83.

Clark, R. Davis, and Touchberry, R. W. "Effect of Body Weight and Age at Calving on Milk Production in Holstein Cattle," Journal of Dairy Science, XLV (December, 1962), pp. 1500-1510.

Clarke, D. C., and Hassler, J. B. Pricing Fat and Skim Components of Milk. Agricultural Experiment Station Bulletin 737. Berkeley, California: University of California, 1953.

Dantzig, George B. Linear Programming and Extensions. Princeton: Princeton University Press, 1963.

Dreyfus, Stuart E. "A Generalized Equipment Replacement Study," Journal of the Society for Industrial and Applied Mathematics, VIII (September, 1960).

Dunbar, R. S., and Henderson, C. R. "Heritability of Fertility in Dairy Cattle," Journal of Dairy Science, XXXVI (October, 1953), pp. 1063-71.

d'Epenoux, F. "A Probabilistic Production and Inventory Problem," Management Science, X (October, 1963), pp. 98-108.

Faris, J. Edwin. "Analytical Techniques Used in Determining the Optimum Replacement Pattern," Journal of Farm Economics, XLII (November, 1960), pp. 755-66.

Faustman, Martin. "Berechnung des Wertes, welchen Waldboden, sowie noch nicht haubare Holzbestände für die Waldwirtschaft besitzen," Allg. Forst und Jagd Zeitung, XXV (1849), pp. 441-55.

Frick, G. E., and Henry, W. F. Production Efficiency on New England Dairy Farms. V. Adjustments in Obtaining Dairy Herd Replacements. Agricultural Experiment Station Bulletin. Durham, New Hampshire: University of New Hampshire, August, 1956.

Fritz, G. R., McGilliard, L. D., and Madden, D. E. "Environmental Influences of Regression Factors for Estimating 305-Day Production from Part Lactations," Journal of Dairy Science, XLIII (August, 1960), pp. 1108-17.

Gaffney, M. Mason. Concepts of Financial Maturity of Timber and Other Assets. Department of Agricultural Economics A. E. Information Series No. 62. Raleigh, N.C.: North Carolina State College, 1957.

Hadley, G. Linear Algebra. Readings, Massachusett: Addison-Wesley Publishing Co., 1961.

________. Nonlinear and Dynamic Programming. Readings, Massachusett: Addison-Wesley Publishing Co., 1964.

Heady, Earl O., et al. "Milk Production Functions Incorporating Variables for Cow Characteristics and Environment," Journal of Farm Economics, XLVI (February, 1964), pp. 1-19.

Howard, Ronald A. Dynamic Programming and Markov Processes. 2d printing. Cambridge, Massachusett: The M. I. T. Press, 1960.

Inskeep, E. K., Tyler, W. J., and Casida, L. E. "Hereditary Variation in Conception Rates of Holstein-Friesian Cattle," Journal of Dairy Science, XLIV (October, 1961), pp. 1857-62.

Jenkins, Keith B., and Halter, Albert N. A Multi-State Stochastic Replacement Decision Model. Agricultural Experiment Station Technical Bulletin No. 67. Corvallis, Oregon: Oregon State University, 1963.

Johansson, Ivar, and Hansson, Artur. "Causes of Variation in Milk and Butterfat Yields of Dairy Cows," Kungl. Lantbruksakademiens Tidsskrift Journal of the Royal Swedish Academy, LXXIX (No. 6 1/2, 1940).

Kemeny, John G., and Snell, J. Laurie. Finite Markov Chains. Princeton: D. Van Nostrand Co., 1960.

Kendrick, J. F. Standardizing Dairy-Herd-Improvement-Association Records in Proving Sires. U.S.D.A. Agricultural Research Service, Dairy Husbandry Research Branch, ARS-52-1.

Knight, Frank H. Risk, Uncertainty and Profit. Boston: Houghton Mifflin Co., 1921.

Laben, R. C., and Herman, H. A. Genetic Factors Affecting Milk Production in a Selected Holstein-Friesian Herd. Agricultural Experiment Station Research Bulletin No. 459. Columbia, Missouri: University of Missouri, 1950.

Legates, J. E., and Grinnels, C. D. "Genetic Relationships in Resistance to Mastitis in Dairy Cattle," Journal of Dairy Science, XXXV (October, 1952), pp. 829-33.

Lush, Jay L. "Inheritance of Susceptibility to Mastitis," Journal of Dairy Science, XXXIII (February, 1950), pp. 121-25.

Lush, Jay L., and Shrode, Robert R. "Changes in Milk Production with Age and Milking Frequency," Journal of Dairy Science, XXXIII (May, 1950), pp. 338-57.

Madden, D. E., Lush, J. L., and McGilliard, L. D. "Relations between Parts of Lactation and Producing Ability of Holstein Cows," Journal of Dairy Science, XXXVIII (November, 1955), pp. 1264-71.

Madden, D. E., McGilliard, L. D., and Ralston, N. P. "Relations between Test-Day Milk Production of Holstein Cows," Journal of Dairy Science, XLII (February, 1959), pp. 319-26.

Marschak, J. "Money and the Theory of Assets," Econometrica, VI (October, 1938), p. 324.

Menge, A. C., et al. "Variation and Association among Postpartum Reproduction and Production Characteristics in Holstein-Friesian Cattle," Journal of Dairy Science, XLV (February, 1962), pp. 233-41.

Mitchell, R. G., Corley, E. L., and Tyler, W. J. "Heritability, Phenotypic and Genetic Correlations between Type Ratings and Milk and Fat Production in Holstein-Friesian Cattle," Journal of Dairy Science, XLIV (August, 1961), pp. 1502-10.

Morrison, Frank B. Feeds and Feeding. 22nd edition. Ithaca, New York: The Morrison Publishing Co., 1956.

O'Bleness, G. V., and Van Vleck, L. D., "Reasons for Disposals of Dairy Cows from New York Herds," Journal of Dairy Science, XLV (September, 1962) pp. 1087-93.

Parzen, Emanuel. Stochastic Processes. San Francisco: Holden-Day Inc., 1962.

Preinreich, Gabriel A. D. "The Economic Life of Industrial Equipment," Econometrica, VIII (January, 1940), pp. 12-44.

Rendel, J. M., et al. "The Inheritance of Milk Production Characteristics," The Journal of Agricultural Science, XLVIII (April, 1957), pp. 426-31.

Scheffé, Henry. The Analysis of Variance. New York: John Wiley & Sons, 1959.

Schrode, R. R., et al. "Changes in Repeatability with Changes in Herd Environment," Journal of Dairy Science, XLIII (September, 1960), p. 1343.

Searle, S. R., and Henderson, C. R. "Establishing Age-Correction Factors Related to the Level of Herd Production," Journal of Dairy Science, XLII (May, 1959), pp. 824-35.

Shultis, Arthur, Forker, Olan D., and Appleman, Robert D. California Dairy Farm Management. Agricultural Experiment Station Circular No. 417 (revised). Berkeley, California: University of California, January, 1963.

Smith, Vernon L. Investment and Production. Cambridge, Massachusett: Harvard University Press, 1961.

Snedecor, George W. Statistical Methods. 5th ed. Ames, Iowa: Iowa State College Press, 1956.

Terborgh, George, Dynamic Equipment Policy. New York: McGraw-Hill Book Co., 1949.

Tintner, Gerhard. "A Contribution to the Nonstatic Theory of Production," in Studies in Mathematical Economics and Econometrics, edited by O. Lange, F. McIntyre, and T. O. Yntema. Chicago: University of Chicago Press, 1942, pp. 92-109.

Trimberger, George W., and Davis, H. P. "Predictability of Breeding Efficiency in Dairy Cattle from Their Previous Conception Rate and from Their Heredity," Journal of Dairy Science, XXVIII (September, 1945), pp. 659-669.

Van Vleck, L. D., and Henderson, C. R. "Estimates of Genetic Parameters of Some Functions of Part Lactation Milk Records," Journal of Dairy Science, XLIV (June, 1961), p. 1078.

Van Vleck, L. D., and Henderson, C. R. "Regression Factors for Extending Part Lactation Milk Records," Journal of Dairy Science, XLIV (June, 1961), pp. 1085-92.

Western Dairy Journal, February, 1961 - September, 1964.

Winder, J. W. L., and Trant, G. I. "Comments on 'Determining the Optimum Replacement Pattern'," Journal of Farm Economics, XLIII (November, 1961), pp. 939-951.

Young, C. W., Legates, J. E., and Lecce, J. G. "Genetic and Phenotypic Relationships between Clinical Mastitis, Laboratory Criteria, and Udder Height," Journal of Dairy Science, XLIII (January, 1960), pp. 54-62.

Appendix A

TABLE A.1. Age-corrected herd averages of pounds 305 days fat-corrected milk

Year	ME-herd	MA-herd	Year	ME-herd	MA-herd
1931	7210	8210	1947	10520	11800
1932	7740	8070	1948	11070	11650
1933	8580	8900	1949	11030	11800
1934	8410	8340	1950	10700	10320
1935	8900	10040	1951	10360	10570
1936	8790	10210	1952	9000	10110
1937	8660	10490	1953	9220	10260
1938	7850	9950	1954	9530	10210
1939	8370	9710	1955	9100	11510
1940	8360	9730	1956	9920	11480
1941	9230	9860	1957	10050	10850
1942	9860	10050	1958	10730	11400
1943	10850	11370	1959	11360	11210
1944	10810	11830	1960	12120	11290
1945	10310	11650	1961	12830	10730
1946	10170	11820	1962	13030	10920

[a] Only the first five lactations of each cow were used when calculating the herd average. Lactation records were excluded if the lactation length was less than 260 days. The year averages given are for lactations initiated in the given year. For example, if a cow calved in December, 1953, the following lactation was counted as a 1953-lactation. Age-correction factors used are given on page 144.

