II. "On a Meteoric Stone found at Makariwa, near Invercargill, New Zealand." By G. H. F. Ulrich, F.G.S., Professor of Mining and Mineralogy in the University of Dunedin, N.Z. Communicated by Professor J. W. Judd, F.R.S. Received December 14, 1892.

(Abstract.)

The specimen described in this memoir was found in the year 1879 in a bed of clay which was cut through in making a railway at Invercargill, near the southern end of the Middle Island of New Zealand. Originally this meteorite appears to have been about the size of a man's fist, and to have weighed 4 or 5 lbs., but it was broken up, and only a few small fragments have been preserved. The stone evidently consisted originally of an intimate admixture of metallic matter (nickel iron) and of stony material, but much of the metallic portion has undergone oxidation. Microscopic examination of thin sections shows that the stony portion, which is beautifully chondritic in structure, contains olivine, enstatite, a glass, and probably also magnetite, and through these stony materials the nickel iron and troilite are distributed. The specific gravity of portions of the stone was found to vary between 3·31 and 3·54, owing to the unequal distribution of the metallic particles. A partial chemical examination of this meteorite was made by the author and Mr. James Allen, but the complete analysis has been undertaken by Mr. L. Fletcher, F.R.S., of the British Museum. The analysis, which when finished will be communicated to this Society, has gone so far as to show that the percentage mineral composition of the Makariwa meteorite may be expressed approximately by the following numbers: nickel iron 1, oxides of nickel and iron 10, troilite 6, enstatite 39, olivine 44.

III. "On Operators in Physical Mathematics. Part I." By Oliver Heaviside, F.R.S. Received December 15, 1892.

Connexion between a Flux and a Force through an Operator.

1. In the investigation of physical questions we often have to answer such a question as this: Given a force f, a function of the time, acting at one place in a connected system, find the effect F, of some given type, produced by the force at its own or some other place. Or it may be that it is not an impressed force that is given, but displacement of some kind. Or, in order to produce mathe-

matical simplicity, we may have a space-distribution of force or of displacement given, whose effect is required.

To answer the question, we may investigate the general differential equation of the system, find its solution (series, integrals, &c.), and then introduce special values of constants or of functions to limit the generality of the problem, and bring the solution to satisfy the required conditions. Details may differ according to circumstances, but this may serve to describe the usual process.

2. There is, however, a somewhat different way of regarding the question. We may say that we have no special concern with the general solution which would express the disturbance anywhere due to initial energy throughout the system; but that we have simply a connected system, a given point (for example) of which is subjected to impressed force, communicating energy to the system, and we only want to know the effects due to this force itself. Since, therefore, the connexions are definite, we must have some definite connexion between the "flux" F and the "force" f, say

$$F = Yf, \qquad (1)$$

where Y is a differentiating operator of some kind, a function of d/dt, the time-differentiator, for instance, when the connexions are of a linear nature. Here f is some given function of the time, and Y indicates the performance upon f of certain operations, whose result should be to produce the required function F.

3. An important point to be noted here is that there is, or should be, no indefiniteness about the above equation. The operator Y should be so determined as to fully eliminate all indeterminateness, and so that the equation contains in itself the full expression of the connexion between the force and the flux, without any auxiliary conditions, or subsequent limitations, except what may be implicitly involved in the equation itself.

Determinateness of a Solution through the Operator.

4. But as soon as we come to distinctly recognize this determinateness of connexion, another point of important significance presents itself. It should be possible to find F completely from f through the operator Y without ambiguity and without external assistance. That is to say, an equation of the form (1) not only expresses a problem, but also its solution. It may, indeed, not be immediately interpretable, but require conversion to some other form before its resultant meaning can be seen. But it is, for all that, a particular form of the solution, usually a condensed form, though sometimes it may be of far greater complexity than the full ordinary solution. In this respect the nature of the function f is of controlling importance.

We need not assert that the determinateness of F from equation (1) is true for all forms of the function Y that may be written down arbitrarily; but that it is true in the forms presenting themselves in dynamical problems seems to be necessitated.

5. We have, therefore, presented to us the problem of solving this equation for any particular form of Y that occurs. This may be very easy and obvious, or it may be excessively difficult and obscure. In the latter case it may be so merely because we do not know how to do it. Then we should find out. As our argument is that Y finds F from f definitely, there should be definite rules for the manipulation of the operator Y, or of the expression Yf, for its conversion to the form of an ordinary mathematical function, which will be the solution in the usual sense, freed from differentiating operations. We may find how to work by experiment. For, if two different methods lead to different results, one of which we find to be correct by independent tests, we can safely assert that one of the methods was partly wrong, whilst the other may have been wholly correct. So by practice we may come to know something about it.

6. Again, the function Y, regarded as an algebraical function, may admit of different forms of expression. These are algebraically equivalent, but to what extent they may be equivalent in their analytical aspects—for instance, one series involving differentiations equivalent to another involving integrations, and leading to results which are either identical or equivalent—cannot be safely said beforehand. It is, in its generality, a rather difficult and obscure matter. In special cases I find that forms of Y which are algebraically equivalent are also analytically equivalent; but I have not succeeded in determining the amount of latitude that is permissible in the purely algebraical treatment of operators. No doubt there are definite limitations, but they have to be found. I have, however, extensively employed the algebraical treatment experimentally,* subject to independent tests for guidance. It proved itself to be a powerful (if somewhat uncertain) kind of mathematical machinery. We may, for example, do in a line or two, work whose verification by ordinary methods may be very lengthy. On the other hand, the very reverse may be the case. I have, however, convinced myself that the subject is one that deserves to be thoroughly examined and elaborated by mathematicians, so that the method may be brought into general use in mathematical physics, not to supplant ordinary methods, but to supplement them; in short, to be used when it is found to be useful. As regards the theory of the subject, it is interesting in an unusual degree, and the interest is heightened by the mystery that envelops certain parts of it.

