# OPERATIONAL METHODS IN MATHEMATICAL PHYSICS.

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§ 1. This essay-review of Jeffreys' very welcome and valuable Tract with the above title * has been written at the editor's request. Many readers of the *Gazette* must have heard of Heaviside's operational method of solving the equations of dynamics and mathematical physics. If they have tried to learn about them from Heaviside's own works, they have attempted a difficult task. Nothing more obscure than his mathematical writings is known to me. A Cambridge Tract is now at their disposal. From it much may be learned; but the air of mystery still—at least in part—remains.

In every course on differential equations the student learns to solve by a symbolical method the linear differential equation with constant coefficients. He writes the equation:

$$a_0\frac{d^n x}{dt^n}+a_1\frac{d^{n-1}x}{dt^{n-1}}+\dots+a_n x=T,$$

as $f(D)x=T$, and he finds a particular integral with the help of the inverse operator.

Part of Jeffreys' Tract deals with an extension of this method. The solution of the linear differential equation with constant coefficients (or equations, if we are concerned with simultaneous equations) is found, satisfying the given initial conditions, and this without the trouble of finding the arbitrary constants of the complementary function and the relation between them. In this section there is no serious difficulty. But I wonder if the time saved in the solution of the equations is worth the labour involved in learning (or teaching) the new method.

The other, and more important, part of the Tract deals with something much more obscure, Heaviside's operational method. What this is will appear from an example taken from his writings,† the simple heat conduction problem, where a solid bounded by the plane $x=0$, and extending to infinity in the direction of the positive axis of $x$, has its plane surface kept at the constant temperature $v_0$, the initial temperature through the solid being zero.

In this problem we have to find $v$ to satisfy:

$$\frac{\partial v}{\partial t}=k\frac{\partial^2 v}{\partial x^2},\quad \text{when } x>0,\ t>0; \qquad \text{1. 1}$$

$$v=v_0,\quad \text{when } x=0,\ t>0, \qquad \text{1. 2}$$

$$v=0,\quad \text{when } x>0,\ t=0. \qquad \text{1. 3}$$

Putting $\dfrac{\partial}{\partial t}=p=kq^2$, we have

$$\frac{\partial^2 v}{\partial x^2}=q^2 v;$$

and a suitable formal solution is

$$\begin{aligned} v&=e^{-qx}v_0\\ &=\left(1-qx+q^2\frac{x^2}{2!}-\dots\right)v_0. \end{aligned} \qquad \text{1. 4}$$

Now $p$ stand for $\dfrac{\partial}{\partial t}$, and Heaviside found that if we interpret $p^{\frac{1}{2}}$, operating on 1, as $(\pi t)^{-\frac{1}{2}}$, and obtain $p^{\frac{3}{2}}$, $p^{\frac{5}{2}}$, ... from $p^{\frac{1}{2}}$ by differentiation, the symbolical

* *Cambridge Tracts in Mathematics and Mathematical Physics.* No. 23, by Harold Jeffreys, (Camb. Univ. Press), 1927. Price 6s. 6d. net.

† *Electromagnetic Theory*, by Oliver Heaviside, vol. 2, p. 13, 1899.

solution given in 1. 4, and similar solutions of other problems of the same kind, do really satisfy the equations from which we start.

In this case,* beginning with $p^{\frac{1}{2}} . 1 = \frac{1}{\sqrt{(\pi t)}}$, we have

$$p^{\frac{3}{2}} . 1 = p . p^{\frac{1}{2}} . 1 = \frac{\partial}{\partial t} \frac{1}{\sqrt{(\pi t)}} = -\frac{1}{2\sqrt{\pi}} t^{-\frac{3}{2}}$$

and $$p^{\frac{5}{2}} . 1 = p . p^{\frac{3}{2}} . 1 = \frac{1 . 3}{2 . 2} \frac{1}{\sqrt{\pi}} t^{-\frac{5}{2}}, \text{ etc.}$$

And $p$, $p^2$, etc., operating on 1, give zero.