TABLE A.2. Regression analysis of pounds 305 days FCM: Herd average excluded as explanatory variable; ME-herd

Lact. No.	No. obs.	Regression coefficient estimates [a]								Condi-tional stand. dev.
		Constant term	Explanatory variables							
			Previous calving interval [b]	Present calving interval [b]	u_1	u_2	u_3	u_4	u_5	
1st	702	6,281.48	. .	5.3539 (1.6157)						1,857.97
2nd	490	4,420.14	7.8265 (1.7285)	4.7593 (1.7294)	.6766 (.0439)					1,632.03
3rd	342	5,998.87	5.2606 (2.3282)	4.9908 (2.2475)	.1494 (.0677)	.6061 (.0615)				1,746.99
4th	239	4,432.18	7.8218 (3.2377)	6.9678 (2.7341)	.2830 (.0882)	.1202 (.0878)	.5177 (.0732)			1,829.58
5th	157	5,768.97	5.8657 (3.9178)	5.5363 (3.8130)	.0630 (.1200)	.0757 (.1137)	.0718 (.1060)	.4705 (.0966)		1,934.28
6th	101	6,053.49	2.6308 (3.9236)	7.4008 (3.9882)	.0930 (.1074)	.0918 (.1089)	- .0896 (.1018)	.0590 (.1291)	.6715 (.1092)	1,477.75

a/ The figures in parentheses are standard deviations of estimates.

b/ Calving intervals are measured in days. The value was set equal to 440 whenever the observed calving interval was 440 days or longer.

TABLE A.3. Regression analysis of pounds 305 days FCM: Herd average included as explanatory variable; ME-herd

Lact. No.	No. obs.	Regression coefficient estimates[a]									Condi-tional stand. dev.
		Constant term	Explanatory variables								
			Herd average	Previous calving interval[b]	Present calving interval[b]	u_1	u_2	u_3	u_4	u_5	
1st	702	-2,186.31	0.8410 (.0466	. . . .	4.9287 (1.3162)						1,513.35
2nd	490	-5,312.89	.9304 (.0530)	7.6698 (1.5664)	4.9725 (1.5636)	0.4126 (.0478)					1,477.17
3rd	342	-5,125.52	1.0506 (.0760)	7.0840 (2.0966)	3.5715 (2.0422)	.0838 (.0647)	.4213 (.0613)				1,582.90
4th	239	-6,178.98	1.0333 (.0908)	7.5552 (2.8659)	6.2900 (2.4442)	.2761 (.0823)	.1993 (.0812)	.3451 (.0719)			1,636.03
5th	157	-3,516.72	.8201 (.1200)	5.6631 (3.6646)	7.0162 (3.6510)	.1380 (.1213)	.2089 (.1138)	.0679 (.1037)	.2941 (.1037)		1,827.64
6th	101	-4,426.80	.8945 (.1084)	6.7019 (3.6711)	5.9784 (3.5568)	.1294 (.1062)	.1731 (.1033)	- .0906 (.0916)	.1334 (.1239)	.4310 (.1048)	1,301.59

a/ The figures in parentheses are standard deviations of estimates.

b/ Calving intervals are measured in days. The value was set equal to 440 whenever the observed calving interval was 440 days or longer.

TABLE A.4. Regression analysis of pounds 305 days FCM: Herd average excluded as explanatory variable; MA-herd

Lact. No.	No. obs.	Regression coefficient estimates[a]								Conditional stand. dev.
		Constant term	Explanatory variables							
			Previous calving interval[b]	Present calving interval[b]	u_1	u_2	u_3	u_4	u_5	
1st	654	6,609.07		6.4309 (1.7622)						1,772.02
2nd	474	3,741.58	13.4477 (2.1339)	2.2788 (1.9745)	.6767 (.0486)					1,756.04
3rd	326	6,025.52	10.9907 (2.5666)	1.0595 (2.5164)	.0485 (.0725)	.4783 (.0613)				1,743.45
4th	210	10,077.03	2.8660 (4.0299)	-1.5049 (3.7047)	.0852 (.1165)	.1981 (.0986)	.3733 (.0858)			2,048.28
5th	139	8,743.78	5.1057 (4.6094)	.2681 (4.4177)	.0820 (.1409)	.0891 (.1329)	.0655 (.1230)	.2498 (.0959)		2,028.65
6th	82	8,526.84	-2.5706 (5.7410)	6.8800 (5.0229)	- .0682 (.1365)	.1685 (.1417)	- .0573 (.1387)	.1805 (.1017)	.4879 (.0986)	1,590.37

a/ The figures in parentheses are standard deviations of estimates.

b/ Calving intervals are measured in days. The value was set equal to 440 whenever the observed calving interval was 440 days or longer.

TABLE A.5. Regression analysis of pounds 305 days FCM: Herd average included as explanatory variable; MA-herd

Lact. No.	No. obs.	Regression coefficient estimates[a]									Conditional stand. dev.
		Constant term	Explanatory variables								
			Herd average	Previous calving interval[b]	Present calving interval[b]	u_1	u_2	u_3	u_4	u_5	
1st	654	- 1,690.83	0.8467 (.0765)	. .	4.0808 (1.6317)						1,626.90
2nd	474	- 6,172.06	1.0189 (.1016)	10.3141 (2.0793)	2.4746 (1.9001)	.6011 (.0316)					1,688.99
3rd	326	- 4,258.18	.9250 (.1299)	10.4267 (2.4561)	2.1598 (2.4132)	.0631 (.0752)	.4476 .0625)				1,669.97
4th	210	- 4,026.71	1.2447 (.1877)	4.3257 (3.7602)	-1.3802 (3.4987)	.0428 (.1249)	.2347 (.0962)	.3280 (.0845)			1,930.74
5th	139	- 4,095.33	1.0728 (.2343)	6.0969 (4.4441)	2.3904 (4.1641)	.1801 (.1524)	.1011 (.1332)	.1385 (.1211)	.1397 (.1043)		1,929.51
6th	82	- 2,170.45	1.0178 (.2381)	.4463 (5.5322)	3.2207 (4.9430)	.0304 (.1583)	.2805 (.1425)	- .1143 (.1337)	.0847 (.1087)	.4482 (.1017)	1,524.78

a/ The figures in parentheses are standard deviations of estimates.

b/ Calving intervals are measured in days. The value was set equal to 440 whenever the observed calving interval was 440 days or longer.

TABLE A.6. Estimated parameters in a conditional multivariate distribution of 305 days FCM a/
ME-herd

Lactation No.	Standard deviation	Correlation coefficients				
		Lactation No.				
		2	3	4	5	6
		Herd average allowed to vary				
1	1,857.97	0.6102	0.4598	0.4902	0.3564	0.3651
2	2,060.02		.6272	.5255	.3911	.3797
3	2,260.72			.6324	.4365	.3438
4	2,478.58				.5846	.4970
5	2,408.66					.7578
6	2,309.13					
		Herd average kept constant				
1	1,513.35	0.3894	0.2235	0.3522	0.2751	0.3424
2	1,603.74		.4155	.3812	.3317	.3863
3	1,744.94			.4311	.2669	.1976
4	1,921.50				.3982	.4066
5	2,052.56					.6200
6	1,758.93					

a/ The variables kept constant are lengths of the preceding and the present calving intervals, in the lower part of the table also the herd average. The estimates in this table are derived from the estimates in tables A.2. and A.3. respectively, by the method described in appendix B.

TABLE A.7. Estimated parameters in a conditional multivariate distribution of 305 days FCM a/
MA-herd

Lactation No.	Standard deviation	Correlation coefficients				
		Lactation No.				
		2	3	4	5	6
		Herd average allowed to vary				
1	1,772.02	0.5639	0.3226	0.2716	0.2071	0.1817
2	2,126.36		.5212	.3869	.2588	.3203
3	2,044.47			.4414	.2451	.2293
4	2,336.50				.3448	.4043
5	2,191.28					.6001
6	2,088.89					
		Herd average kept constant				
1	1,626.90	0.5011	0.2828	0.2204	0.2569	0.2772
2	1,951.66		.4840	.3685	.2811	.4073
3	1,911.12			.4018	.2719	.1982
4	2,160.99				.2629	.2828
5	2,075.72					.5559
6	1,952.78					

a/ The variables kept constant are lengths of the preceding and the present calving intervals, in the lower part of the table also the herd average. The estimates in this table are derived from the estimates in tables A.4. and A.5. respectively, by the method described in appendix B.

TABLE A.8. Expected 305 days FCM and estimated coefficients of variation

Lactation No.	Expected 305 days FCM[a/] in pounds		Estimated coefficients of variation[b/]	
	ME-herd	MA-herd	ME-herd	MA-herd
1	8398	9165	18.0	17.8
2	9161	9992	17.5	19.5
3	9810	10801	17.8	19.5
4	9826	10690	19.6	20.2
5	9851	10956	21.2	18.9
6	9700	10382	18.1	18.8

a/ Expected 305 days FCM's in pounds are derived from the regression equations in tables A.3 and A.5 respectively, and the following values for the explanatory variables:

	ME-herd	MA-herd
Herd average	10288	10894
Preceding calving interval	385	395
Present calving interval	392	400
All u-values	0	0

Since the u-values are set equal to zero, the expected values for 305 days FCM are those expected in a case where no culling for low production takes place. The other explanatory variables are close to the observed herd averages, which in the sample data varied a little between subsamples for different lactation numbers.

b/ The "coefficients of variation" are derived by dividing the estimated standard deviations in the lower parts of tables A.6 and A.7 respectively with the expected 305 days FCM from this table.

