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ON THE NATURE OF MATHEMATICAL PROOFS

Everything that exists exists in some degree, and if it exists in some degree it ought to be measured.

—Mathematicians' Bill of Rights

Bertrand Russell has defined mathematics as the science in which we never know what we are talking about or whether what we are saying is true. Mathematics has been shown to apply widely in many other scientific fields. Hence, most other scientists do not know what they are talking about or whether what they are saying is true.

Thus, providing a rigorous basis for philosophical insights is one of the main functions of mathematical proofs.

Aristotle was among the first philosophers to study mathematical proofs. He invented the sillygism, a device which, because of its absolute uselessness, has interested countless philosophers and logicians. Briefly, by means of a sillygism, one infers a conclusion from a major and minor premise. In fact, logicians are always coming to conclusions. The miracle of it is that they haven't got around to stopping yet.

The major premise makes a statement about a class of things; for instance, "Not all major premises are true." The minor premise says that the thing with which we are concerned is a member of the class; for instance, "The last six words of the first sentence of this paragraph are a major premise." From this we conclude, "It is not always true that not all major premises are true." Such is the overwhelming capacity of logic to inform us of the realities of daily life.

We note, however, that premises are essential for sillygisms. A baleful influence on Aristotelian habits among logicians has been exercised by the recent plethora of signs warning "Keep Off the Premises."

Another function of the mathematical proof is to draw probable inferences from mathematical models of physical systems. For instance, given a sufficient number of physical data, the most desirable mathematical model is probably 36-24-36. *De gustibus non est disputandum.*

With mathematical proof, scientists have succeeded in relating the hitherto disparate fields of thermodynamics and communication engineering in the discipline of information theory. "Information," as technically defined, is proportionate to surprise; the more surprising a message is, the more information it contains. If someone listening over the telephone heard "Hello" at the beginning of a conversation, he would not be very surprised; but his gain of information would be quite large if he were suddenly electrocuted. Great new possibilities in mathematical proof were made available with the development of set theory around the end of the last century and the beginning of this one. A set is any well-defined collection; examples are the country club set, the table please set, permanent hair-set upset. A theorem in set theory recently discovered by this author is eminently worthy of mention here; the proof will be sketched.

THEOREM: A set whose only element is a set may be isomorphic to a set whose only element is a set whose only elements are a subgroup of the group of elements in the set which is the only element of the set with which it is isomorphic (sick).

This intuitively obvious theorem follows rather deviously from the first isomorphism theorem of group theory. The duty of the logician, however, is to find the shortest logical line between hypothesis and conclusion. In his interest we give the following proof by a familiar method.

Hypothesis: We assume the entire existing body of mathematics. Step 1: "By inspection" the theorem follows.

The aesthetically appealing simplicity of this method of proof has made many students revere the power of logic. The beauty of this method is exceeded by that of only one other, practiced by Immanuel Kant and first explicated by this author as "proof by assumption." By assuming the desired conclusion in the hypothesis, the proof is somewhat simplified.

Besides "proof by inspection" and "proof by assumption," we have to consider "proof by induction." Induction is so widespread that even the Army does not hesitate to use it. So mechanical is its application that there exists an electronic device cleverly called an "induction coil." While inductive technique is simple, its results can be both deep and profound.

The inductive principle is based on a set of five axioms stated around the end of the last century by an Italian musician named Piano. Piano was trying to teach his *bambino* (Italian for "child") some arithmetic. The first axiom was that zero is a number. Any idiot knows this, which is why Piano was a musician and not a mathematician. Axioms two, three, and four were on a similar level of sophistication. For the fifth axiom, we must introduce the idea of a property. The numbers 1, 4, 9, and 16 all have the property of being the square of some natural number. If we call this property F, then we can say, "1 is F, 4 is F," and so on. Now let F be an arbitrary property, for instance, "monotonous," "incomprehensible." Piano's fifth axiom was "Every number is F if the property F satisfied the two conditions: (1) zero is F, and (2) if any individual is F, then so is its successor." At this point Piano's *bambino* wet his diaper.

This brings logical systems to mind. A logical system is distinct from a collection of theorems much as a mansion differs from a brickyard: in a logical system each theorem is based upon what has preceded. G. Polya has observed that Euclid's contribution was not in collecting geometrical facts but in arranging them logically. Had he thrown them together randomly he might have been just an ordinary author of high school texts.