Interpreting the symbols in 1. 4 in this way, we have

$$\begin{aligned} v &= \left\{ 1 - \frac{2}{\sqrt{\pi}} \left( \frac{x}{2\sqrt{(kt)}} - \frac{1}{3} \left( \frac{x}{2\sqrt{(kt)}} \right)^3 + \frac{1}{2\,!\,5} \left( \frac{x}{2\sqrt{(kt)}} \right)^5 - \dots \right) \right\} v_0 \\ &= v_0 \left\{ 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{(kt)}}} e^{-u^2} du \right\} \\ &= v_0 \left( 1 - \operatorname{Erf} \frac{x}{2\sqrt{(kt)}} \right), \end{aligned}$$

with the usual notation for the "error function,"

$$\operatorname{Erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du.$$

§ 2. Heaviside himself hardly claimed that he had "proved" his operational method of solving these partial differential equations to be valid. With him † mathematics was of two kinds: *Rigorous* and *Physical.* The former was *Narrow*: the latter *Bold* and *Broad.* And the thing that mattered was that the Bold and Broad Mathematics got the results. "To have to stop to formulate rigorous demonstrations would put a stop to most physico-mathematical enquiries." Only the purist had to be sure of the validity of the processes employed.

Jeffreys (p. 47) agrees that the arguments upon which Heaviside relies are "in many cases suggestive rather than demonstrative." And he seems to think his Tract places the operational method on another plane. But, if I may say so, there is too much of the Bold and Broad School in the work. It leaves me still doubtful if it is wise to make this method one of the tools of the mathematical physicist. There is no room for mystery in mathematics. If we can be clear, let us be so. And for my part I consider the best way of attacking many of these questions is to use contour integrals. It is only in England and America that the mathematical physicist is afraid of the elementary theory of the functions of a complex variable required in this method. And surely he need not indulge this fear in Cambridge. To adopt the words of Heaviside, which Jeffreys takes for the motto of his Tract,—

*Even Cambridge mathematicians deserve justice.*

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* If $p$ stands for $\frac{\partial}{\partial t}$, one would expect $p^{-1}$ to denote integration. Then, for positive integral values of $n$, we would have

$$p^{-n} . 1 = \frac{t^n}{n!} = \frac{t^n}{\Gamma(n+1)}.$$

If this final formula is to hold for fractional values of $n$, we would have

$$p^{-\frac{1}{2}} . 1 = \frac{t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} = 2\sqrt{\frac{t}{\pi}}.$$

And $p^{\frac{1}{2}} . 1 = p . p^{-\frac{1}{2}} . 1 = \frac{1}{\sqrt{(\pi t)}}$.

† Cf. *loc. cit.*, p. 4.

§ 3. Turn now to Chapter I. of the Tract. This deals with the solution of the equations:

$$\left.\begin{array}{l} e_{11}y_1+e_{12}y_2+\dots+e_{1n}y_n=S_1 \\ e_{21}y_1+e_{22}y_2+\dots+e_{2n}y_n=S_2 \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ e_{n1}y_1+e_{n2}y_2+\dots+e_{nn}y_n=S_n \end{array}\right\}, \quad \dots\dots 3.\ 1$$

where the $y$'s are dependent variables, $x$ the independent variable, $e_{rs}$ denotes $a_{rs}\dfrac{d}{dx}+b_{rs}$, where $a_{rs}$ and $b_{rs}$ are constants, and the $S$'s are known functions of $x$. It is not assumed that $a_{rs}=a_{sr}$, $b_{rs}=b_{sr}$, but it is assumed that the determinant formed by the $a$'s is not zero, and that when $x=0$, we have $y=u_1$, and so on, the $u$'s being known constants.

Now let $Q$ stand for the operation of integrating with regard to $x$ from 0 to $x$, so that $Qy=\int_0^x y\,dx$.

Perform the operation $Q$ on both sides of each equation in 3. 1.

Then we have
$$\begin{aligned} Qe_{rs}y_s&=\int_0^x\left(a_{rs}\frac{dy_s}{dx}+b_{rs}y_s\right)dx \\ &=a_{rs}(y_s-u_s)+b_{rs}Qy_s \\ &=f_{rs}y_s-a_{rs}u_s, \quad \dots\dots 3.\ 2 \end{aligned}$$

where
$$f_{rs}=a_{rs}+b_{rs}Q. \quad \dots\dots 3.\ 3$$

Thus the equations 3. 1 and the initial conditions are together equivalent to the equations:

$$\left.\begin{array}{l} f_{11}y_1+f_{12}y_2+\dots+f_{1n}y_n=v_1+QS_1 \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ f_{n1}y_1+f_{n2}y_2+\dots+f_{nn}y_n=v_n+QS_n \end{array}\right\}. \quad \dots\dots 3.\ 4$$

where $v_r=a_{r1}u_1+a_{r2}u_2+\dots+a_{rn}u_n$.