* The reader will find examples in my 'Electrical Papers,' vol. 2, of the treatment of irrational as well as rational operators.

Electromagnetic Operators.

7. Perhaps the best way of beginning the subject, to obtain a good idea of the nature of the operators and the advantages attending their use, is through the theory of a connected system of linear electrical conductors. For the electrical equations seem to be peculiarly fitted for the illustration of abstract dynamical properties in a clear manner, even when quite practical electromagnetic arrangements are concerned. We know that we may, by the application of Ohm's law to every conductor (or to circuits of conductors), express the steady current C_n in a conductor n due to an impressed force e_m in a conductor m by an equation

$$C_n = Y_{mn} e_m, \tag{2}$$

where Y_{mn} is some algebraical function of the resistances of the conductors—usually of all the resistances, although in special cases it may become independent of the values of some of them. Now, suppose it is not the steady current that is wanted, but the variable current when e_m varies. The answer is obviously given by the same equation when the function Y involves only resistances; that is, when there is no storage of electric or magnetic energy, so that Y is a constant not involving d/dt. Then the flux and the force keep pace together, and their ratio does not vary. It is, however, less obvious that the same equation should persist, in a generalized form, when every branch of the system is made to be any electromagnetic arrangement we please which would, in the absence of its connexions with the rest of the system, be a self-contained arrangement. To obtain the generalized form of Y_{mn} we have merely to substitute for the resistances concerned the equivalent resistance operators. That is, instead of $V = RC$, where V is voltage, C current, and R resistance, we have an equation $V = ZC$ in general for every conductor, where Z is the resistance operator appropriate to the nature of the conductor, which may be readily constructed from the electrical particulars. These Z's substituted in Y_{mn} in place of the R's make equation (2) fully express the new connexion between the flux C_n and the force e_m. There is much advantage in working with resistance operators because they combine and are manipulated like simple resistances. Of course (2) is really a differential equation, though not in the form usually given. To make it an ordinary differential equation we should clear of fractions, by performing such operations upon both sides of (2) as shall remove denominators and all inverse operations. It is then spread out horizontally to a great length (usually) and becomes very unmanageable. Also, we lose sight of the essential structure of the operator Y.

8. By arrangements of coils and condensers in our linear system we may construct an infinite variety of resistance operators, and of conductance operators, such as Y_{mn} above. They are, however, always algebraical functions of p, and are finite. If expanded, equation (2) always becomes an ordinary linear equation of a finite number of terms. But if we allow conduction in masses, or dielectric displacement in masses (with allowance for propagation in time), the finite series we were previously concerned with become infinite series. This, at first, appears a complication, but it may be quite the reverse, for an infinite series following an easily recognized law may be more manageable than a finite series. Still, however, the equivalence to ordinary differential equations persists, provided our arrangement is bounded. But when we remove this restriction, and permit free dissipation of energy in space (or equivalently), another kind of operators comes into view. The complexity of the previous, due to the reaction of the boundaries, is removed; simpler forms of operators result, and they do not necessarily admit of the equations taking the form of ordinary differential equations, as they may be of an irrational nature. This brings us necessarily to the study of generalized differentiation, concerning which, more presently.

Operators admitting of Easy Treatment.

9. In the meantime, notice briefly some of the ideas and devices that occur generally in the treatment of operators. First of all, we may obtain the steady state of F due to steady f, when there can be a steady state of F, by simply putting $p = 0$ in the operator Y connecting them, p meaning d/dt. Even when there is no resultant steady state of the flux, as when reflections from a boundary continue for ever, the term $F = Y_0 f$ has its proper place and significance.

Next, we may notice that if the form of Y should involve nothing more than separate differentiations, as in

$$F = (a + bp + cp^2 + \dots)f, \tag{3}$$

then all we have to do is to execute the differentiations to obtain F from f. When f is a continuous function, this presents nothing special. When discontinuous, however, a special treatment may be needed.

In a similar manner, there may be only separate integrations or inverse differentiations indicated in Y, as when

$$F = (a + bp^{-1} + cp^{-2} + \dots)f. \tag{4}$$

Since f is a definite function of the time, so are its successive time-integrals. In this case, f may be discontinuous, and yet present no difficulty. Suppose it is zero before and constant after the moment $t = 0$. Then we shall have

$$p^{-1}f = ft, \quad \&c., \qquad p^{-n}f = f\frac{t^n}{\lfloor n}. \tag{5}$$

A combination of direct and inverse operations, which frequently occurs in the theory of waves, is exemplified in

$$F = \epsilon^{-pr/v}(a + bp^{-1} + cp^{-2} + \ldots)f. \tag{6}$$

Here we may perform the integrations first, getting the result $\phi(t)$ say, and then let the exponential operate, giving, by Taylor's theorem,

$$F = \phi(t - r/v). \tag{7}$$

Or we may let the exponential operator work first, and then perform the integrations. This may be less easy to manage, on account of the changed limits.

