ON THE THEORY OF APPORTIONMENT.

By William R. Thompson.

1. If in an accepted sense, P is the probability that one method of treatment, T_1, is better than a rival, T_2, we may develop a system of apportionment such that the proportionate use of T_1 is $f_{(P)}$, a monotone increasing function, rather than make no discrimination at all up to a certain point and then finally entirely reject one or the other. The only paper * which has so far appeared in his field, as far as I am aware, is one by myself in a recent issue of *Biometrika*. In this paper I have considered the case of choice between two such rival *treatments*,† and for symmetry suggested that $f_{(Q)} \equiv 1 - f_{(P)}$ where $Q = 1 - P$. Then the *risk* of assignment to T_1 when it is not the better is $Q \cdot f_{(P)}$, while the corresponding risk for T_2 is $P \cdot f_{(Q)}$. Accordingly, I suggested further that we set $f_{(P)} = P$, which is a necessary and sufficient condition that these two risks be equal. Their sum, the *total risk*, is then $2PQ$.

A special case was considered wherein the result of use of T_i at any given trial is either *success* or *failure*, the probability of failure being an unknown, p_i, *à priori* (independently for $i = 1, \cdot \cdot \cdot, k$) equally likely to lie in either of any two equal intervals in the possible range, $(0, 1)$. It is further assumed that for a given T_i we have an experience of exactly n_i independent trials, the number of *successes* being s_i and of *failures* being $r_i \equiv n_i - s_i$; and the probability of obtaining such a sample is

$$\binom{n_i}{r_i} \cdot p_i^{r_i} \cdot q_i^{s_i} \text{ where } q_i = 1 - p_i.$$

Restricting consideration to the case, $k = 2$, dropping the subscript *one* and using a prime instead of subscript *two*, then it was shown that

$$(1) \qquad P = \psi_{(r,s,r',s')} \equiv \frac{\sum_{\alpha=0}^{r'} \binom{r + r' - \alpha}{r} \cdot \binom{s + s' + 1 + \alpha}{s}}{\binom{n + n' + 2}{n + 1}}.$$

Now, it is well known that the probability, $\bar{P}$, that by drawing at random

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* W. R. Thompson, *Biometrika*, vol. 25 (1933), pp. 285-294.

† By *treatment* we imply a special mode of dealing with *individuals* of a given class of things.

without replacements from a mixture of W *white* and B *black* balls we shall encounter w white before b black is given by

$$(2) \qquad \bar{P} = \frac{\sum_{\alpha=0}^{h} \binom{W}{w+\alpha} \cdot \binom{B}{b-1-\alpha}}{\binom{W+B}{w+b-1}},$$

where $h = \text{Min}(b-1, W-w)$. The object of the present paper is *first*, to show exactly how ψ may be expressed in the form of (2) and thus make possible the use of a machine based on this principle in the apportionment, and thereby avoid an enormous amount of calculation where tables are not available; and *second*, to develop a complete statement of the group, G, of substitutions of the arguments of $\psi_{(a_1,a_2,a_3,a_4)}$ which leave ψ invariant, and also those of the set, A, which change the value to $1-\psi_{(a_1,a_2,a_3,a_4)}$. The application of these substitutions to give a convenient form for calculation * of ψ or for other purposes is obvious. On this account the ψ-function is a convenient form for expression † of the incomplete hypergeometric series, as in the case of two problems considered by Pearson,‡ where for certain original variables which we may denote by a, b, c, and d we may express § a required probability by $\psi_{(a,b,c,d-1)}$.