TABLE A.9. Definition of production variables v_f's, a sequence of which form a Markov process, and comparison with variables v_f^*'s based on least squares regression coefficients

ME-herd

$$v_1 = 0.41263\ u_1$$
$$v_1 = 0.41263\ u_1$$

$$v_2 = 0.08383\ u_1 + 0.42125\ u_2$$
$$v_2 = 0.08383\ u_1 + 0.42125\ u_2$$

$$v_3 = 0.06134\ u_1 + 0.30824\ u_2 + 0.34509\ u_3$$
$$v_3 = 0.27608\ u_1 + 0.19925\ u_2 + 0.34509\ u_3$$

$$v_4 = -0.03344\ u_1 + 0.18132\ u_2 + 0.16712\ u_3 + 0.29405\ u_4$$
$$v_4 = 0.13796\ u_1 + 0.20893\ u_2 + 0.06785\ u_3 + 0.29405\ u_4$$

$$v_5 = -0.08790\ u_1 + 0.06412\ u_2 + 0.11286\ u_3 + 0.12329\ u_4 + 0.43102\ u_5$$
$$v_5 = 0.12940\ u_1 + 0.17314\ u_2 - 0.09059\ u_3 + 0.13336\ u_4 + 0.43102\ u_5$$

MA-herd

$$v_1 = 0.60108\ u_1$$
$$v_1 = 0.60108\ u_1$$

$$v_2 = 0.06310\ u_1 + 0.44762\ u_2$$
$$v_2 = 0.06310\ u_1 + 0.44762\ u_2$$

$$v_3 = 0.03370\ u_1 + 0.23907\ u_2 + 0.32803\ u_3$$
$$v_3 = 0.04275\ u_1 + 0.23466\ u_2 + 0.32803\ u_3$$

$$v_4 = 0.01747\ u_1 + 0.13354\ u_2 + 0.18238\ u_3 + 0.13974\ u_4$$
$$v_4 = 0.18013\ u_1 + 0.10108\ u_2 + 0.13853\ u_3 + 0.13974\ u_4$$

$$v_5 = -0.06340\ u_1 + 0.08713\ u_2 + 0.11878\ u_3 + 0.07595\ u_4 + 0.44815\ u_5$$
$$v_5 = 0.03036\ u_1 + 0.28047\ u_2 - 0.11427\ u_3 + 0.08466\ u_4 + 0.44815\ u_5$$

a/ The u's are the deviations in pounds between observed values of 305 days FCM and the "within lactation number" regression lines, when length of preceding and present calving intervals and herd average are included as explanatory variables in the regression equations. It would give equivalant results to define production variables as linear combinations of observed 305 days FCM for each lactation, after these observed values had been corrected for variation in the given explanatory variables. The coefficients in the linear combinations would have been the same as in this table.

TABLE A.10. Unconditional and conditional variances of u_f [a/]

Lactation No. (f)	$V(u_f)$	$V(u_f \mid v_{f-1})$	$V(u_f \mid v^*_{f-1})$
ME-herd			
1	2,290,233		
2	2,571,982	2,182,039	2,182,039
3	3,044,812	2,505,572	2,505,572
4	3,692,158	2,768,531	2,676,596
5	4,212,985	3,422,019	3,340,279
6	3,093,843	1,901,415	1,694,142
MA-herd			
1	2,646,816		
2	3,808,963	2,852,675	2,852,675
3	3,652,381	2,788,793	2,788,793
4	4,669,856	3,727,921	3,727,753
5	4,308,594	3,779,859	3,722,992
6	3,813,359	2,545,666	2,324,958

a/ The u's are deviations in pounds between observed values of 305 days FCM and the "within lactation number" regression lines, when length of preceding and present calving interval and herd average are included as explanatory variables in the regression equations.

TABLE A.11. Production class intervals a/

Class No.	ME-herd			MA-herd		
	Class interval	Mean value	Freq. b/	Class interval	Mean value	Freq. b/
			v_1			
1	+ ∞ to + 1000.0	+ 1275.5	.0546	+ ∞ to + 1500.0	+ 1930.9	.0625
2	+ 1000.0 to + 600.0	+ 780.0	.1137	+ 1500.0 to + 900.0	+ 1167.3	.1162
3	+ 600.0 to + 200.0	+ 389.9	.2061	+ 900.0 to + 300.0	+ 583.6	.2008
4	+ 200.0 to − 200.0	0	.2512	+ 300.0 to − 300.0	0	.2410
5	− 200.0 to − 600.0	− 389.9	.2061	− 300.0 to − 900.0	− 583.6	.2008
6	− 600.0 to − 1000.0	− 780.0	.1137	− 900.0 to − 1500.0	− 1167.3	.1162
7	− 1000.0 to − ∞	− 1275.5	.0546	− 1500.0 to − ∞	− 1930.9	.0625
			v_2			
1	+ ∞ to + 1248.9	+ 1556.2	.0445	+ ∞ to + 1381.5	+ 1795.8	.0686
2	+ 1248.9 to + 749.3	+ 965.7	.1093	+ 1381.5 to + 828.9	+ 1076.7	.1176
3	+ 749.3 to + 249.8	+ 482.7	.2131	+ 828.9 to + 276.3	+ 538.3	.1969
4	+ 249.8 to − 249.8	0	.2662	+ 276.3 to − 276.3	0	.2338
5	− 249.8 to − 749.3	− 482.7	.2131	− 276.3 to − 828.9	− 538.3	.1969
6	− 749.3 to − 1248.9	− 965.7	.1093	− 828.9 to − 1381.5	− 1076.7	.1176
7	− 1248.9 to − ∞	− 1556.2	.0445	− 1381.5 to − ∞	− 1795.8	.0686

TABLE A.11. Continued

Class No.	ME-herd Class interval	ME-herd Mean value	ME-herd Freq.[b/]	MA-herd Class interval	MA-herd Mean value	MA-herd Freq.[b/]
			v_3			
1	+ ∞ to + 1344.7	+ 1785.9	.0809	+ ∞ to + 1191.0	+ 1667.7	.1099
2	+ 1344.7 to + 806.8	+ 1051.1	.1197	+ 1191.0 to + 714.6	+ 938.5	.1209
3	+ 806.8 to + 268.9	+ 525.6	.1892	+ 714.6 to + 238.2	+ 469.3	.1723
4	+ 268.9 to – 268.9	0	.2204	+ 238.2 to – 238.2	0	.1938
5	– 268.9 to – 806.8	– 525.6	.1892	– 238.2 to – 714.6	– 469.3	.1723
6	– 806.8 to – 1344.7	– 1051.1	.1197	– 714.6 to – 1191.0	– 938.5	.1209
7	– 1344.7 to – ∞	– 1785.9	.0809	– 1191.0 to – ∞	– 1667.7	.1099
			v_4			
1	+ ∞ to + 1395.5	+ 1782.8	.0583	+ ∞ to + 1242.9	+ 1534.0	.0437
2	+ 1395.5 to + 837.3	+ 1084.5	.1149	+ 1242.9 to + 745.8	+ 952.7	.1089
3	+ 837.3 to + 279.1	+ 542.1	.2037	+ 745.8 to + 248.6	+ 476.3	.2136
4	+ 279.1 to – 279.1	0	.2462	+ 248.6 to – 248.6	0	.2676
5	– 279.1 to – 837.3	– 542.1	.2037	– 248.6 to – 745.8	– 476.3	.2136
6	– 837.3 to – 1395.5	– 1084.5	.1149	– 745.8 to – 1242.9	– 952.7	.1089
7	– 1395.5 to – ∞	– 1782.8	.0583	– 1242.9 to – ∞	– 1534.0	.0437

TABLE A.11. Continued

Class No.	ME-herd			MA-herd		
	Class interval	Mean value	Freq.[b]	Class interval	Mean value	Freq.[b]
			v_5			
1	+ ∞ to + 1582.2	+ 2072.6	.0737	+ ∞ to + 1643.5	+ 2154.8	.0708
2	+ 1582.2 to + 949.3	+ 1233.0	.1186	+ 1643.5 to + 986.1	+ 1283.6	.1198
3	+ 949.3 to + 316.4	+ 616.5	.1937	+ 986.1 to + 328.7	+ 640.0	.1946
4	+ 316.4 to − 316.4	0	.2280	+ 328.7 to − 328.7	0	.2296
5	− 316.4 to − 949.3	− 616.5	.1937	− 328.7 to − 986.1	− 640.0	.1946
6	− 949.3 to − 1582.2	− 1233.0	.1186	− 986.1 to − 1643.5	− 1283.6	.1198
7	− 1582.2 to − ∞	− 2072.6	.0737	− 1643.5 to − ∞	− 2154.8	.0708

[a] The intervals are defined in terms of v-values, where the v-variables are defined as in table A.9. Class intervals are arbitrarily defined and are chosen so as to facilitate derivation of transition probabilities. Class means and relative frequencies are derived based on the assumption about normal distribution.

[b] Relative frequencies of cows falling in the given production classes in a population where no culling for low production has taken place.