Let $\Delta$ stand for the operational determinant formed by the $f$'s,

*i.e.*
$$\Delta=\left|\begin{array}{l} f_{11},\ f_{12},\ \dots,\ f_{1n} \\ f_{21},\ f_{22},\ \dots,\ f_{2n} \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ f_{n1},\ f_{n2},\ \dots,\ f_{nn} \end{array}\right|. \quad \dots\dots 3.\ 5$$

If $\Delta$ is expanded by the ordinary rules of algebra, and equal powers of $Q$ collected, we obtain a polynomial in $Q$. The term independent of $Q$ is the determinant formed by the $a$'s, and, by hypothesis, this does not vanish. Let $F_{rs}$ be the co-factor of $f_{rs}$ in $\Delta$. Then $F_{rs}$ is also a polynomial in $Q$.

Operate on the first equation of 3. 4 with $F_{1s}$, on the second with $F_{2s}$, etc., and add. The only term in the sum which does not vanish is

$$\sum_{r=1}^{n}(F_{rs}f_{rs})\,y_s \quad \text{or} \quad \Delta y_s.$$

Therefore
$$\Delta y_s=\sum_{r=1}^{n}F_{rs}(v_r+QS_r), \quad \dots\dots 3.\ 6$$

And
$$\begin{aligned} y_s&=\frac{1}{\Delta}\left\{\sum_1^n F_{rs}(v_r+QS_r)\right\} \\ &=\frac{\phi(Q)}{\Delta(Q)}+\sum_{r=1}^{n}\frac{\psi_r(Q)}{\Delta Q}QS_r, \quad \dots\dots 3.\ 7 \end{aligned}$$

where $\phi(Q)$ and $\psi(Q)$ are polynomials in $Q$, whose degree is ordinarily one less than that of $\Delta(Q)$.

The formal solution of 3. 7 can now be treated in the same way as the particular integral of $f(D)\,y=X$ is found in elementary work.

The operator on the right can be expanded in ascending powers of $Q$, and evaluated term by term; of course on the understanding that, if this gives rise to an infinite series, the $S$'s are such that this series converges.

Alternatively the expressions $\dfrac{\phi(Q)}{\Delta(Q)}$, etc., can be broken up into partial fractions. Since the determinant formed by the $a$'s is not zero, we may denote it by $A$, and $\Delta(Q)$ takes the form $A(1-\alpha_1 Q)\,(1-\alpha_2 Q)\,\ldots\,(1-\alpha_n Q)$ where the $\alpha$'s are ordinarily different.

A fraction of the type $\dfrac{1}{1-\alpha Q}$, operating upon a constant $L$, will give a term $Le^{\alpha x}$; and, operating upon $QS$, it will give a term $e^{\alpha x}\int_0^x Se^{-\alpha x}dx$.

The case of repeated roots $\alpha$ is discussed in the text. And the argument is throughout quite similar to the usual treatment of the equation $f(D)\,y=X$.

§ 4. If now, with Heaviside, we write $p^{-1}$ for $Q$, the equations 3. 4 become

$$\left.\begin{array}{l}(a_{11}+b_{11}p^{-1})y_1+(a_{12}+b_{12}p^{-1})y_2+\ldots(a_{1n}+b_{1n}p^{-1})y_n=v_1+p^{-1}S_1\\ (a_{21}+b_{21}p^{-1})y_1+(a_{22}+b_{22}p^{-1})y_2+\ldots(a_{2n}+b_{2n}p^{-1})y_n=v_2+p^{-1}S_2\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ (a_{n1}+b_{n1}p^{-1})y_1+(a_{n2}+b_{n2}p^{-1})y_2+\ldots(a_{nn}+b_{nn}p^{-1})y_n=v_n+p^{-1}S_n\end{array}\right\}. \quad 4.\ 1$$

Multiply throughout by $p$, as if it were a mere number. Then we have

$$\left.\begin{array}{l}(a_{11}p+b_{11})y_1+(a_{12}p+b_{12})y_2+\ldots+(a_{1n}p+b_{1n})y_n=pv_1+S_1\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ (a_{n1}p+b_{n1})y_1+(a_{n2}p+b_{n2})y_2+\ldots+(a_{nn}p+b_{nn})y_n=pv_{12}+S_n\end{array}\right\}. \quad \ldots\ldots 4.\ 2$$

Solve the $n$ equations of 4. 2 by ordinary algebra, and we obtain a solution identical with that of 3. 7, except that $p^{-1}$ will take the place of $Q$, and both numerator and denominator will be multiplied by the same power of $p$.