Two important fundamental cases, which constitute working formulæ, are

$$F = \frac{p}{p-a}f, \quad \text{and} \quad F = \frac{p}{p+a}f, \tag{8}$$

with unit operand, that is, $f = 0$ before and constant after $t = 0$. Here we may expand in inverse powers of p, getting, in the first case

$$F = (1 + ap^{-1} + a^2p^{-2} + \ldots) = \epsilon^{at}, \tag{9}$$

and in the second case ϵ^{-at}. The latter expresses the effect of a unit impulse in a system having one degree of freedom, with friction, as when an impulsive voltage acts upon a coil.

Solutions for Simple Harmonic, Impulsive, and Continued Forces.

10. A very important case, admitting of simple treatment, occurs when the force is simple periodic, or a sinusoidal function of the time. It may happen that the resulting state of F is also sinusoidal. For this to occur, there must be dissipation of energy, to allow the initial departure from the simple periodic state to subside. We then have $p^2 = -n^2$ applied to F as well as f, where $n/2\pi$ is the frequency; so that the substitution of ni for p in Y brings equation (1) to the form

$$F = (Y_0 + Y_1 i)f = (Y_0 + Y_1 n^{-1}p)f, \tag{10}$$

where Y_0 and Y_1 are functions of n^2. We now find F by a simple direct operation. This case is so important because its application is so general, and its execution usually presents no difficulties, whilst the interpretation of the result may be valuable and instructive physically.

A continued constant force of unit strength, commencing when $t = 0$, may be represented by

$$p^0 = \left(\tfrac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\sin nt}{n} dn\right), \tag{11}$$

using a well-known integral. We may apply this to equation (1), if desired, and obtain a particular form of solution. And from (11) we see that a unit impulse is represented by

$$p^1 = \frac{1}{\pi}\int_0^\infty \cos nt \,.\, dn \tag{12}$$

acting at the moment $t = 0$. This is, of course, the basis of Fourier's theorem. But, instead of the application of the fully developed Fourier's theorem, it is more convenient to use (12) itself. Thus, when f_0 is an impulse acting when $t = 0$, we have the equation

$$F = Ypf_0,$$

pf_0 representing the force. So, by (12),

$$F = \frac{f_0}{\pi}\int_0^\infty Y \cos nt . dn \tag{13}$$

gives us a particular form of the solution arising from an impulse. Take $p = ni$ in Y to convert the quantity to be integrated to an algebraical form.

Since a continuously varying force may be represented by a succession of infinitesimal impulses, we see that a single time-integration applied to (13), f_0 being then a function of the time, gives us a form of solution of the equation $F = Yf$, for any kind of f and Y that can occur. It is, however, a theoretical rather than a practical form of solution. For it usually happens that the definite integral is quite unamenable to evaluation. The same may be often said of the solution (13) for an impulse, and in such cases it may be questioned whether the form $F = Yf$ itself is not just as plain and intelligible. In fact, in certain cases, a very good way to solve (or evaluate) a solution in the form of a definite integral is to undo it, or convert it to the symbolical form $F = Yf$, and then solve it by any way that may be feasible. Nevertheless, it is interesting to know that we may have a full solution, and the definite integrals are sometimes practically workable, or may be transformed to easier kinds.

Partial Fractions and Normal Solutions.

11. There is also the method of partial fractions. It is not always applicable, and is especially inapplicable when the removal of boundaries drives the roots of the determinantal equation into contiguity. But the application is very wide, nevertheless. Put $Z = Y^{-1}$; then the solution of $F = Yf$, when f is constant, starting when $t = 0$, is

$$F = f\left\{\frac{1}{Z_0} + \Sigma \frac{\epsilon^{pt}}{p\,(dZ/dp)}\right\}, \tag{14}$$

where Z_0 is the steady Z, got by taking $p = 0$ in Z, and the summation ranges over the roots of the equation $Z = 0$, considered as an algebraical equation in p. That is, p is entirely algebraical in (14). Similarly, the effect of an impulse f_0 is represented by

$$F = f_0 \Sigma \frac{\epsilon^{pt}}{dZ/dp}, \tag{15}$$

and from this again, by time-integration, we can obtain an expression for the effect due to any varying f, which may be quite as unmanageable as the previous definite integral for the same. On the other hand, (14) and (15) furnish the most direct and practical way of investigating certain kinds of problems, whether there be but a few or an infinite number of degrees of freedom. This method is the real foundation of all formulæ for the expansion of arbitrary functions in series of normal functions. For, find the impressed forcive that would keep up the arbitrary state. We may then apply the above to every element of the forcive to find its effect, and by integration throughout the system get the arbitrary functions expanded in normal functions. Or, without reference to impressed force, find the differential equation connecting any element of the initial state and the effect it produces later. It will be of a form similar to our $F = Yf$, and it may be similarly solved by a series, which contains the expression of the expansion of the initial state in the proper functions.

Or we may investigate the normal functions themselves, and employ their proper conjugate property to obtain the expansion representing any initial state. But this method does not apply very naturally to equations of the form we are considering.

Decomposition of an Operator into a Series of Wave Operators.