TABLE A.12. Transition probabilities with respect to production classes; ME-herd [a]

From class	To class 1	2	3	4	5	6	7
				From v_1 to v_2			
1	.2387	.2896	.2757	.1458	.0427	.0069	.0006
2	.1115	.2259	.3104	.2328	.0954	.0212	.0028
3	.0540	.1553	.2867	.2889	.1589	.0477	.0085
4	.0229	.0925	.2294	.3104	.2294	.0925	.0229
5	.0085	.0477	.1589	.2889	.2867	.1553	.0540
6	.0028	.0212	.0954	.2328	.3104	.2259	.1115
7	.0006	.0069	.0427	.1458	.2757	.2896	.2387
				From v_2 to v_3 [b]			
1	.7007	.2246	.0660	.0083	.0004	.0000	.0000
2	.2945	.3652	.2544	.0759	.0095	.0005	.0000
3	.0723	.2334	.3653	.2476	.0721	.0089	.0004
4	.0086	.0682	.2405	.3654	.2405	.0682	.0086
				From v_3 to v_4 [b]			
1	.4832	.3558	.1416	.0186	.0008	.0000	.0000
2	.1214	.3629	.3781	.1246	.0126	.0004	.0000
3	.0230	.1702	.4104	.3145	.0762	.0056	.0001
4	.0024	.0428	.2411	.4274	.2411	.0428	.0024
				From v_4 to v_5 [b]			
1	.4648	.2850	.1759	.0611	.0118	.0014	.0000
2	.2041	.2823	.2897	.1637	.0508	.0086	.0008
3	.0800	.1903	.3015	.2631	.1265	.0334	.0052
4	.0237	.0935	.2287	.3082	.2287	.0935	.0237

TABLE A.12. Continued

---

a/ The transition probabilities in this table are derived from the covariance matrix estimate represented in the lower part of table A.6 by using the definitions of v-variables in table A.9 and the definition of production classes in table A.10. The method for derivation of transition probabilities is explained on pp. 90 - 93.

b/ Because of the symmetry, transition probabilities from classes 5, 6, and 7 can be read from the rows for classes 3, 2, and 1, respectively, by reading the rows in opposite direction. This is seen from the transition probability matrix for the transition from v_1 to v_2, which is reproduced in complete form in this table.

TABLE A.13. Transition probabilities with respect to production classes; MA-herd [a]

From class	To class						
	1	2	3	4	5	6	7
	From v_1 to v_2						
1	.3425	.2756	.2258	.1137	.0350	.0066	.0008
2	.1667	.2379	.2813	.2010	.0867	.0225	.0039
3	.0821	.1708	.2712	.2600	.1506	.0527	.0126
4	.0348	.1034	.2202	.2832	.2202	.1034	.0348
5	.0126	.0527	.1506	.2600	.2712	.1708	.0821
6	.0039	.0225	.0867	.2010	.2813	.2379	.1667
7	.0008	.0066	.0350	.1137	.2258	.2756	.3425
	From v_2 to v_3 [b]						
1	.7115	.2012	.0726	.0135	.0012	.0000	.0000
2	.3200	.3277	.2422	.0911	.0173	.0017	.0000
3	.0984	.2297	.3277	.2381	.0881	.0164	.0016
4	.0171	.0851	.2339	.3278	.2339	.0851	.0171
	From v_3 to v_4 [b]						
1	.3797	.4899	.1276	.0028	.0000	.0000	.0000
2	.0183	.3526	.5533	.0751	.0007	.0000	.0000
3	.0006	.0683	.5395	.3710	.0206	.0000	.0000
4	.0000	.00405	.1855	.6209	.1855	.00405	.0000
	From v_4 to v_5 [b]						
1	.4494	.2779	.1828	.0712	.0163	.0022	.0002
2	.2148	.2699	.2775	.1664	.0581	.0118	.0015
3	.0911	.1889	.2870	.2542	.1315	.0396	.0077
4	.0302	.0997	.2236	.2930	.2236	.0997	.0302

TABLE A.13. Continued

---

a/ The transition probabilities in this table are derived from the covariance matrix estimate represented in the lower part of table A.7, by using the definitions of v-variables in table A.9 and the definition of production classes in table A.10. The method for derivation of transition probabilities is explained on pp. 90-93.

b/ Because of symmetry, transition probabilities from classes 5, 6, and 7 can be read from the rows for classes 3, 2, and 1, respectively, by reading the rows in opposite direction. This is seen from the transition probability matrix for the transition from v_1 to v_2, which is reproduced in complete form in this table.

TABLE A.14. Estimated relationships between 305 days FCM and FCM for separate months when production is not influenced by a new pregnancy; ME-herd

Month of lactation	Lactation No.			
	1st		2nd - 6th	
	No. obs.	Regression	No. obs.	Regression[b/]
Incompl.	135	+ 1270 + 0.045 X[a/]	243	
1 - 3	135	+ 5740 + 0.287 X	243	
4	135	+ 960 + 0.097 X	243	
5	135	+ 730 + 0.092 X	243	
6	135	- 720 + 0.109 X	243	
7	120	- 950 + 0.104 X	218	
8	102	- 970 + 0.101 X	186	
9	82	- 1500 + 0.102 X	153	
10	68	- 1630 + 0.101 X	123	- 238 + 0.083 X
11	53	- 910 + 0.088 X	88	- 194 + 0.072 X
12	37	- 1600 + 0.090 X	55	+ 103 + 0.036 X
13	19	- 3190 + 0.103 X	41	+ 298 + 0.003 X

a/ X stands for 305 days FCM after correction for length of present calving interval.

b/ Some of the estimates belonging in this column were lost after the uses which should be made of them were finished.

TABLE A.15. Estimated regression coefficients for monthly FCM regressed on 305 days FCM for months when production is influenced by a new pregnancy: ME-herd, 1st. lactation

Month of lactation	Calving interval, days							
	320–350	351–380	381–411	412–441	442–472	473–502	503–533	534–
7	0.090							
8	0.072	0.111						
9	0.080	0.110	0.098					
10	0.072	0.025	0.109	0.066				
11	. .	. .	0.041	0.139	0.137			
12	. .	. .	. .	0.113	0.153	0.126		
13	. .	. .	. .	. .	0.134	0.095	0.013	
14	. .	. .	. .	. .	. .	0.016	0	0.074
15	. .	. .	. .	. .	. .	. .	0.010	0.066
16	. .	. .	. .	. .	. .	. .	. .	0

TABLE A.16. Estimated regression coefficients for monthly FCM regressed on 305 days FCM for months when production is influenced by a new pregnancy: ME-herd, 2nd. - 6th. lactation

Month of lact- ation	Calving interval, days								
	320-350	351-380	381-411	412-441	442-472	473-502	503-533	534-563	564-
7	0.111								
8	0.129	0.118							
9	0.073	0.134	0.109						
10	0.070	0.023	0.080	0.120					
11	. .	. .	0.027	0.063	0.074				
12	. .	. .	. .	0.056	0.054	0.128			
13	. .	. .	. .	. .	0.002	0.102	0.086		
14	. .	. .	. .	. .	. .	0.056	0.053	-0.004	
15	. .	. .	. .	. .	. .	. .	0	-0.022	0
16	. .	. .	. .	. .	. .	. .	. .	0	0.004

TABLE A.17. Assumed relationship between 305 days FCM and FCM for separate months; for use in the replacement models: 1st lactation

Month of lactation	Calving interval		
	12 months	15 months	18 months
1 - 4	591 + 0.40 X[b]	591 + 0.40 X	591 + 0.40 X
5	42 + 0.10 X	42 + 0.10 X	42 + 0.10 X
6	3 + 0.10 X	3 + 0.10 X	3 + 0.10 X
7	- 31 + 0.10 X	- 31 + 0.10 X	- 31 + 0.10 X
8	- 94 + 0.10 X	- 73 + 0.10 X	- 73 + 0.10 X
9	- 167 + 0.10 X	- 109 + 0.10 X	- 109 + 0.10 X
10	- 119 + 0.10 X[c]	- 142 + 0.10 X	- 142 + 0.10 X
11		- 194 + 0.10 X	- 173 + 0.10 X
12		- 276 + 0.10 X	- 220 + 0.10 X
13		- 315 + 0.10 X[c]	- 268 + 0.10 X
14			0 + 0.06 X
15			177 + 0.03 X
16			344 + 0.00 X[c]

a/ The assumed relationships in this table are based on the empirical findings partly presented in tables A.14 and A.15, but with an "evening out" of regression coefficients and with months defined as months from the day of freshening.

b/ X stands for 305 days FCM in pounds after correction for length of the present calving interval.

c/ Some part of the expected production at the end of the lactation may actually be produced during the 11th, the 14th, and the 17th month respectively, while for convenience it is added to the production the month before in this table.

TABLE A.18. Assumed relationship between 305 days FCM and FCM for separate months; for use in the replacement models: 2nd - 6th lactation [a]

Month of lactation	Calving interval		
	12 months	15 months	18 months
1 - 4	1269 + 0.40 X [b]	1269 + 0.40 X	1269 + 0.40 X
5	61 + 0.10 X	61 + 0.10 X	61 + 0.10 X
6	- 21 + 0.10 X	- 21 + 0.10 X	- 21 + 0.10 X
7	- 103 + 0.10 X	- 103 + 0.10 X	- 103 + 0.10 X
8	- 218 + 0.10 X	- 195 + 0.10 X	- 195 + 0.10 X
9	- 362 + 0.10 X	- 277 + 0.10 X	- 277 + 0.10 X
10	- 400 + 0.10 X [c]	- 345 + 0.10 X	- 345 + 0.10 X
11		- 299 + 0.085 X	- 276 + 0.085 X
12		- 247 + 0.065 X	- 162 + 0.065 X
13		- 316 + 0.055 X [c]	- 141 + 0.055 X
14			188 + 0.010 X
15			183 + 0.000 X [c]

[a] The assumed relationships in this table are based on the empirical findings partly presented in tables A.14 and A.16, but with an "evening out" of regression coefficients and with months defined as months from the day of freshening.

[b] X stands for 305 days FCM in pounds after correction for length of the present calving interval.

[c] Some part of the expected production at the end of the lactation may actually be produced during the 11th, the 14th, and the 16th month respectively, while for convenience it is added to the production the month before in this table.