We write this solution as

$$y_s=\frac{1}{\Delta}\left\{\sum_{r=1}^{n}F_{rs}(pv_r+S_r)\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots 4.\ 3$$

where $F_{rs}$ and the determinant $\Delta$ are not to be confused with the same operational symbols in 3. 6.

The operators on the right of 4. 3 are of the form $f(p)/F(p)$, where $f(p)$ and $F(p)$ are polynomials in $p$, $F(p)$ being of degree $n$ and $f(p)$ of the same or lower degree. Resolving $F(p)$ into its $n$ factors, supposed all different, we have the algebraical identity,

$$\frac{f(p)}{pF(p)}=\frac{f(0)}{F(0)p}+\sum_\alpha\frac{f(\alpha)}{\alpha F'(\alpha)}\frac{1}{p-\alpha},$$

whence

$$\frac{f(p)}{F(p)}=\frac{f(0)}{F(0)}+\sum_\alpha\frac{f(\alpha)}{\alpha F'(\alpha)}\frac{p}{p-\alpha}.$$

If this operates on unity, the term $\dfrac{p}{p-\alpha}$ gives rise to $e^{\alpha x}$, and *

$$\frac{f(p)}{F(p)}\cdot 1=\frac{f(0)}{F(0)}+\sum_\alpha\frac{f(\alpha)}{\alpha F'(\alpha)}e^{\alpha x}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots 4.\ 4$$

If it operates on $e^{\mu x}$, we replace $e^{\mu x}$ by $\dfrac{p}{p-\mu}\cdot 1$.

* This is usually known as Heaviside's Expansion Theorem, or the Partial Fraction Rule. Cf. *loc. cit.*, p. 127.

Then $\frac{f(p)}{F(p)} \cdot e^{\mu x} = \frac{pf(p)}{(p-\mu)F(p)} \cdot 1$

$$= p\left\{\frac{f(\mu)}{F(\mu)}\frac{1}{p-\mu} + \sum_{a}\frac{f(a)}{(a-\mu)F'(a)}\frac{1}{p-a}\right\} \cdot 1$$

$$= \frac{f(\mu)}{F(\mu)}e^{\mu x} + \sum_{a}\frac{f(a)}{(a-\mu)F'(a)}e^{ax}.$$

If $S$ is expressed as a linear combination of exponentials, we can apply this rule to each separately. Thus the solution applies to practically all functions known to physics.

It will be seen that in this section we replace $d/dx$ in the original equations of 3. 1 by $p$, and to the right of each equation add the result of dropping the $b$'s on the left and replacing the $y$'s by their initial values. Then we proceed to solve these equations by ordinary algebra, and interpret our result by certain simple rules.

In this argument, in which I have followed the Tract pretty closely, there does not seem any lack of rigour, though there is just a little mystery left about $p$, for it is only $p^{-1}$ which has been defined. It is obvious that in this discussion $p$ is not just the operation of differentiation, for in that case $pv_r$, etc., would vanish.

If the reader wishes a proof on other lines, he has in Chapter II. an independent discussion based on Bromwich's interpretation of a function of $p$ as a contour integral.* His proof for these dynamical equations is completely satisfactory.

§ 5. This work can be at once extended to equations of a higher order by breaking them up into equations of the first order.

For example, take the equation

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = X, \quad \text{.................................5. 1}$$

where $a$, $b$ are constants, and $X$ is a function of $x$. The solution is also to satisfy $y = y_0$ and $\frac{dy}{dx} = y_1$, when $x = 0$.

Introduce a new variable $z$ given by $z = \frac{dy}{dx}$.

Then 5. 1 is replaced by the two equations:

$$\left.\begin{aligned} \frac{dz}{dx} + az + by &= X \\ z - \frac{dy}{dx} &= 0 \end{aligned}\right\}. \quad \text{.................................5. 2}$$

Operating on these with $Q$, we have

$$\left.\begin{aligned} (z - y_1) + aQz + bQy &= QX \\ Qz - (y - y_0) &= 0 \end{aligned}\right\}. \quad \text{............................5. 3}$$

Thus

$$\left.\begin{aligned} (p + a)z + by &= py_1 + X \\ z - py &= -py_0 \end{aligned}\right\}. \quad \text{............................5. 4}$$

Solving 5. 4 by algebra, we have

$$(p^2 + ap + b)y = (p^2 + ap)y_0 + py_1 + X,$$

and

$$y = \frac{(p^2 + ap)y_0 + py_1 + X}{p^2 + ap + b}. \quad \text{............................5. 5}$$

Let the roots of $p^2 + ap + b = 0$ be $\alpha$ and $\beta$.