12. There is also another method which contrasts remarkably with the previous, viz., to decompose the operator Y into a series of other operators of a certain type expressing the propagation of waves. This is best illustrated by an example. Suppose the question is, given an impressed force acting at one part of a long telegraph circuit, find the effect produced. One way would be to first find the effect due to a simple periodic force; from this the effect of an impulse follows; and from the latter the effect due to any f. A second way is by means of the normal functions, either through the conjugate property or by partial fractions. Lastly, we may decompose the operator Y into operators of the form which would exist were the circuit infinitely long, so that the effect of terminal reflections and absorptions does not appear. Say we have

$$F = (Y_0 + Y_1 + Y_2 + \dots)f. \qquad (16)$$

Then $F = Y_0 f$ will represent the initial wave from the source f, whilst the rest will express the succeeding reflected waves from the terminations of the circuit. The operators Y_0, &c., may be all of the same type, so that it suffices to solve $F = Y_0 f$, that is, convert it to an ordinary algebraic functional form, to obtain that form of the complete solution which has the greatest physical meaning, inasmuch as it shows in detail the whole march of F in terms of f. So does the solution in terms of normal functions, but not immediately, because the successive waves are expressed in the form of an infinite series of vibrating systems. Their resultant effect cannot be seen at once. We might, indeed, almost say that the form of solution in successive (or simultaneous) waves was *the* solution, being of the most explicit nature. Should, however, the impressed force be of a distributed nature, of the type suggested by a normal function, for example, then clearly it is the expression in terms of waves that becomes complex and unnatural. We also see that, although a direct transformation from one form of solution to another may be wholly impracticable algebraically, yet it may be readily carried out through the function Y as intermediary.

Treatment of an Irrational Operator. Solutions in Ascending Series.

13. The above general remarks are necessarily very sketchy. Some of the matters mentioned may be returned to, but the object of the preceding is merely to prepare the mind of the reader for the more transcendental matter to follow. Let us now consider how to treat irrational operators directly, without the assistance of definite integrals. The first form that presented itself to me was that exhibited by

$$Y = \left(\frac{K + Sp}{R + Lp}\right)^{\frac{1}{2}}, \qquad (17)$$

where p is d/dt and R, S, K, L are constants. It occurs in the theory of a submarine cable or other telegraph circuit, and in other problems.

R and L are themselves differentiating operators of complicated form in general, or, more strictly, $R+Lp$ is a resistance operator, say R'', of very complex form. But it is quite sufficient to take the form $R+Lp$, where R is the effective resistance and L the inductance per unit length of circuit. S and K mean the permittance and leakage conductance per unit length.

Now we may readily obtain the simple periodic solution out of (17), by the before-mentioned substitution $p = ni$; and in doing so we may use the general operator R'', for that will then assume the form $R+Lp$. From this solution a wholly uninterpretable definite integral can be derived to express the effect of an impulse or of a steady impressed force. The question was, how to obtain a plain understandable solution from (17) itself to show the effect of a steady force. To illustrate, we may here take merely the case in which $K = 0$, whilst R and L are constants, because the inclusion of K (to be done later) considerably complicates the results. We have then to solve

$$F = \left(\frac{p}{a+p}\right)^{\frac{1}{2}}, \tag{18}$$

where a is a constant and $p = d/dt$. The operand is understood to be unity, that is, $f = 0$ before and $=1$ after $t = 0$. It is needless to write unit operands, and it facilitates the working to omit them. Now, the first obvious suggestion is to employ the binomial theorem to expand the operator. This may be done either in rising or in descending powers of p. Try first descending powers, since by experience with rational operators we know that that way works. We have

$$F = (1+ap^{-1})^{-\frac{1}{2}} = 1-\tfrac{1}{2}\frac{a}{p}+\frac{1\,.\,3}{2^2\lfloor 2}\frac{a^2}{p^2}-\frac{1\,.\,3\,.\,5}{2^3\lfloor 3}\frac{a^3}{p^3}+\dots\,. \tag{19}$$

The integrations, being separated from one another, can be immediately carried out through $p^{-n} = t^n/\lfloor n$, giving the result

$$F = 1-\frac{at}{2}+\frac{1\,.\,3}{(\lfloor 2)^2}\left(\frac{at}{2}\right)^2-\frac{1\,.\,3\,.\,5}{(\lfloor 3)^2}\left(\frac{at}{2}\right)^3+\dots\,, \tag{20}$$

or, which is the same,

$$F = \epsilon^{-\frac{at}{2}}\,I_0\left(\frac{at}{2}\right), \tag{21}$$

where I_0 is the well-known cylinder function. Now, that this result is correct may be tested independently, viz., by its correctly satisfying the differential equation concerned and the imposed conditions. We therefore obtain some confidence in the validity of the process employed.

In a precisely similar manner we may show that

$$\left(\frac{p}{p-a}\right)^{\frac{1}{2}} = \epsilon^{\frac{at}{2}} I_0\left(\frac{at}{2}\right). \tag{22}$$

Further modifications are confirmatory. Thus, by making use of

$$f(p)\epsilon^{at} = \epsilon^{at} f(p+a), \tag{23}$$

we can shift ϵ^{at} and similar functions back and forth. Also using the stock formulæ

$$\frac{p}{p-a} = \epsilon^{at}, \qquad \frac{p}{p+a} = \epsilon^{-at}, \tag{23a}$$

we have the following transformations,

$$\left(\frac{p}{a+p}\right)^{\frac{1}{2}} = \epsilon^{-at}\,\epsilon^{at}\left(\frac{p}{a+p}\right)^{\frac{1}{2}} = \epsilon^{-at}\left(\frac{p-a}{p}\right)^{\frac{1}{2}}\epsilon^{at}$$

$$= \epsilon^{-at}\left(\frac{p-a}{p}\right)^{\frac{1}{2}}\left(\frac{p}{p-a}\right) = \epsilon^{-at}\left(\frac{p}{p-a}\right)^{\frac{1}{2}}. \tag{24}$$

In this we may use the result (22), and so come round to (21) again.