TABLE A.19. Observed frequencies of lactations with known calving intervals

Lact. No.	Calving interval, days												
	-319	320 -350	351 -380	381 -411	412 -441	442 -472	473 -502	503 -533	534 -563	564 -594	595 -624	625 - up	Sum
ME-herd													
1	29	145	147	108	76	54	36	26	7	13	1	12	654
2	18	115	129	77	42	30	21	15	11	3	3	5	469
3	7	57	94	54	34	21	19	6	4	2	4	3	305
4 - 6	22	90	124	90	47	33	16	7	10	5	4	3	451
MA-herd													
1	11	56	160	111	89	69	36	24	13	32[a/]			601
2	9	66	118	79	61	40	17	13	12	13			428
3	7	37	97	56	44	25	12	12	8	14			312
4 - 6	11	46	144	80	42	31	23	16	17	18			428

a/ In the MA-herd, the figures in this column are number of lactations with calving intervals from 564 days and up.

TABLE A.20. Observed frequencies of lactations with unknown calving intervals (last lactation of each cow)

Lact. No.	Minimal length of calving interval, days[a]											No inf.[b]	Not bred[c]
	351	381	412	442	473	503	534	564	595	625	Sum		
ME-herd													
1	8	6	3	4	6	3	5	4	1	12	52	44	46
2	10	10	11	6	5	7	4	2	1	6	62	42	34
3	7	8	8	8	10	7	4	3	1	5	61	25	25
4–6	18	15	17	5	14	10	10	4	8	9	110	25	63
MA-herd													
1	12	11	9	13	12	12	9	38			116		18
2	18	11	15	13	11	10	11	24			113		29
3	13	13	20	6	4	10	6	17			89		17
4–6	16	26	23	17	14	10	7	29			142		67

a/ Information in herd books shows that attempts were made to breed the cow but that she was replaced before freshening.

b/ Information missing in the herd books.

c/ Information in herd books suggests that the cow was not bred because removal already was decided.

TABLE A.21. Estimated probability distribution of calving intervals in populations where all cows are kept at least until 625 days after last freshening

Calving interval, days	ME-herd		MA-herd
	1st - 2nd lact.	3rd - 6th lact.	All lact.
- 350	0.2384	0.1832	0.1086
351 - 380	0.2202	0.2358	0.2391
381 - 411	0.1528	0.1642	0.1569
412 - 441	0.1018	0.1006	0.1215
442 - 472	0.0758	0.0715	0.0911
473 - 502	0.0552	0.0550	0.0529
503 - 533	0.0436	0.0246	0.0438
534 - 563	0.0218	0.0327	0.0383
564 - 594	0.0219	0.0192	0.0229
595 - 624	0.0058	0.0294	0.0194
625 -	0.0627	0.0838	0.1055

Note:

In the replacement models, it is assumed that the calving interval will be either 12 months, 15 months, 18 months, or longer. The assumed probability of each alternative is derived from the figures in this table by summing the estimated probabilities for the following intervals:

12 months	C.I.	- 411 days
15 "	"	412 - 502 days
18 "	"	503 - 594 days
longer	"	595 -

TABLE A.22. Number observed lactations and number involuntary removals classified by level of production

	Production level[a]		
	Low	Medium	High
ME-herd			
Total[b]	656	517	751
Involuntary removals[c]	42	50	57
Porportion involuntary removals	0.064	0.097	0.076
MA-herd			
Total[b]	615	610	570
Involuntary removals[c]	63	61	52
Proportion involuntary removals	0.102	0.100	0.091

a/ Classification is based on 305 days FCM for the previous lactation. All first lactations are excluded since they have no previous record.

b/ Any case where a cow has freshened is counted as a lactation. The data include all lactations from the second to the seventh inclusive.

c/ The number of cases where a cow removed before next freshening for one of the reasons classified as "involuntary". See page 173 for a list of such reasons.

TABLE A.23. Number observed lactations and number involuntary removals classified by time period

	Time period						
	30-35	36-40	41-45	46-50	51-55	56-60	61-
	ME-herd						
Total[a/]	185	238	385	457	515	650	385
Involuntary removals[b/]	2	19	17	47	52	44	31
Proportion involuntary removals	0.011	0.080	0.044	0.103	0.101	0.068	0.081
	MA-herd						
Total[a/]	132	212	314	543	573	592	239
Involuntary removals[b/]	3	15	34	37	48	39	29
Proportion involuntary removals	0.023	0.071	0.108	0.068	0.084	0.066	0.121

[a/] Any case where a cow has freshened is counted as a lactation. The data include all lactations from the first to the seventh inclusive.

[b/] The number of cases where a cow was removed before next freshening for one of the reasons classified as "involuntary". See page 173 for a list of such reasons.

TABLE A.24. Number observed lactations and number involuntary removals classified by lactation number

	Lactation number						
	1	2	3	4	5	6	7
ME-herd							
Total[a/]	891	656	477	325	219	150	97
Involuntary removals[b/]	63	36	40	28	18	16	11
Proportion involuntary removals	0.071	0.055	0.084	0.086	0.082	0.107	0.113
MA-herd							
Total[a/]	810	603	435	318	211	144	84
Involuntary removals[b/]	29	28	41	44	25	25	13
Proportion involuntary removals	0.036	0.046	0.094	0.138	0.118	0.174	0.155

a/ Any case where a cow has freshened is counted as a lactation.

b/ The number of cases where a cow was removed before next freshening for one of the reasons classified as "involuntary". See page 173 for a list of such reasons.

TABLE A.25. Probabilities of involuntary removals estimated as functions of time period and of lactation number

Parameter[a/]	ME-herd	MA-herd
$\hat{\beta}$	0.0067	0.0266
$\hat{\alpha}_j$:		
1930-35	- 0.0070	- 0.0804
1936-40	0.0596	- 0.0303
1941-45	0.0247	0.0253
1946-50	0.0829	- 0.0161
1951-55	0.0823	- 0.0035
1956-60	0.0512	0.0501
1961-	0.0646	0.0276

a/ The probability p_{ij} that for the j'th time period a cow which has started the i'th lactation will be involuntary removed during that lactation is assumed to be:

$$p_{ij} = \alpha_j + \beta L_i, \text{ where } L_i \text{ is lactation number.}$$

TABLE A.26. Assumed monthly probabilities of involuntary removals

Lactation No.	ME-herd	MA-herd
1	0.006060	0.004102
2	0.006622	0.006227
3	0.007183	0.008352
4	0.007744	0.010477
5	0.008306	0.012602
6	0.008867	0.014728

TABLE A.27. Assumed relationship between milk production and feed consumption

1st lactation		
FCM, lb/month	<535	≥535
Grain, lb/month[a]	92	-140 + 0.433 x FCM
Alfalfa hay, lb/month	730	730
2nd - 6th lactation		
FCM, lb/month	<580	≥580
Grain, lb/month[a]	61	-191 + 0.433 x FCM
Alfalfa hay, lb/month	840	840

a/ Grain or a suitable concentrate mix.

TABLE A.28. Alternative price sets used in the replacement models

	Price alternative			
	1[a]	2[b]	3[c]	4[d]
	$	$	$	$
Items sold:				
Milk per 100 lb	4.70	3.40	4.70	5.20
Calves per head	18.-	18.-	18.-	18.-
Replaced cows per head:				
1st lact.	115.-	115.-	115.-	115.-
1st - 2nd lact.[e]	115.-	115.-	115.-	115.-
2nd - 3rd lact.	115.-	115.-	115.-	115.-
3rd - 4th lact.	112.-	112.-	112.-	112.-
4th - 5th lact.	109.-	109.-	109.-	109.-
5th - 6th lact.	105.-	105.-	105.-	105.-
Items purchased:				
Grain per ton	66.-	66.-	66.-	62.-
Hay per ton	24.-	24.-	24.-	30.-
Replacement heifers per head	225.-	225.-	175.-	225.-

a/ Price situation taken to be representative for the Central Valley, and with milk price set equal to a blend price.

b/ Central Valley prices but with milk price set equal to the price of production milk.

c/ Central Valley prices but replacement prices set $ 50.- lower per head.

d/ Price situation taken to be representative for the Los Angelos area, and with milk price set equal to a blend price.

e/ Cows which are sold at least seven months after one freshening and not more than seven months after the next freshening. For the sixth lactation the price is assumed valid until ten months after the sixth freshening.

TABLE A.29. Expected 305 days FCM in pounds for present and next lactation when production class is given; ME-herd

Production class	Lactation No.				
	1	2	3	4	5
			Present lactation		
1	12796	13978	14303	14815	14733
2	11595	12705	13153	13486	13246
3	10650	11664	12330	12453	12154
4	9705	10623	11508	11421	11062
5	8760	9582	10686	10389	9970
6	7815	8541	9863	9356	8878
7	6614	7268	8713	8027	7391
			Next lactation[a]		
1	11742	12927	13060	12736	13008
2	11246	12337	12325	12038	12168
3	10856	11854	11800	11495	11552
4	10466	11371	11274	10953	10935
5	10076	10888	10748	10411	10318
6	9686	10405	10223	9868	9702
7	9190	9815	9488	9170	8862

[a] Figures given here are for the case where the present calving interval is 12 months. For 15 and 18 months calving intervals, the figures should be increased with (440 - 365) (= 75) times the regression coefficient of 305 days FCM on length of the previous calving interval. For the ME-herd, these regression coefficients are allowed to vary between lactation numbers and are assumed to be equal to the estimates given in table A.3.

TABLE A.30. Expected 305 days FCM in pounds for present and next lactation when production class is given; MA-herd

Production class	Lactation No.				
	1	2	3	4	5
			Present lactation		
1	12325	13664	13645	14350	14706
2	11055	12161	12360	13056	13206
3	10084	11035	11534	11997	12098
4	9113	9910	10707	10937	10996
5	8142	8785	9880	9877	9894
6	7171	7659	9054	8818	8786
7	5901	6156	7769	7524	7286
			Next lactation[a]		
1	11634	12307	12409	12337	12524
2	10870	11588	11680	11756	11653
3	10287	11049	11210	11279	11009
4	9703	10511	10741	10803	10369
5	9119	9973	10272	10327	9729
6	8536	9434	9802	9850	9085
7	7772	8715	9073	9269	8214

a/ Figures given here are for the case where the present calving interval is 12 months. For 15 and 18 months calving intervals, the figures should be increased with (440 - 365) (= 75) times the regression coefficient of 305 days FCM on length of the previous calving interval. For the MA-herd, this regression coefficient is assumed to be 8 lb FCM per day for all lactation numbers.