* The revived interest in Heaviside's operational method is due chiefly to a paper by Bromwich on "Normal Co-ordinates in Dynamical Systems," *Proc. London Math. Soc.* (2), vol. 15, 1916, and to other papers of his in which the method is freely used.

Then from 5. 5,

$$y=p\left\{\frac{(p+a)y_0+y_1}{(p-\alpha)(p-\beta)}\right\}+\frac{1}{(\alpha-\beta)}\left\{\frac{1}{p-\alpha}-\frac{1}{p-\beta}\right\}X.$$

Therefore

$$y=\frac{1}{\alpha-\beta}\left[\left\{\frac{p}{p-\alpha}(y_1-\beta y_0)-\frac{p}{p-\beta}(y_1-\alpha y_0)\right\}+\left\{\frac{1}{p-\alpha}-\frac{1}{p-\beta}\right\}X\right]. \quad ..5.6$$

Now $$\frac{p}{p-\alpha}\cdot 1=\frac{1}{1-\alpha Q}\cdot 1=e^{\alpha x}.$$

And $$\frac{1}{p-\alpha}X=\frac{Q}{1-\alpha Q}X=e^{\alpha x}\int_0^x Xe^{-\alpha x}dx.$$

Thus

$$y=\frac{1}{(\alpha-\beta)}\left[(y_1-\beta y_0)e^{\alpha x}-(y_1-\alpha y_0)e^{\beta x}+e^{\alpha x}\int_0^x Xe^{-\alpha x}dx-e^{\beta x}\int_0^x Xe^{-\beta x}dx\right]. \quad 5.7$$

Worked-out examples of the solution of equations of this type are given on p. 14 of the Tract. It will be seen that to solve

$$\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by=X,$$

subject to the initial conditions, we write down what Jeffreys calls the *subsidiary* equation

$$(p^2+ap+b)y=(p^2+ap)y_0+py_1+X,$$

and proceed thereafter according to quite simple rules. The work is certainly much shorter than when the usual method is followed.

§ 6. If Heaviside's operational method were simply that described and proved in §§ 3-4, it would not have been the cause of so much debate. But we have pointed out in § 1 that it was used by him in the treatment of the partial differential equations of mathematical physics. Indeed most of his researches in electric waves are carried out with its aid.

Jeffreys still leaves much that is mysterious in this connection. He deals in Chapters IV. and V. with the equations $\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}$ of wave motion and $\frac{\partial v}{\partial t}=k\frac{\partial^2 v}{\partial x^2}$ of heat conduction. In the case of the former he says (p. 41), " we are led by our previous rules to consider the subsidiary equation "

$$(\sigma^2-c^2p^2)y=\sigma^2 f(x)+\sigma F(x),$$

where $\sigma$ stands for $\frac{\partial}{\partial t}$, $p$ for $\frac{\partial}{\partial x}$, and the initial values of $y$ and $\frac{\partial y}{\partial t}$ are $f(x)$ and $F(x)$. And p. 53, which he devotes to the " proof," does not help us much.

Again in the case of the heat equation (p. 55), he writes down the subsidiary equation

$$\sigma v-k\frac{\partial^2 v}{\partial x^2}=\sigma v_0,$$

where $\sigma=\frac{\partial}{\partial t}$, and $v=v_0$, when $t=0$. His justification of the operational method is eleven lines on p. 66.

Bromwich, on whose work he relies, admits that he has not given a complete proof of the validity of his solutions in the case of continuous systems. In his own words * all that he has done is to " establish an analogy " between the formulae he uses for the operational method in the solution of the partial differential equations and that which he has proved for the dynamical equations and discrete systems.

* Cf. *loc. cit.*, *Proc. London Math. Soc.* (2), vol. 15, 1916, p. 421. But see also pp. 438 *et seq.*

In this matter I think Jeffreys' Tract is subject to criticism. Heaviside was quite open about it. He belonged to the Bold and Broad School. Jeffreys, we take it, to be of the Cambridge School, in which Mathematics is Mathematics. It is true that he obtains correct solutions of the problems he discusses. But it seems to me that too little is said in justification of the method employed.