From the above we see that

$$I_0(at) = \epsilon^{at}\left(\frac{p}{2a+p}\right)^{\frac{1}{2}} = \epsilon^{-at}\left(\frac{p}{p-2a}\right)^{\frac{1}{2}}; \tag{25}$$

and further, by shifting the exponentials to the right, to make them the operands (instead of t^0),

$$I_0(at) = \left(\frac{p-a}{p+a}\right)^{\frac{1}{2}}\epsilon^{at} = \left(\frac{p+a}{p-a}\right)^{\frac{1}{2}}\epsilon^{-at}; \tag{26}$$

and now further again, by employing (23a) in place of the exponentials, we obtain

$$I_0(at) = \left(\frac{p-a}{p+a}\right)^{\frac{1}{2}}\frac{p}{p-a} = \frac{p}{(p^2-a^2)^{\frac{1}{2}}}, \tag{27}$$

which is an entirely different kind of operator, since the square of p occurs under the radical sign, instead of the first power. But (27) may be readily tested and found to be not wanting. For expand by the binomial theorem, thus,

$$\frac{p}{(p^2-a^2)^{\frac{1}{2}}} = \left(1-\frac{a^2}{p^2}\right)^{-\frac{1}{2}} = 1+\tfrac{1}{2}\frac{a^2}{p^2}+\frac{1\,.\,3}{2^2\lfloor 2}\frac{a^4}{p^4}+\dots\,. \tag{28}$$

This may be immediately integrated, giving as result the series

$$I_0(at) = 1+\frac{(at)^2}{2^2}+\frac{(at)^4}{2^2 4^2}+\dots\,, \tag{29}$$

the well-known formula for $I_0(at)$ in rising powers of the square of the variable, as required.

Transformation to a Descending Series.

14. There is such a perfect harmony in all the above transformations, without a single hitch, that you are tempted at first to think that you may do whatever you like with the operators in the way of algebraical transformation. There is a considerable amount of truth in this, but it is not wholly true. I shall show later some far more comprehensive and surprising transformations effected by simple means. At the same time I should emphasize the necessity of caution and of frequent verification, for no matter how sweetly the algebraical treatment of operators may work sometimes, it is subject at other times (owing to our ignorance) to the most flagrant failures.

But in the above we only utilized one way of effecting the binomial expansion. There is a second way, viz., in ascending powers of the differentiator. The two forms are algebraically equivalent so far as the convergency allows, but we have, so far, no reason to suppose that they are analytically equivalent. But on examination we find that they are. Thus, using the first of (25) and expanding, we get

$$I_0(at) = \epsilon^{at}\left(\frac{p}{2a+p}\right)^{\frac{1}{2}} = \epsilon^{at}\left(1+\frac{p}{2a}\right)^{-\frac{1}{2}}\left(\frac{p}{2a}\right)^{\frac{1}{2}}$$

$$= \epsilon^{at}\left(1-\frac{p}{4a}+\frac{1\,.\,3}{2}\left(\frac{p}{4a}\right)^2 - \dots\right)\left(\frac{p}{2a}\right)^{\frac{1}{2}}. \qquad (30)$$

Here the operand is t^0 or unity. Or we may make it $(p/2a)^{\frac{1}{2}}$ if we please. If we know its value, as a function of t, the rest of the work is easy, as it consists merely of differentiations. But nothing that has gone before gives any information as to the meaning of $p^{\frac{1}{2}}$, let alone its value. We may, however, find it indirectly. We may prove independently that when at is very big, $I_0(at)$ tends to be represented by $\epsilon^{at}(2\pi at)^{-\frac{1}{2}}$. From this we conclude that the value of $p^{\frac{1}{2}}$ must be $(\pi t)^{-\frac{1}{2}}$. Then (30) becomes

$$I_0(at) = \epsilon^{at}\left(1-\frac{p}{4a}+\frac{1\,.\,3}{2}\left(\frac{p}{4a}\right)^2 - \dots\right)\frac{1}{(2\pi at)^{\frac{1}{2}}}, \qquad (30a)$$

and now performing the rest of the differentiations, we arrive at

$$I_0(at) = \frac{\epsilon^{at}}{(2\pi at)^{\frac{1}{2}}}\left\{1+\frac{1}{8at}+\frac{1^2 3^2}{\underline{|2}(8at)^2}+\frac{1^2 3^2 5^2}{\underline{|3}(8at)^3}+\dots\right\}, \qquad (31)$$

which, on test, is found to be equivalent to the ascending series (29). Of course only the convergent part of the series can be utilized for calculating the value of the function. It is, however, the series for practical use when *at* is big enough to make calculation by the convergent ascending series very lengthy. Stop when the convergency of (31) ceases. The result will be too big. Leave out the last counted term, and the result is too small. Counting only half the last convergent term, the result is nearly right, being a little too big. There seems no possible way of hitting the exact value. But still, when *at* is big, we can get quite close enough, to four or more figures, or any other number we please when *at* is sufficiently increased.

Fractional Differentiation.