TABLE A.31. Optimal replacement decisions and present values under an infinite planning horizon:
ME-herd; price alternative 1

A. Absolute present value of state 1[a/] $ 4,315.48

B. Optimal replacement decisions, and relative present values of other states in dollars.[b/]

Production class	Lactation No. 1	2	3	4	5
	12 months C.I.				
1	315.86	355.59	320.80	281.57	232.70
2	271.59	301.96	266.71	235.77	194.40
3	235.49	251.37	226.96	199.59	166.40
4	202.02	203.01	189.55	165.41	138.40
5	169.46	159.57	155.38	134.41	110.30
6	140.39	120.54	125.79	R 112.00	R 109.00
7	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
	15 months C.I.				
1	344.15	315.53	319.49	281.55	239.37
2	289.33	254.34	258.95	227.72	191.67
3	246.32	196.77	214.20	185.04	156.17
4	204.37	141.52	171.38	143.59	120.27
5	164.27	R 115.00	131.70	R 112.00	R 109.00
6	125.61	R 115.00	R 115.00	R 112.00	R 109.00
7	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
	Unknown C.I.				
1	243.90	221.77	188.77	175.20	156.01
2	209.61	183.01	158.45	146.57	128.81
3	177.45	147.35	136.21	123.82	R 109.00
4	148.84	R 115.00	R 115.00	R 112.00	R 109.00
5	121.06	R 115.00	R 115.00	R 112.00	R 109.00
6	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
7	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00

[a/] Absolute present value has been derived in the way explained on page 206. State 1 represents the process immediately before a heifer is purchased, therefore, the present value does not include the value of the animal.

[b/] R stands for "replace", no notation for "keep". The relative values given are the differences between present values of the given states and present value of state 1. Therefore, they can be said to represent the value to the dairy producer of an animal with the given characteristics.

TABLE A.32. Optimal replacement decisions and present values under an infinite planning horizon:
ME-herd; price alternative 2

A. Absolute present value of state 1[a/] $ 1,934.28

B. Optimal replacement decisions, and relative present values of other states in dollars.[b/]

Production class	Lactation No. 1	2	3	4	5
	12 months C.I.				
1	275.88	294.31	267.38	236.37	197.51
2	247.91	261.34	234.06	208.35	174.51
3	224.78	229.77	209.02	185.66	157.61
4	202.48	198.64	184.87	163.57	140.71
5	180.34	168.90	161.81	142.93	123.81
6	159.68	139.02	140.87	123.37	R 109.00
7	134.19	R 115.00	R 115.00	R 112.00	R 109.00
	15 months C.I.				
1	295.61	278.11	269.55	239.60	205.04
2	261.36	240.30	231.09	206.35	175.94
3	233.95	203.84	203.67	179.45	153.94
4	206.76	168.00	175.70	152.45	131.64
5	180.12	132.78	149.08	126.74	R 109.00
6	153.68	R 115.00	123.51	R 112.00	R 109.00
7	117.53	R 115.00	R 115.00	R 112.00	R 109.00
	Unknown C.I.				
1	219.23	200.59	172.89	161.47	145.36
2	197.52	176.84	154.55	143.93	128.96
3	177.61	154.66	140.63	129.91	116.66
4	159.20	132.65	127.69	115.93	R 109.00
5	141.08	R 115.00	R 115.00	R 112.00	R 109.00
6	122.78	R 115.00	R 115.00	R 112.00	R 109.00
7	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00

[a/] Absolute present value has been derived in the way explained on page 206. State 1 represents the process immediately before a heifer is purchased, therefore, the present value does not include the value of the animal.

[b/] R stands for "replace", no notation for "keep". The relative values given are the differences between present values of the given states and present value of state 1. Therefore, they can be said to represent the value to the dairy producer of an animal with the given characteristics.

TABLE A.33. Optimal replacement decisions and present values under an infinite planning horizon:
ME-herd; price alternative 3

A. Absolute present value of state 1[a]	$ 4,471.33

B. Optimal replacement decisions, and relative present values of other states in dollars.[b]

Production class	Lactation No.				
	1	2	3	4	5
			12 months C.I.		
1	286.06	328.40	297.63	263.36	221.03
2	243.36	257.57	244.43	218.32	182.73
3	206.96	226.51	206.02	183.30	154.73
4	177.62	180.57	170.72	150.80	126.73
5	147.47	140.47	139.43	121.74	R 109.00
6	121.01	R 115.00	R 115.00	R 112.00	R 109.00
7	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
			15 months C.I.		
1	312.86	285.38	294.67	261.52	225.64
2	259.54	224.98	234.98	208.41	177.94
3	218.20	168.88	191.51	166.85	142.44
4	178.30	115.96	150.73	127.01	R 109.00
5	140.54	R 115.00	R 115.00	R 112.00	R 109.00
6	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
7	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
			Unknown C.I.		
1	224.44	203.70	176.61	164.91	148.17
2	191.06	165.37	146.62	136.56	120.97
3	159.71	130.50	124.88	114.24	R 109.00
4	132.20	R 115.00	R 115.00	R 112.00	R 109.00
5	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
6	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
7	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00

a/ Absolute present value has been derived in the way explained on page 206. State 1 represents the process immediately before a heifer is purchased, therefore, the present value does not include the value of the animal.

b/ R stands for "replace", no notation for "keep". The relative values given are the differences between the present values of the given states and the present value of state 1. Therefore, they can be said to represent the value to the dairy producer of an animal with the given characteristics.

TABLE A.34. Optimal replacement decisions and present values under an infinite planning horizon:
ME-herd: price alternative 4

A. Absolute present value of state 1[a] $ 4,887.21

B. Optimal replacement decisions, and relative present values of other states in dollars.[b]

Production class	Lactation No. 1	2	3	4	5
	12 months C.I.				
1	326.45	373.26	336.33	295.11	243.84
2	275.53	310.78	273.38	241.84	198.74
3	234.64	252.49	227.78	200.18	165.74
4	196.99	197.70	185.55	161.27	132.64
5	160.71	149.41	147.97	126.60	R 109.00
6	128.62	R 115.00	115.93	R 112.00	R 109.00
7	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
	15 months C.I.				
1	358.77	325.57	332.73	292.75	249.12
2	295.53	254.41	262.28	230.15	193.22
3	246.20	188.29	210.87	181.06	151.72
4	198.55	125.62	162.58	134.12	109.72
5	153.31	R 115.00	118.70	R 112.00	R 109.00
6	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
7	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
	Unknown C.I.				
1	251.22	224.74	191.37	177.51	157.71
2	211.26	179.47	156.00	144.00	125.71
3	174.48	138.28	130.31	117.72	R 109.00
4	141.67	R 115.00	R 115.00	R 112.00	R 109.00
5	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
6	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
7	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00

[a] Absolute present value has been derived in the way explained on page 206. State 1 represents the process immediately before a heifer is purchased, therefore, the present value does not include the value of the animal.

[b] R stands for "replace", no notation for "keep". The relative values given are the differences between the present values of the given states and the present values of state 1. Therefore, they can be said to represent the value to the dairy producer of an animal with the given characteristics.

TABLE A.35. Optimal replacement decisions and present values under an infinite planning horizon:
MA-herd; price alternative 1

A. Absolute present value of state 1 a/ $ 3,721.32

B. Optimal replacement decisions, and relative present values of other states in dollars. b/

Production class	Lactation No. 1	2	3	4	5
	12 months C.I.				
1	368.66	376.70	335.57	298.74	247.60
2	312.85	317.38	278.67	258.22	210.00
3	269.01	267.33	244.56	223.32	182.10
4	226.17	218.11	209.87	189.91	154.40
5	185.19	172.05	177.02	158.94	126.70
6	147.11	125.06	144.64	128.26	R 109.00
7	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
	15 months C.I.				
1	400.48	378.00	343.47	304.28	261.13
2	333.87	309.78	274.28	255.80	214.43
3	281.84	251.33	233.23	214.36	179.43
4	230.88	193.86	193.07	173.93	144.23
5	181.04	138.25	154.07	135.04	R 109.00
6	131.81	R 115.00	116.54	R 112.00	R 109.00
7	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
	Unknown C.I.				
1	255.30	226.61	205.21	193.14	173.41
2	216.93	188.36	170.81	165.03	145.11
3	187.19	156.50	148.54	141.16	123.91
4	158.09	124.89	126.20	117.78	R 109.00
5	128.78	R 115.00	R 115.00	R 112.00	R 109.00
6	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
7	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00

a/ Absolute present value has been derived in the way explained on page 206. State 1 represents the process immediately before a heifer is purchased, therefore, the present value does not include the value of the animal.

b/ R stands for "replace", no notation for "keep". The relative values given are the differences between the present values of the given states and the present value of state 1. Therefore, they can be said to represent the value to the dairy producer of an animal with the given characteristics.