§ 7. I take as an example his solution of the problem of pp. 59 and 60 by the operational method, and in the next section I give the solution by the method of contour integrals and the standard path adopted by myself. His problem requires the solution of the following :

$$\left.\begin{aligned} &\frac{\partial v}{\partial t}=k\frac{\partial^2 v}{\partial x^2}, \text{ when } 0<x<l, \text{ and } t>0 \quad &\text{.........7. 1}\\ &\frac{\partial v}{\partial x}-hv=0, \text{ when } x=0, \text{ and } t>0 \quad &\text{.........7. 2}\\ &v=v_0, \text{ when } x=l, \text{ and } t>0 \quad &\text{.........7. 3}\\ &v=v_0, \text{ when } 0<x<l, \text{ and } t=0 \quad &\text{.........7. 4}\end{aligned}\right\}$$

The equation 7. 1 is replaced by the subsidiary equation

$$\frac{\partial^2 v}{\partial x^2}-q^2 v=-q^2 v_0, \quad \text{.........7. 5}$$

where

$$\frac{\partial}{\partial t}=\sigma=kq^2.$$

Then 7. 3 and 7. 5 suggest

$$v=(1-A\sinh q(l-x))v_0,$$

and 7. 4 shows that $A$ should satisfy

$$qA\cosh ql-hv_0(1-A\sinh ql)=0.$$

Thus we have the operational solution

$$v=\left(1-\frac{h\sinh q(l-x)}{q\cosh ql+h\sinh ql}\right)v_0. \quad \text{.........7. 6}$$

The values of $\sigma$ which satisfy $q\cosh ql+h\sinh ql=0$ are real and negative, and one form of the solution is given by the Partial Fraction Rule of § 4, except that in this case the denominator has an infinite number of zeros.

Again, we may write 7. 6 as

$$v=\left[1-\frac{he^{-qx}}{q+h}\left(1-e^{-2q(l-x)}\right)\left(1-\frac{q-h}{q+h}e^{-2ql}+\dots\right)\right]v_0. \quad \text{.........7. 7}$$

If the length $l$ is great enough to make the terms in $e^{-2ql}$, etc., negligible, this reduces to the first two terms,

$$v=\left[1-\frac{h}{q+h}e^{-qx}\right]v_0. \quad \text{.........7. 8}$$

Now $q=\sqrt{\frac{\sigma}{k}}$ and Bromwich's Rule (Ch. II. § 2. 1), with an obvious change in the notation, gives

$$\frac{e^{-qx}}{q+h}=\frac{1}{2\pi l}\int_{c-\iota\infty}^{c+\iota\infty}\frac{\sqrt{k}}{\sqrt{\mu}+h\sqrt{k}}\exp\left(\mu t-x\sqrt{\frac{\mu}{k}}\right)\frac{d\mu}{\mu}.$$

Put $\lambda^2=\mu$ and take the corresponding path in the $\lambda$-plane.

The solution is then found to be

$$v=v_0\left\{\operatorname{Erf}\frac{x}{2\sqrt{(kt)}}+\exp(h^2kt+hx)\left(1-\operatorname{Erf}\frac{x+2hkt}{2\sqrt{(kt)}}\right)\right\}. \quad \text{.........7. 9}$$

§ 8. Now dropping all talk about operational methods and returning to ordinary mathematics and contour integrals, let us find the solution of the problem of § 7.

It is convenient in all these heat problems to have the initial temperature throughout the solid zero, so we put $v = v_0 + u$ in the equations 7. 1 to 7. 4.

Thus we have

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \text{ when } 0 < x < l, \text{ and } t > 0 && \text{......8. 1}\\ &\frac{\partial u}{\partial x} - hu = hv_0, \quad \text{when } x = 0, \text{ and } t > 0 && \text{......8. 2}\\ &u = 0, \quad \text{when } x = l, \text{ and } t > 0 && \text{......8. 3}\\ &u = 0, \quad \text{when } 0 < x < l, \text{ and } t = 0 && \text{......8. 4}\end{aligned}\right\}$$

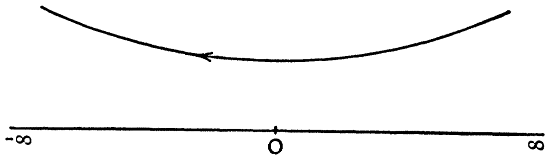
Equations 8. 1 and 8. 3 are satisfied by

$$Ae^{-k\alpha^2 t}\sin\alpha(l - x).$$

From 8. 2 we see that $A$ should satisfy

$$A(\alpha\cos\alpha l + h\sin\alpha l) = -hv_0. \quad \text{......8. 5}$$

Then we take the standard path $(P)$ * of Fig. 1 in the $\alpha$-plane chosen so that at infinity on the right the argument of $\alpha$ lies between 0 and $\frac{1}{4}\pi$, and on the left between $\frac{3}{4}\pi$ and $\pi$.