15. Knowing in the above manner $p^{\frac{1}{2}}$, the values of $p^{\frac{3}{2}}$, $p^{\frac{5}{2}}$, &c., follow by complete differentiations. But although, on the basis of the above, a considerable amount of work may be done, and extensions made, yet it is desirable to stop for a moment. For the whole question of generalized differentiation is raised. The operator $p^{\frac{1}{4}}$ presents itself in analogous problems, along with $p^{\frac{5}{4}}$, &c. We want a general method of treating p^n, when n is not confined to be integral. Notice, however, in passing a remarkable peculiarity of the above investigation. If we had put $L = 0$ in (17), as well as $K = 0$, we should have had the form $Y = p^{\frac{1}{2}}$ to consider at the beginning, with no evident means of treating it. By taking, on the other hand, a more general case, as we did, we avoided the fractional differentiation altogether, and easily obtained a convergent solution, viz., (21), through (18), (19), and (20). It is not always that we simplify by generalizing.

The sum total of the whole information contained in my mathematical library on the subject of generalized differentiation is contained in the remark made on p. 197 of the second part of Thomson and Tait's 'Natural Philosophy,' paragraph (n), relating to the process by which spherical harmonics of any degree may be derived from the reciprocal of a distance :—"The investigation of this generalized differentiation presents difficulties which are confined to the evaluation of P_s, and which have formed the subject of interesting mathematical investigations by Liouville, Gregory, Kelland, and others."

I was somewhat struck with this remark when I first read it, in trying to plough my way through the fertile though rather heavy field of Thomson and Tait, but as the subject was no sooner mentioned than it was dropped, it passed out of mind. Nor did the absence of any reference to the subject in other mathematical works, and in papers concerning mathematical physics generally, tend to preserve my recollection of the remark. Only when the subject was forced upon my attention in the above manner did I begin to investigate it, and not having access to the authorities quoted, I was compelled to work it out myself. I cannot say that my results are quite the same, though there must, I think, be a general likeness. I can, however, say that it is a very interesting subject, and deserves to be treated in works on the Integral Calculus, not merely as a matter concerning differentiation, but because it casts light upon mathematical theory generally, even upon the elements thereof. And as regards the following brief sketch, however imperfect it may be, it has at least the recommendation of having been worked out in a mind uncontaminated by the prejudices engendered by prior knowledge acquired at second hand. I do not say it is the better for that, however.

Differentiation Generalized.

16. The question is, what is the meaning of ∇^n, if ∇ signify d/dx, when n has any value? This is, no doubt, partly a matter of convention; but apart from all conventions, there must be fundamental laws involved. Now observe that the effect of a whole differentiation ∇ upon the function x^n is to lower the degree by unity. This applies universally when n is not integral. When it is integral, there seem to be exceptions. But we can scarcely suppose that there is a real breach of continuity in the property. We also observe that a whole differentiation ∇ multiplies by the index, making $\nabla x^n = nx^{n-1}$; and again there are apparent exceptions. Now the first thing to do is to get rid of the exceptions. Next, the obvious conclusion from one ∇ lowering the index by unity, ∇^2 by two, and so on, is that ∇^n lowers the degree n times, whether n be integral or fractional. Further, since

$$\nabla \frac{x^n}{\underline{|n}} = \frac{x^{n-1}}{\underline{|n-1}}, \tag{32}$$

when n is positively integral, and $\underline{|n}$ is the factorial function $1.2.3....n$; and, similarly,

$$\nabla^n \frac{x^n}{\underline{|n}} = 1, \tag{33}$$

whatever positive integer n may be, it is in agreement with the previous to define generalized differentiation by the last equation, for all values of n, provided we simultaneously define $\underline{|n}$ to be given by

$$\underline{|n} = n\underline{|n-1}, \tag{34}$$

for all values of n from $-\infty$ to $+\infty$, and to agree with the factorial function when n is integral, that is, $\underline{|1} = 1$, $\underline{|2} = 1.2$, $\underline{|3} = 1.2.3$, &c. We shall still call $\underline{|n}$ the factorial function, and $(\underline{|n})^{-1}$ the inverse factorial.

Now, by the above

$$\nabla \frac{x}{\lfloor\!\underline{1}} = \frac{x^0}{\lfloor\!\underline{0}} = 1\,, \tag{35}$$

if x be positive, as we shall suppose throughout. We conclude that $\lfloor\!\underline{0} = 1$.

Further differentiations give

$$\nabla x^0 = \frac{x^{-1}}{\lfloor\!\underline{-1}}, \qquad \nabla^2 x^2 = \frac{x^{-2}}{\lfloor\!\underline{-2}}, \quad \&c. \tag{36}$$

We, therefore, conclude that $(\lfloor\!\underline{-1})^{-1} = 0$, $(\lfloor\!\underline{-2})^{-1} = 0$, &c., or that the inverse factorial function vanishes for all integral negative values of n. We therefore know the value of the inverse factorial for all integral values of the variable, and a rough curve can be readily drawn. Say

$$y = \frac{1}{\lfloor\!\underline{n}}, \tag{37}$$

y being the ordinate, and n the abscissa. It has evidently a hump between $n = 0$ and $n = 1$, is positive for all + values of n, asymptotically tending to the n axis as n is increased, and is oscillatory on the other side of the origin. This is not demonstrative, but only highly probable so far.

The Inverse Factorial Function.