TABLE A.36. Present values under an infinite planning horizon if the replacement policy deviates from the optimal: ME-herd; price alternative 1; no voluntary replacement

A. Absolute present value of state 1[a/] $ 4,224.09

B. Relative present values of other states in dollars.[b/]

Production class	Lactation No. 1	2	3	4	5
	12 months C.I.				
1	327.82	368.86	332.03	290.54	238.14
2	280.31	313.71	276.22	243.61	199.84
3	240.17	260.22	233.61	205.00	171.84
4	200.90	206.01	190.99	165.93	143.84
5	160.55	151.18	147.98	126.53	115.74
6	121.55	93.51	104.39	86.19	85.74
7	71.16	31.96	43.75	33.82	42.04
	15 months C.I.				
1	356.87	330.13	331.48	291.35	245.76
2	298.95	267.49	269.28	236.45	198.06
3	251.98	207.13	221.79	191.42	162.56
4	204.46	146.24	173.96	145.28	126.66
5	156.81	84.14	125.76	98.94	90.26
6	108.58	20.35	76.42	51.12	52.56
7	44.10	- 51.02	8.77	- 10.67	- 2.34
	Unknown C.I.				
1	251.95	230.49	194.60	180.17	159.65
2	215.95	190.95	163.64	151.13	132.45
3	181.69	153.79	140.34	127.47	112.25
4	150.06	116.41	117.00	103.58	91.85
5	118.21	77.62	93.27	79.27	71.15
6	85.83	37.59	69.21	54.46	49.45
7	43.36	- 9.14	35.27	20.77	16.35

a/ Absolute present value has been derived in the way explained on page 206. State 1 represents the process immediately before a heifer is purchased, therefore, the present value does not include the value of the animal.

b/ R stands for "replace", no notation for "keep". The relative values given are the differences between the present values of the given states and the present value of state 1. Therefore, they can be said to represent the value to the dairy producer of an animal with the given characteristics.

TABLE A.37. Replacement decisions and present values under an infinite planning horizon for a replacement policy which deviates from the optimal:
ME-herd; price alternative 1; intensive culling

A. Absolute present value of state 1[a/] $ 4,203.47

B. Replacement decisions, and relative present values of other states in dollars.[b/]

Production class	Lactation No. 1	2	3	4	5
	12 months C.I.				
1	333.08	374.09	336.51	293.70	239.95
2	285.43	319.26	281.68	247.51	201.65
3	245.85	265.10	240.07	210.57	173.65
4	208.60	209.14	198.70	175.20	145.65
5	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
6	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
7	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
	15 months C.I.				
1	362.34	335.77	336.19	294.79	247.89
2	304.27	273.45	274.93	240.57	200.19
3	257.86	212.44	228.39	197.17	164.69
4	212.28	149.86	181.76	154.58	128.79
5	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
6	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
7	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
	Unknown C.I.				
1	255.31	233.82	196.82	181.90	160.87
2	219.21	194.44	166.22	153.13	133.67
3	185.27	156.93	143.29	130.09	113.47
4	154.69	R 115.00	R 115.00	R 112.00	R 109.00
5	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
6	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
7	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00

[a/] Absolute present value has been derived in the way explained on page 206. State 1 represents the process immediately before a heifer is purchased, therefore, the present value does not include the value of the animal.

[b/] R stands for "replace", no notation for "keep". The relative values given are the differences between the present values of the given states and the present value of state 1. Therefore, they can be said to represent the value to the dairy producer of an animal with the given characteristics.

TABLE A.38. Replacement decisions and present values under an infinite planning horizon for a replacement policy which deviates from the optimal:
ME-herd; price alternative 1; replacement policy as for the MA-herd

A. Absolute present value of state 1[a/] $ 4,306.61

B. Replacement decisions, and relative present values of other states in dollars.[b/]

Production class	Lactation No.				
	1	2	3	4	5
			12 months C.I.		
1	316.67	356.77	321.82	282.36	233.17
2	271.93	302.94	267.52	236.54	194.87
3	235.34	252.03	227.33	200.34	166.87
4	201.33	203.13	189.37	166.11	138.87
5	168.33	158.83	154.73	135.05	110.77
6	139.02	119.25	124.83	106.72	R 109.00
7	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
			15 months C.I.		
1	345.04	316.80	320.58	282.42	239.91
2	289.76	255.44	259.83	228.56	192.21
3	246.27	197.56	214.66	185.86	156.71
4	203.81	141.79	171.31	144.37	120.81
5	163.28	89.85	131.17	106.02	R 109.00
6	124.39	R 115.00	95.00	R 112.00	R 109.00
7	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
			Unknown C.I.		
1	244.48	222.53	189.29	175.64	156.32
2	209.95	183.68	158.89	147.00	129.12
3	177.53	147.85	136.49	124.24	108.92
4	148.64	113.24	114.91	102.14	R 109.00
5	120.62	R 115.00	R 115.00	R 112.00	R 109.00
6	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
7	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00

[a/] Absolute present value has been derived in the way explained on page 206. State 1 represents the process immediately before a heifer is purchased, therefore, the present value does not include the value of the animal.

[b/] R stands for "replace", no notation for "keep". The relative values given are the differences between the present values of the given states and the present value of state 1. Therefore, they can be said to represent the value to the dairy producer of an animal with the given characteristics.

TABLE A.39. Replacement decisions and present values under an infinite planning horizon for a replacement policy which deviates from the optimal:
MA-herd; price alternative 1; replacement policy as for the ME-herd

A. Absolute present value of state 1[a/] $ 3,709.08

B. Replacement decisions, and relative present values of other states in dollars.[b/]

Production class	Lactation No. 1	2	3	4	5
12 months C.I.					
1	369.40	377.99	336.53	299.32	248.24
2	312.86	318.26	279.45	258.51	210.64
3	268.34	267.45	244.90	223.54	182.74
4	224.90	217.38	209.06	190.23	155.04
5	183.58	170.78	174.22	159.45	127.34
6	145.52	124.26	140.08	R 112.00	R 109.00
7	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
15 months C.I.					
1	401.34	379.38	344.53	304.97	261.87
2	334.02	310.77	275.16	256.21	215.17
3	281.34	251.59	233.70	214.71	180.17
4	229.79	193.30	192.44	174.39	144.97
5	179.62	R 115.00	151.53	R 112.00	R 109.00
6	130.40	R 115.00	R 115.00	R 112.00	R 109.00
7	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
Unknown C.I.					
1	255.83	227.37	205.82	193.58	173.87
2	217.14	188.94	171.34	165.36	145.57
3	187.11	156.76	148.89	141.46	R 109.00
4	157.74	R 115.00	R 115.00	R 112.00	R 109.00
5	128.29	R 115.00	R 115.00	R 112.00	R 109.00
6	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00
7	R 115.00	R 115.00	R 115.00	R 112.00	R 109.00

a/ Absolute present value has been derived in the way explained on page 206. State 1 represents the process immediately before a heifer is purchased, therefore, the present value does not include the value of the animal.

b/ R stands for "replace", no notation for "keep". The relative values given are the differences between the present values of the given states and the present value of state 1. Therefore, they can be said to represent the value to the dairy producer of an animal with the given characteristics.

TABLE A.40. Probabilities of calving, and probability distribution of replacement causes, under given replacement policies

	Replacement policy as defined by table				
	A.31	A.32	A.35	A.36	A.37
Herd	ME	ME	MA	ME	ME
Calving No.:					
1st	1.000	1.000	1.000	1.000	1.000
2nd	0.801	0.852	0.763	0.856	0.536
3rd	0.571	0.652	0.541	0.728	0.337
4th	0.412	0.489	0.387	0.584	0.227
5th	0.276	0.365	0.279	0.464	0.159
6th	0.179	0.230	0.159	0.366	0.103
Sum	3.239	3.588	3.129	3.998	2.361
Replacement cause:					
Accidents, diseases	0.266	0.307	0.282	0.361	0.175
"Sterility"[a]	0.130	0.195	0.247	0.305	0.110
Low production[b]	0.150	0.111	0.150	. .	0.569
Low production + breeding trouble[c]	0.290	0.177	0.186	. .	0.052
Reached end of 6th lactation	0.164	0.210	0.135	0.334	0.094
Sum	1.000	1.000	1.000	1.000	1.000

a/ Cows which are replaced 10 months after last freshening because they have not conceived.

b/ Cases where all cows in a given production class are replaced even if the calving interval is only 12 months.

c/ Cases where cows with 15 months or unknown calving intervals are replaced, while cows with 12 months calving interval in the same production class are retained.

Appendix B

ESTIMATION OF PARAMETERS IN A MULTIVARIATE NORMAL DISTRIBUTION BASED ON A SAMPLE FROM A POPULATION SUBJECT TO CULLING

1. The Problem

The problem is to estimate parameters in the stochastic model (4.3) for a given population of dairy cows, when $x_{f\alpha}$ denotes 305 days production of fat-corrected milk (FCM) for the f'th lactation of the α'th cow.[1/] The population is a conceptually infinite population of dairy cows consisting of all cows of a given breed and under given management conditions which can be born and raised to the age of first freshening from a given genetic stock. The parameters describe the distribution of population 305 days FCM as it would be if all cows were allowed to finish F lactation cycles. Since in practice many cows are removed before the end of F lactation cycles, some of the variables $x_1,\ldots\ldots,x_F$ are conceptual, representing yields which could occur if the α'th cow was allowed to finish F lactation cycles.

We want to arrive at unbiased estimates of the parameters in this conceptual F-variate distribution. In any random sample from the defined population, however, some cows are removed from the herds for some reason before F lactation cycles are finished.[2/] If the sample is selected so that it consists only of cows which have completed all F lactation cycles, the sample will be biased since it is usually cows for

---

1/ See page 69 or page 296.

2/ The cows actually born and raised to the age of first freshening are assumed to represent a random sample from the population defined here. From these cows again, we may draw a random subsample which is also a random sample from the original population.

which 305 days FCM for the first lactations deviate negatively from the population averages which are culled at an early time.

In various studies where the purpose has been to develop "age-correction factors" to be used in connection with sire proving programs and studies of genetic relationships, the problem caused by culling has been recognized and various methods introduced to correct for this bias. Most of these methods are based on a comparison of paired consecutive records of the same cows.[1/] If not corrected for, culling in the population from which the sample is drawn may result in bias both with respect to estimated average production under given values of the explanatory variables (with other words, in the location of the regression lines), and with respect to the estimated variance around the regression line. Most previous studies have been concerned only with bias in estimated average production. For the purpose of this study, we want to correct for bias also in estimated variances. The estimation method which is proposed below will do this.