*The path (P) in the α-plane*

FIG. 1.

It will be seen that, when we take the integral over this path $(P)$,

$$u = -\frac{hv_0}{\iota\pi}\int e^{-k\alpha^2 t}\frac{\sin\alpha(l - x)}{\alpha\cos\alpha l + h\sin\alpha l}\frac{d\alpha}{\alpha}, \quad \text{......8. 6}$$

satisfies 8. 1, 8. 2, and 8. 3.

We have still to prove that this value of $u$ vanishes when $t = 0$; that is, we have to show that

$$\int_P \frac{\sin\alpha(l - x)}{\alpha\cos\alpha l + h\sin\alpha l}\frac{d\alpha}{\alpha} \quad \text{......8. 7}$$

is zero.

Take the closed circuit of Fig. 2, consisting of the path $(P)$, and the arc of a circle, centre at the origin, lying above this path.

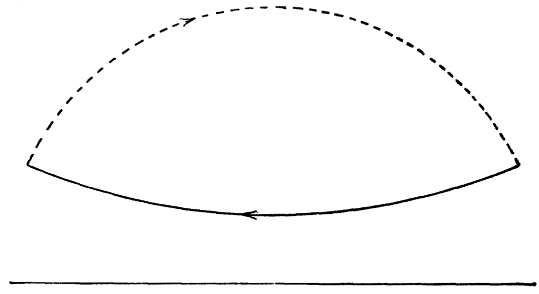



FIG. 2.

* In this section I follow the method and use the diagrams of Chapter XI. of my book on *Conduction of Heat* (Ed. 1921). See also Chapter X. §§ 80-90.

The roots of $\alpha \cos \alpha l + h \sin \alpha l = 0$ are infinite in number, and all lie on the real axis in the $\alpha$-plane, to each positive root $\alpha_r$ corresponding a negative root $-\alpha_r$.

There are thus no poles of the integrand of 8. 7 in the closed contour of Fig. 2, and the integral round it vanishes. But, when the radius of the circle tends to infinity, the integral over the circular arc vanishes.

It follows that

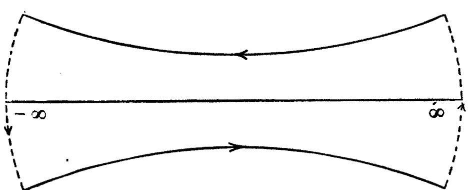
$$\int_P \frac{\alpha \sin \alpha (l-x)}{\alpha \cos \alpha l + h \sin \alpha l} \frac{d\alpha}{\alpha}$$

is zero.

Thus we have established that the function given by 8. 6 satisfies all the conditions of our problem.

We have now only to express this solution in real terms.

(i) As the integrand of 8. 6 is an odd function of $\alpha$, on dividing by 2 we can replace the path $(P)$ of Fig. 1 by the path $(Q)$ of Fig. 3, since the integral over the dotted arcs vanishes at infinity.




*The path (Q) in the α-plane*

FIG. 3.

In this way we have, from 8. 6,

$$u = -\frac{hv_0}{2\iota\pi} \int_Q e^{-k\alpha^2 t} \frac{\sin \alpha (l-x)}{\alpha \cos \alpha l + h \sin \alpha l} \frac{d\alpha}{\alpha}$$

$$= -hv_0 \left[ \frac{l-x}{1+hl} + 2 \sum_{r=1}^{\infty} \frac{\sin \alpha_r (l-x)}{(1+hl) \cos \alpha_r l - \alpha_r l \sin \alpha_r l} \frac{e^{-k\alpha_r^2 t}}{\alpha_r} \right], \quad \text{........8. 8}$$

by the Theory of Residues.

This result corresponds to that given by Heaviside's Partial Fraction Rule.