2. We begin by considering the function, $\bar{N}_{(r,s,r',s')}$ of four rational integers $\geqq 0$, defined by

$$(3) \qquad \bar{N}_{(r,s,r',s')} \equiv \sum_{\alpha=0}^{\alpha \leq s,r'} \binom{r+r'+1}{r+1+\alpha} \cdot \binom{s+s'+1}{s-\alpha},$$

and extend this definition to include

$$(4) \qquad \bar{N}_{(r,s,-1,s')} \equiv 0 \equiv \bar{N}_{(r,-1,r',s')}, \quad \text{and}$$
$$\bar{N}_{(r,s,r',-1)} \equiv \binom{r+s+r'+1}{r'} \equiv \bar{N}_{(-1,r',s,r)}.$$

Now, in the previous paper,¶ I have defined an *N-function* identical with $\bar{N}$ for the arguments in (4) and otherwise equal to the numerator of the right member of (1). Obviously,

* B. H. Camp, *Biometrika*, vol. 17 (1925), pp. 61-67.

† W. R. Thompson, *loc. cit.*

‡ Karl Pearson, *Philosophical Magazine*, Series 6, vol. 13 (1907), pp. 365-378; *Biometrika*, vol. 20A (1928), pp. 149-174.

§ W. R. Thompson, *loc. cit.*

¶ W. R. Thompson, *loc. cit.*

$$\bar{N}_{(r,s,r',s')} \equiv \bar{N}_{(s',r',s,r)} \equiv \binom{n+n'+2}{n+1} - \bar{N}_{(s,r,s',r')},$$

as has been proved for the N-function,* and

(5) $$\bar{N}_{(r,s,0,s')} \equiv N_{(r,s,0,s')};$$

and we may verify readily by (3) that *in general*

(6) $$\bar{N}_{(r,s,r',s')} \equiv \bar{N}_{(r,s-1,r',s')} + \bar{N}_{(r,s,r',s'-1)},$$

which relation was shown in my first paper to hold for the N-function also. Accordingly, by complete induction we may demonstrate that

(7) $$\bar{N}_{(r,s,r',s')} \equiv N_{(r,s,r',s')},$$

and therefore

(8) $$\psi_{(r,s,r',s')} \equiv \frac{\sum_{\alpha=0}^{\alpha\le s,r'} \binom{r+r'+1}{r+1+\alpha} \cdot \binom{s+s'+1}{s-\alpha}}{\binom{r+s+r'+s'+2}{r+s+1}},$$

By a simple rearrangement of factors after expressing the binomial coefficients in (8) by factorial numbers we may obtain

(9) $$\psi_{(r,s,r',s')} \equiv \frac{\sum_{\alpha=0}^{\alpha\le s,r'} \binom{r+s+1}{r+1+\alpha} \cdot \binom{r'+s'+1}{r'-\alpha}}{\binom{r+r'+s+s'+2}{r+r'+1}}$$

which is the equivalent of the expression in (2) if we set $W = r + s + 1$, $B = r' + s' + 1$, $w = r + 1$ and $b = r' + 1$, which is the required relation. Furthermore, (8) and (9) give

(10) $$\psi_{(r,s,r',s')} \equiv \psi_{(r,r',s,s')},$$

i. e., $\psi_{(a_1,a_2,a_3,a_4)}$ is invariant under the substitution $(2, 3)$, which therefore belongs to the group, G. Now, by the identities of (10) and (23) of the previous paper,† we have obviously established that $(1, 4)(2, 3)$ is also in G, and that $(1, 2)(3, 4)$ changes ψ to $1 - \psi$ and is therefore an element of the set A. On the other hand if $a_1 = 3$, $a_2 = 2$, $a_3 = 1$, and $a_4 = 0$, the substitution $(1, 3)$ brings a change in value of ψ from $9/14$ to $13/14$, and therefore $(1, 3)$ belongs neither to G nor A. Now, if the four arguments are all different they may be arranged in 24 different ways; whence, if m is the

* W. R. Thompson, *loc. cit.*

† W. R. Thompson, *loc. cit.*

number of different substitutions in the group, G, then $24/4 \geqq m \equiv 0 \pmod 4$. Accordingly, we have established the fact that the complete group leaving ψ invariant is generated by the two transpositions, $(2,3)$, and $(1,4)$; i. e.,

$$G = [(2,3),(1,4)]. \tag{11}$$

Moreover, the set of substitutions, A, changing ψ to $1-\psi$ may be represented in the form,

$$A = \{g \cdot (1,2)(3,4)\} \tag{12}$$

where g is an element of G.