To illustrate the various methods of proof discussed above, we give an extended example of a logical system. (For the first theorem and lemma of this system, which I propose to call "the pejorative calculus," I am indebted to Professor Lee M. Sonneborn, Fine Topologist, of the University of Kansas. Dr. Sonneborn is initially known among his students as "L.M.S.F.T." The rest of the system is presented for the first time in this paper.)

The Pejorative Calculus

LEMMA I. *All horses are the same color* (by induction).

Proof: It is obvious that one horse is the same color. Let us assume the proposition, $P(k)$, that k horses are the same color and show this to imply that $k + 1$ horses are the same color. Given the set of $k + 1$ horses, we remove one horse; then the remaining k horses are the same color, by hypothesis. We remove another horse and replace the first; the k horses, by hypothesis, are again the same color. We repeat this until by exhaustion the $k + 1$ sets of k horses each have shown to be the same color. It follows then that, since every horse is the same color as every other horse, $P(k)$ entails $P(k + 1)$. But since we have shown $P(1)$ to be true, P is true for all succeeding values of k; i.e., all horses are the same color.

THEOREM I. *Every horse has an infinite number of legs* (proof by intimidation).

Proof: Horses have an even number of legs. Behind they have two legs, and in front they have fore legs. This makes six legs, which is certainly an odd number of legs for a horse. But the only number that is both odd and even is infinity. Therefore horses have an infinite number of legs. Now to show that this is general, suppose that somewhere there is a horse with a finite number of legs. But that is a horse of another color, and, by the lemma, that does not exist.

COROLLARY I. *Everything is the same color.*

Proof: The proof of Lemma I does not depend at all on the nature of the object under consideration. The predicate of the antecedent of the universally quantified conditional "for all x, if x is a horse, then x is the same color," namely, "is a horse," may be generalized to "is anything" without affecting the validity of the proof; hence, "for all x, if x is anything, x is the same color." (Incidentally, x is the same color even if x isn't anything, but we do not prove that here.)

COROLLARY II. *Everything is white.*

Proof: If a sentential formula in x is logically true, then any particular substitution instance of it is a true sentence. In particular then, "for all x if x is an elephant, then x is the same color" is true. Now it is manifestly axiomatic that white elephants exist (for proof by blatant assertion consult Mark Twain, "The Stolen White Elephant"). Therefore, all elephants are white. By Corollary I everything is white.

THEOREM II. *Alexander the Great did not exist and he had an infinite number of limbs.*

Proof: We prove this theorem in two parts. First, we note the obvious fact that historians always tell the truth (for historians always take a stand, and, therefore, they cannot lie). Hence, we have the historically true sentence, "If Alexander the Great existed, then he rode a black horse Bucephalus." But we know by Corollary II everything is white; hence Alexander could not have ridden a black horse. Since the consequent of the conditional is false in order for the whole statement to be true, the antecedent must be false. Hence, Alexander the Great did not exist.

We also have the historically true statement that Alexander was warned by an oracle that he would meet death if he crossed a certain river. He had two legs; and "fore-warned is four-armed." This gives him six limbs, an even number, which is certainly an odd number of limbs for a man. Now the only number that is even and odd is infinity; hence, Alexander had an infinite number of limbs. But suppose he had a finite number of limbs. Then it would be possible to put his limbs in

a one-to-one correspondence with the natural numbers, an operation which we shall call "limbing"; and there would exist a last limb, and we should be able to limb it. But only an infinite series approaches a limb it. Hence he had an infinite number of limbs.

We have proved: Alexander the Great did not exist and he had an infinite number of limbs.

It is not to be imagined from this merely compendious account of the nature of mathematical proofs that everything has been proved. Witness the celebrated paradox of Euler's little liver lemma concerning the four cooler problem. Specifically, we cite two unproved examples. The first is the famous Goldbrick conjecture from the theory of numbers, which states that every prime number is expressible as the sum of two even numbers. No counter-example has been found to this seemingly artless assertion, and the search for its proof has occupied mathematicians for centuries.

The second example is a generalization well-known, even if only intuitively, to practically the whole uncivilized world. It is Chisholm's famous first law: "If something can go wrong, it will."

Nor is it to be thought that there are not other types of proofs, which in print shops are recorded on proof sheets. There is the bullet proof and the proof of the pudding. Finally, there is 200 proof, a most potent spirit among mathematicians and people alike.