(ii) Again, we can write 8. 6 in the form

$$u = -\frac{hv_0}{\iota\pi} \int_P e^{-k\alpha^2 t} \frac{e^{\iota\alpha x}}{h - \iota\alpha} \frac{1 - e^{2\iota\alpha(l-x)}}{1 + \dfrac{h+\iota\alpha}{h-\iota\alpha} e^{2\iota\alpha l}} \frac{d\alpha}{\alpha}. \quad \text{..................8. 9}$$

For a first approximation, $l$ being large, we have

$$u = -\frac{hv_0}{\iota\pi} \int_P e^{-k\alpha^2 t + \iota\alpha x} \frac{d\alpha}{\alpha (h - \iota\alpha)}$$

$$= -\frac{v_0}{\iota\pi} \int_P e^{-k\alpha^2 t + \iota\alpha x} \left( \frac{1}{\alpha} - \frac{1}{\alpha + \iota h} \right) d\alpha. \quad \text{.....................8. 10}$$

We take the two parts of this integral separately.

For the first, as in Fig. 4, we can deform the path $(P)$ into the real axis in the $\alpha$-plane and a semicircle (vanishing in the limit) at the origin.

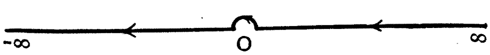



FIG. 4.

It follows that

$$\frac{1}{\iota\pi}\int_P e^{-k\alpha^2 t+\iota\alpha x}\frac{d\alpha}{\alpha}=1-\frac{2}{\pi}\int_0^\infty e^{-k\alpha^2 t}\frac{\sin\alpha x}{\alpha}\,d\alpha$$

$$=1-\operatorname{Erf}\frac{x}{2\sqrt{(kt)}}.^{*} \qquad \text{8. 11}$$

For the second, a similar deformation of the path ($P$) into the line $\alpha+\iota h=0$ and a small semicircle at $(0, -\iota h)$, shows that

$$\frac{1}{\iota\pi}\int e^{-k\alpha^2 t+\iota\alpha x}\frac{d\alpha}{\alpha+\iota h}=e^{h^2kt+hx}\left(1-\frac{2}{\pi}\int_0^\infty e^{-k\beta^2 t}\frac{\sin\beta(x+2hkt)}{\beta}\,d\beta\right)$$

$$=e^{h^2kt+hx}\left(1-\operatorname{Erf}\frac{x+2hkt}{2\sqrt{(kt)}}\right). \qquad \text{8. 12}$$

Thus from 8. 10, 8. 11, 8. 12:

$$v=u+v_0=v_0\left(\operatorname{Erf}\frac{x}{2\sqrt{(kt)}}+\exp(h^2kt+hx)\right)\left(1-\operatorname{Erf}\frac{x+2hkt}{2\sqrt{(kt)}}\right), \qquad \text{8. 13}$$

as in 7. 9.

In Chapter XI. of my book on *Conduction of Heat* a number of examples will be found worked out in this way. The application to the general case follows when the Green's Function, expressed by similar contour integrals,† is used.

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The following notes from Dr. Jeffreys and Dr. Bromwich explain themselves:—

I think Prof. Carslaw has missed a point in his third paragraph, where he describes part of the tract as dealing with an extension of the usual method of finding a particular integral by means of the expansion of the inverse operator $1/f(D)$. The method I give is not an extension of this older method, but a substitute for it. Whereas in the older method the fundamental operator is $D$ ($=d/dx$), in mine it is $Q$ or $p^{-1}$, which by definition means definite integration. $p$ as such is not defined, because it never occurs in actual solutions. The principal advantage is the much greater generality of the method. Whereas it is the normal occurrence for an expansion of an operator in powers of $Q$ to give an intelligible and unique answer, this happens with a series in powers of $D$ only in freak cases where the series happens to terminate, or where the function operated on is a polynomial or an exponential, and special treatment is necessary in every type of application to show that the result actually satisfies the differential equation. In fact I think that no possible difficulty in dealing with the new method would excuse persisting in the obscurity of the old one, quite apart from the fact that it avoids the troublesome solution of the terminal conditions to determine the so-called arbitrary constants.

With regard to the comparative desirability of direct operational methods and complex integrals, I think they both have their spheres of application. It must be noticed that the operational method, when applied to sets of ordinary differential equations, is a special case of the method developed by Caqué, Fuchs, and Baker for linear equations even with variable coefficients, which is a practical method of the highest importance on its own account. It should be better known, partly because it gives a direct proof that solutions can actually be found to satisfy given terminal conditions, and partly because it tends to correct the idea of the average student that the solution of a differential equation consists necessarily and entirely in getting a formal answer in finite terms. I have the highest respect for complex integrals in their place, but their application to differential equations with variable

* Cf. my book on *Fourier's Series and Integrals* (Ed. 1921) Ex. 13 p. 195.

† Cf. *loc. cit.*, Chapter X