By the aid of (11) and (12) we may prove and state in simple form certain relations,* and prior to any use of the ψ-function obtain the most convenient arrangement for the work; and in tabulations only 3 values need be listed for each combination of the four arguments without loss of completeness, namely $\psi_{(a,b,c,d)}$, $\psi_{(a,b,d,c)}$, and $\psi_{(a,c,d,b)}$. We may readily verify also that if two of these arguments are equal then two of the three values are sufficient, if three of the arguments are equal or there are two pairs of equal arguments then one value is enough, and if $a=b=c=d$ then none is needed in order to evaluate ψ in a simple manner by means of (11) and (12). By use of the N-function as previously suggested † instead of ψ intabulation in a systematic process with increasing arguments we may list only values of this reduced form of table; e. g., $a \geqq d \geqq c \geqq b > 0$ with the relations given in (6) and (7) and

$$N_{(r,s,r',0)} \equiv \binom{r+s+r'+2}{r'+1} - \binom{r+r'+1}{r} \tag{13}$$

$$\text{and} \quad N_{(r,0,r',s')} \equiv \binom{r+r'+1}{r}.$$

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* W. R. Thompson, *loc. cit.*

† Thus we may obtain readily, the relation,

$$\psi_{(r,s,r',s')} \equiv \psi_{(r-1,s,r',s')} - \frac{(s+s'+1)\cdot\binom{r+r'}{r}\cdot\binom{s+s'}{s}}{(r+s+1)\binom{r+s+r'+s'+2}{r+s+1}},$$

and simply from limit relations previously established,

$$I_{p(r+1,s+1)} \equiv I_{p(r,s+1)} - \binom{n}{s} p^r \cdot q^{s+1}$$

$$\equiv I_{p(r+1,s)} + \binom{n}{r} p^{r+1} \cdot q^s$$

where $q = 1-p$, and $I_{x(u,v)} \equiv \dfrac{B_{x(u,v)}}{B_{1(u,v)}}$.

3. For my own purposes I constructed a rough machine based on the probability relation (9) as follows:

I took the cover of a square cardboard box, which I cut and bent along the diagonal forming a box having the shape of an isosceles triangle with 45° base angles. In this I placed $n + n' + 2$ balls as used in bearings. Of these $n' + 1$ had been made dull by a copper sulphate bath. I shall call these *black* and the others *white*. I then shuffled these balls in the box, and at random allowed them all * to line up along the long side or hypotenuse of the box. This alignment I regarded as a draft proceeding from left to right. Here the advantage of a prior arrangement of the arguments of ψ so as to make the number of balls to be scanned as small as possible is apparent. The critical condition was to encounter $r + 1$ *white* before $r' + 1$ *black* balls.

I supposed now that I was considering a case of the sort where I have to assign individuals to one of two methods of treatment,† T_1 and T_2, in proportion based on the ψ-function of the accumulated evidence in the conventional r, s, r', s' form. I then gave certain values to p_1 and p_2 to govern the chance of *failure* when T_1 and T_2 were tried, respectively; but otherwise acted as if p_1 and p_2 were unknown. Starting with no experience, then $r = s = r' = s' = 0$, I placed $r + s + 1 = 1$ white and $r' + s' + 1 = 1$ black ball in the box, and shuffled. After alignment then T_1 was chosen if the white ball was at the left and otherwise T_2 was chosen. The treatment chosen, T_i, was *tried* by the corresponding probability, p_i, and the result recorded in new values of r, s, r', s'; i. e., if T_2 were tried with *success* these new values then were $0, 0, 0, 1$; if with failure then they would have been $0, 0, 1, 0$. Similar remarks hold if T_1 were chosen. I then added a ball, white if T_1 had been tried and otherwise black. These three balls were now shuffled and aligned at random. As before, if the critical condition of encountering $r + 1$ white before $r' + 1$ black balls were met then the treatment, T_1 was used at this turn, and otherwise T_2. The result of the treatment indicated was noted and new values of r, s, r', s' obtained, and another ball added to the box according to the criterion described for the last turn, and so on until a given number of trials had been made.