17. Now seek an algebraical function with equidistantly spaced roots on one side (either side) only of the origin. The function

$$(1-n)\left(1-\frac{n}{2}\right)\left(1-\frac{n}{3}\right)\ldots\left(1-\frac{n}{r}\right) \tag{38}$$

vanishes at $n = 1, 2, 3$, &c., up to r. It has no other roots, and is positive when n is negative. Also, its value at $n = 0$ is 1. Similarly the function

$$(1+n)\left(1+\frac{n}{2}\right)\left(1+\frac{n}{3}\right)\ldots\left(1+\frac{n}{r}\right) \tag{39}$$

vanishes at $n = -1, -2$, &c., up to $-r$; is unity at $n = 0$, and is positive when n is positive. These functions are identically the same as

$$1-n+\frac{n(n-1)}{\lfloor\!\underline{2}}-\frac{n(n-1)(n-2)}{\lfloor\!\underline{3}}+\ldots\pm\frac{n(n-1)\ldots(n-r+1)}{\lfloor\!\underline{r}} \tag{40}$$

and

$$1+n+\frac{n(n+1)}{\lfloor\!\underline{2}}+\frac{n(n+1)(n+2)}{\lfloor\!\underline{3}}+\ldots+\frac{n(n+1)\ldots(n+r-1)}{\lfloor\!\underline{r}} \tag{41},$$

where (40) corresponds to (38) and (41) to (39). At first sight, therefore, these functions might represent the inverse factorial, positive or negative, on making r infinite, for the value at the origin is correct, and the vanishing points are equidistantly spaced with unit step all the way to infinity on one side only of the origin. But something else happens when r is made infinite. The value of (38), by (40), becomes $(1-1)^n$, meaning the binomial expansion in rising powers of the second 1. It is, therefore, zero for *all* positive and infinity for all negative values of n. Similarly, (39) becomes $(1-1)^{-n}$, which is zero for all negative and infinity for all positive values of n. That is, from vanishing at detached points, the functions vanish all the way between them as well. Besides, apart from this, we cannot have the value of $\underline{|n}$ correct when n is integral.

We may, however, readily set the matter right. To get rid of the infinity on one side and vanishing all over on the other side of the origin, multiply the functions (38), (40) by r^n and (39), (41) by r^{-n}. Take

$$\frac{1}{\underline{|n}} = r^{-n}(1-1)^{-n} = r^{-n}(1+n)\left(1+\frac{n}{2}\right)\left(1+\frac{n}{3}\right)\ldots, \tag{42}$$

$$\frac{1}{\underline{|-n}} = r^{n}(1-1)^{n} = r^{n}(1-n)\left(1-\frac{n}{2}\right)\left(1-\frac{n}{3}\right)\ldots\ldots \tag{43}$$

We now satisfy all the requirements of the case, and when r is infinite make the inverse factorial curve (37) be a continuous curve from $-\infty$ to $+\infty$, subject to (34), in agreement with the known values when n is positively integral, and harmonizing with the generalized differentiation in (33).

Multiplying (42) and (43) together, we obtain

$$\frac{1}{\underline{|n}}\,\frac{1}{\underline{|-n}} = (1-n^2)\left(1-\frac{n^2}{4}\right)\left(1-\frac{n^2}{9}\right)\ldots = \frac{\sin n\pi}{n\pi}. \tag{44}$$

The multiplication therefore brings all the equidistant roots into play, on both sides of the origin.

This gives us the value of $\underline{|-n}$ in terms of $\underline{|n}$. Only the values of $\underline{|n}$ from $n = 0$ to $n = 1$ need be calculated, since (34) or (44) gives all the rest. But if we take $n = \frac{1}{2}$ in (44) we obtain

$$\underline{|\tfrac{1}{2}}\ \underline{|-\tfrac{1}{2}} = \tfrac{1}{2}(\underline{|-\tfrac{1}{2}})^2 = \tfrac{1}{2}\pi,$$

therefore

$$\underline{|-\tfrac{1}{2}} = \pi^{\frac{1}{2}}, \tag{45}$$

a fundamental result. We now know the value of $\underline{|n+\tfrac{1}{2}}$ when n is any integer. and this brings the matter down to the determination of $\underline{|n}$ from $n = 0$ to $n = \frac{1}{2}$, for which a formula may be used.

Interpretation of Vanishing Differential Coefficients.

18. It should be noted that when we say that

$$\nabla^n \frac{x^n}{\lfloor n} = 1, \tag{46}$$

for all values of n, the right member is really $x^0/\lfloor 0$, and means 0 on the left, and 1 on the right side of the origin of x. That is, it is the limiting form of the function $x^n/\lfloor n$, when n is infinitely small *positive*. It is convenient in the treatment of equations of the form $F = Yf$ to have the function f zero up to a certain point, with consequently F also zero, and then begin to act. Similarly, the expression $\nabla x^0/\lfloor 0$, or $\nabla 1$ or $x^{-1}/\lfloor -1$, although it has the value zero for all positive values of x, is infinite at the origin. But its total amount is finite, viz., 1. Imagine the unit amount of a quantity spread along an infinitely long line to become all massed at the origin. Its linear density will, in the limit, be represented by, as (12) is derived from (11),

$$\nabla 1 = \frac{1}{\pi}\int_0^\infty \cos mx \, dx. \tag{47}$$

It is zero except at $x = 0$. But its integral is still finite, being $\nabla^0 1$ or 1. If we draw the curve $y = x^n/\lfloor n$, with n infinitely small, consisting of two straight lines, with a rounded corner, the curve derived from it by one differentiation will nearly represent the function $\nabla 1$, being nearly all heaped up close to the origin, and of integral amount 1. Similarly $\nabla^2 1$ means a double infinite point, $\nabla^3 1$ a triple infinite point, and so on. But it is the function $\nabla 1$ that is most useful in connexion with differentiating operations, whilst the others are less prominent.