2. Estimation Procedure

The model is specified as:

$$\underline{x} = \begin{bmatrix} x_{1\alpha} \\ x_{2\alpha} \\ \cdot \\ \cdot \\ \cdot \\ x_{F\alpha} \end{bmatrix} = \begin{bmatrix} \underline{\gamma}_1' \underline{z}_{1\alpha} \\ \underline{\gamma}_2' \underline{z}_{2\alpha} \\ \cdot \\ \cdot \\ \cdot \\ \underline{\gamma}_F' \underline{z}_{F\alpha} \end{bmatrix} + \begin{bmatrix} u_{1\alpha} \\ u_{2\alpha} \\ \cdot \\ \cdot \\ \cdot \\ u_{F\alpha} \end{bmatrix} \qquad (4.3)$$

1/ For example, see the discussion in Lush and Shrode, loc.cit.

where the vectors $\underline{\gamma}_f$ $(f = 1,\ldots,F)$ consist of population parameters and the vector $\underline{u}_\alpha$ has a multivariate normal distribution with:

$$E(\underline{u}_\alpha) = \underline{0}$$

$$E(\underline{u}_\alpha \underline{u}_\alpha') = \Sigma$$

In order to get unbiased estimates of the population parameters, the following procedure is suggested:

Draw a sample so that all cows which have freshened at least once have an equal chance to enter the sample. We will assume that the measurement of x_1 is available on all cows in the sample.

Let N_r be the number of cows in the sample which have finished r lactation periods. We have

$$N_1 \geq N_2 \geq \ldots\ldots\ldots \geq N_F$$

We can arrive at estimates of the population parameters by a stepwise procedure, whereby parameters in the r-variate distribution consisting of the first r lactation records are estimated from the sample of N_r cows plus previous estimates of parameters in the $(r-1)$-variate distribution consisting of the first $r-1$ lactation records.

The estimation procedure will start by estimating the vector $\underline{\gamma}_1$ and the variance σ_{11} from the sample of N_1 cows, which is assumed to be an unbiased sample from the population. Having estimates of these parameters, we can estimate the vector $\underline{\gamma}_2$, the variance σ_{22} and the covariance σ_{12} from the sample of N_2 cows plus the estimates already arrived at for $\underline{\gamma}_1$ and σ_{11}. Having estimates for these parameters, we can estimate the vector $\underline{\gamma}_3$, the variance σ_{33} and the covariances σ_{31} and σ_{32} from the sample of N_3 cows plus the estimates already arrived at for parameters in the bivariate distribution of the first two lactation records. We can continue in this way until all parameters in the F-variate distribution are estimated.

Denote the $(r-1)$-dimensional vector consisting of the first $(r-1)$ elements in the vector $\underline{u}_\alpha$ as $\underline{u}_{(r-1)\alpha}$.

Partition the covariance matrix $\boldsymbol{\Sigma}$ as indicated below:

$$\left[\begin{array}{cccc|c|cc}
\sigma_{11} & \sigma_{12} & \cdots\cdots & \sigma_{1,r-1} & \sigma_{1,r} & \cdots\cdots & \sigma_{1,F} \\
\sigma_{21} & \sigma_{22} & \cdots\cdots & \sigma_{2,r-1} & \sigma_{2,r} & \cdots\cdots & \sigma_{2,F} \\
\vdots & & & & & & \\
\sigma_{r-1,1} & \sigma_{r-1,2} & & \sigma_{r-1,r-1} & \sigma_{r-1,r} & & \sigma_{r-1,F} \\
\hline
\sigma_{r,1} & \sigma_{r,2} & & \sigma_{r,r-1} & \sigma_{r,r} & & \sigma_{r,F} \\
\hline
\sigma_{r+1,1} & \sigma_{r+1,2} & & \sigma_{r+1,r-1} & \sigma_{r+1,r} & & \sigma_{r+1,F} \\
\vdots & & & & & & \\
\sigma_{F,1} & \sigma_{F,2} & & \sigma_{F,r-1} & \sigma_{F,r} & & \sigma_{F,F}
\end{array}\right]$$

where the $(r-1)x(r-1)$ -dimensional upper left-hand corner is denoted $\boldsymbol{\Sigma}_{(r-1)}$, and the $1x(r-1)$ -dimensional vector $\left[\sigma_{r,1} \ \ \sigma_{r,2} \ \cdots\cdots \ \sigma_{r,r-1}\right]$ is denoted $\underline{\sigma}_{(r)}$.

Denote as $\underline{\beta}_{(r)}$ the $1x(r-1)$ -dimensional vector of regression coefficients in the regression of u_r on $u_1, u_2, \ldots\ldots, u_{r-1}$. In a multivariate normal distribution:

$$\underline{\beta}_{(r)} = \underline{\sigma}_{(r)} \boldsymbol{\Sigma}^{-1}_{(r-1)} \tag{B,1}$$

In the conditional distribution of u_r given $u_1, u_2, \ldots\ldots, u_{r-1}$, the expected value of u_r is

$$E(u_r | u_1, u_2, \dots, u_{r-1}) = \beta_{(r)} \underline{u}_{(r-1)} \qquad (B.2)$$

and the variance, denoted $\sigma_{rr \cdot 1,2,\dots,r-1}$ is

$$\sigma_{rr \cdot 1,2,\dots,r-1} = \sigma_{rr} - \sigma_{(r)} \Sigma^{-1}_{(r-1)} \sigma'_{(r)} \qquad (B.3)$$

If, for the α'th cow, we have observed $x_{1\alpha}, x_{2\alpha}, \dots, x_{r-1,\alpha}$, and we know the vectors of explanatory variables $\underline{z}_{1\alpha}, \underline{z}_{2\alpha}, \dots, \underline{z}_{r\alpha}$, the expected value of $x_{r\alpha}$ is:

$$\begin{aligned} E(x_{r\alpha} | x_{1\alpha}, x_{2\alpha}, \dots, x_{r-1,\alpha}) &= \gamma'_r \underline{z}_{r\alpha} + E(u_{r\alpha}) \\ &= \gamma'_r \underline{z}_{r\alpha} + \beta_r \underline{u}_{(r-1)\alpha} \qquad (B.4) \end{aligned}$$

The vector $\underline{u}_{(r-1)\alpha}$ consists of the deviations between the observed and the expected values of x_i $(i = 1, \dots, r-1)$:

$$\underline{u}_{(r-1)\alpha} = \begin{bmatrix} u_{1\alpha} \\ u_{2\alpha} \\ \cdot \\ \cdot \\ u_{r-1,\alpha} \end{bmatrix} = \begin{bmatrix} x_{1\alpha} & - & \gamma'_1 \underline{z}_{1\alpha} \\ x_{2\alpha} & - & \gamma'_2 \underline{z}_{2\alpha} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ x_{r-1,\alpha} & - & \gamma'_{r-1} \underline{z}_{r-1,\alpha} \end{bmatrix} \qquad (B.5)$$

The vector $\underline{u}_{(r-1)\alpha}$ can be derived if we know the value of the vectors γ_f $(f = 1, \dots, r-1)$. We do not know the true values of these vectors but have derived estimates of them. Using these estimates, we can estimate $\underline{u}_{(r-1)\alpha}$, and can then estimate γ'_r and β_r in (B.4) from the sample of N_r cows by ordinary least squares method.[1/] The error mean square in the analysis will have $N_r - q - (r-1)$ degrees of freedom, where q is the number of elements in the vector χ_r. The error mean square will be an estimate of $\sigma_{rr \cdot 1,2,\ldots,r-1}$.

From the estimates of $\Sigma_{(r-1)}$ which we have from before and the estimates of $\beta_{(r)}$ and $\sigma_{rr \cdot 1,2,\ldots,r-1}$ from this analysis, we can arrive at estimates of $\sigma_{(r)}$ and σ_{rr}.

We get from (B.1):[2/]

$$\hat{\sigma}_{(r)} = \hat{\beta}_{(r)} \hat{\Sigma}_{(r-1)} \tag{B.6}$$

We get from (B.3):

$$\hat{\sigma}_{rr} = \hat{\sigma}_{rr \cdot 1,2,\ldots,r-1} + \hat{\sigma}_{(r)} \hat{\Sigma}^{-1}_{(r-1)} \hat{\sigma}'_{(r)} \tag{B.7}$$

The estimation requires a sample such that $N_r \geq q + r - 1$. In fact, since it is evident that the distributional properties of the estimators are quite complicated and that derivation of confidence intervals would be very difficult, we would prefer to work with a large sample so that we can feel fairly confident that the errors of estimates are small.

1/ This estimation method raises new problems since regression theory assumes that the independent variables are determined without error. The consequences of error in the determination of the u'es have not been determined. It seems possible that error in the u'es will bias the estimates of the elements in the vector β_r downward. As N_r increases and approach infinity, the errors in the u'es will go to zero. It appears, therefore, that the estimates arrived at are consistent even if they may not be unbiased.

2/ The top script ^ denotes estimates of parameters.

It is assumed above that the measurement of the first lactation record is available on all cows in a random sample. If this is not the case, we may divide the first lactation record in two or more parts, where each part consists of the production within a given time interval of the first lactation. We can define a vector:
$\left[s_1 \quad s_2 \ldots s_s \quad x_2 \ldots x_F \right]$, where the s -es are production records for separate time intervals of the first lactation. If we can assume that this vector has a distribution of the same kind as is assumed for the vector $\underline{x}$ in (4.3) and if the measurement of at least s_1 is available on all cows in the sample, then the procedure explained above can be used as before.