In the accompanying table values of p_1 and p_2 used in such experiments are given together with the final results—the total number of trials, $n + n'$;

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* As a matter of fact it is not necessary that all the balls be lined up. The object is simply to quickly establish a random draft order.

† By *treatment* we imply a special mode of dealing with *individuals* of a given class of things.

the number of these wherein the *conventionally* worse method (T_1) was used, n; and the number of *failures*, r, among these n trials.

To make the table quite clear, take the numbers in the second row. Here we have the record of four parallel experiments wherein T_2 was governed by a condition such that failure might be expected about half the time and T_1 to fail always. The total number of trials, $n + n' = 40$, and the number of these systematically allotted to T_1 was $n = 5, 9, 7$, and 5 in the respective experiments, and r, of course, had the same values here. The relatively small value of these even in so small a total number of trials, indicates strikingly the rapidity with which this systematic apportionment between the rival treatments, T_1 and T_2, tends to favor the better, even though prior knowledge as to the fact that T_2 is the better is disregarded.

Although the machine used is extremely crude, all the results obtained were extremely favorable. A more carefully constructed machine along the same lines might give even better results. I have conducted a few additional experiments with this simple box, in which I have deliberately arranged an unfavorable start. I was greatly pleased to note the rapidity with which the machine brought about a reversal of favor to the better method, T_2, as the experiments proceeded.

4. The system of apportionment which we have examined admits a simple extension to the general case of k rival *treatments*, (T_i). As defined in § 1, we let p_i represent the unknown probability of failure by treatment T_i, and our experience with this treatment to consist of r_i failures and s_i successes, where $i = 1, \cdots, k$. Now, if we place $r_i + s_i + 1$ balls of a kind, C_i; for $i = 1, \cdots, k$; in our box, shuffle and draw as before, then we note that the probability of drawing $r_i + 1$ of the i-th *kind* before $r_j + 1$ of the j-th kind is independent of the presence of the balls of other kinds and identical with P_{ij} where $i \neq j$ and

$$P_{ij} \equiv \psi_{(r_i, s_i, r_j, s_j)}. \tag{14}$$

Thus we see that the probability that $r_i + 1$ balls of the i-th kind be so drawn before $r_j + 1$ of the j-th kind, where $i \neq j = 1, \cdots, k$ is *exactly* P_i defined by the relation

$$P_i \equiv \prod_{i \neq j=1}^{k} P_{ij} \equiv 2 \prod_{j=1}^{j=k} P_{ij}. \tag{15}$$

Arbitrarily, as in the case $k = 2$, we may apportion *individuals* among the k rival *treatments* by assigning to each T_i the portion, f_i, or making the chance of this assignment equal f_i, respectively. We may thus arbitrarily take $f_i = P_i$, which may be calculated or we may use the machine, as we have seen that a unique answer is given at each turn just as in the special case, considered previously. Unlike that case, however, we are unable to state that P_i is the probability that T_i is the best of the k rivals; but its composition in (15) indicates that it may well serve the proposed purpose.

TABLE.

p_1	p_2	Total Trials $(n + n')$	Trials of T_1 (n)	Failures of T_1 (r)	Approx. $(n \cdot p_2)$*
1	0	20	2, 1, 1, 1	2, 1, 1, 1	0
1	1/2	40	5, 9, 7, 5	5, 9, 7, 5	2, 4, 3, 2
1/2	0	40	6, 2, 3, 5	2, 2, 2, 2	0
3/4	1/4	100	3, 4	3, 3	1, 1
1	3/4	100	14, 10	14, 10	10, 7
3/4	1/2	100	23, 14	17, 11	8, 5
1/2	1/4	100	10, 13	5, 6	2, 3
1/4	0	100	4, 6	1, 1	0

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* *Expectation* of loss in the same n had T_2 been used.