But when n is taken to be infinitely small negative in $y = x^n/\lfloor n$, then y drops from ∞ to 1 near the origin, or the corner is turned the other way. That is, the function x^n is unstable when n is zero. It is the difference of the curves $y = x^n$ with n infinitely small positive, and the same with n infinitely small negative, that makes the logarithmic function when infinitely magnified. But we should try to keep away from the logarithm in the algebraical treatment of operators.

Connexion between the Factorial and Gamma Functions.

18A. It will be seen by (42) that our factorial function is the gamma function of Euler somewhat modified and extended. Thus, when n is greater than -1 we have

$$\lfloor n = \Gamma(n+1), \tag{48}$$

and this is also expressed by the definite integral

$$\int_0^\infty \frac{\epsilon^{-x} x^n}{\lfloor n} = 1. \tag{49}$$

But when n is less than -1 we have the oscillatory curve of the inverse factorial, given by (44). We cannot use the definite integral to express $\lfloor n$ when n is less than -1. The one-sided reckoning of the gamma function expressed in $\lfloor n = \Gamma(n+1)$ is so exceedingly inconvenient in generalized differentiation that the factorial function had better be used constantly. For completeness and reference, we may add the general formula. Take the logarithm of (42) and arrange the terms suitably, and we obtain

$$\log \lfloor n = -nC + \frac{n^2}{2} S_2 - \frac{n^3}{3} S_3 + \ldots., \tag{50}$$

where

$$S_m = 1 + \frac{1}{2^m} + \frac{1}{3^m} + \ldots., \tag{51}$$

and

$$C = S_1 - \log r = 0{\cdot}5772\,; \tag{52}$$

it being understood in (42) and (52) that r is made infinite.

From (50) we may obtain a series for the inverse factorial in rising powers of n. Thus,

$$\frac{1}{\lfloor n} = 1 + Cn + (C^2 - S_2)\frac{n^2}{2} + (C^3 - 3\,CS_2 + 2\,S_3)\frac{n^3}{\lfloor 3}$$

$$+ (C^4 - 6\,C^2 S_2 + 8\,CS_3 + 3\,S_2{}^2 - 6\,S_4)\frac{n^4}{\lfloor 4} + \ldots. \;. \tag{53}$$

As before remarked, only the value of $\lfloor n$ from $n = 0$ to $\frac{1}{2}$ needs to be calculated. Any number of special formulæ for $\lfloor n$ may be obtained from algebraical expansions involving this function.

A Suggested Cosine Split.

19. The above split of the function $(\sin n\pi)/n\pi$ into $1/\lfloor n$ and $1/\lfloor -n$, suggests other similar splits. In passing, one may be briefly noticed, the cosine split. Thus, take

$$f(n) = (r - \tfrac{1}{2})^{-n}\left(1 + \frac{n}{\frac{1}{2}}\right)\left(1 + \frac{n}{1\frac{1}{2}}\right)\left(1 + \frac{n}{2\frac{1}{2}}\right)\ldots.\left(1 + \frac{n}{r - \frac{1}{2}}\right), \tag{54}$$

and let $f(-n)$ be the same with the sign of n changed. Then, when $r = \infty$, we shall have

$$f(n) f(-n) = \cos \pi x. \tag{55}$$

By changing n to $n+1$ we find that

$$f(n+1) = f(n)\,\frac{2n+2r+1}{(2n+1)(r-\frac{1}{2})}\,, \tag{56}$$

so that when $r = \infty$,

$$f(n+1) = \frac{f(n)}{n+\frac{1}{2}}\,, \tag{57}$$

or

$$(n-\tfrac{1}{2})\,f(n) = f(n-1). \tag{58}$$

When $n = 0$, (54) gives $f(0) = 1$. Then (58) gives $f(n)$ for any integral n. Thus $f(1) = 2$, $f(2) = \frac{4}{3}$, $f(3) = \frac{8}{15}$, $f(4) = \frac{16}{3.5.7}$, &c. And on the negative side we have $f(-\frac{1}{2}) = 0$, $f(-1) = -\frac{1}{2}$, $f(-1\frac{1}{2}) = 0$, $f(-2) = \frac{3}{4}$, $f(-2\frac{1}{2}) = 0$, $f(-3) = -\frac{15}{8}$, &c. The curve is similar to that of the inverse factorial, but with a much bigger hump on the positive side, near $n = 1$. But I have not found any use for this cosine split, and we may now return to the other one.

The Exponential Theorem Generalized.

20. Although we cannot, owing to its limited applicability, use Euler's integral to express $\underline{|n}$ generally, we may employ it when found convenient, within its own range, and supplement the information it gives by other means. Thus, we know that

$$1 = \int_0^\infty \frac{\epsilon^{-x}x^n}{\underline{|n}}\,dx, \tag{59}$$

when n is over -1. Now the indefinite integral may be exhibited in two different ways, say

$$\int \frac{\epsilon^{-x}x^n}{\underline{|n}}\,dx = \epsilon^{-x}\left(\frac{x^{n+1}}{\underline{|n+1}} + \frac{x^{n+2}}{\underline{|n+2}} + \frac{x^{n+3}}{\underline{|n+3}} + \dots\right) = w_1, \tag{60}$$

in ascending powers of x multiplied by the exponential function, and by

$$\int \frac{\epsilon^{-x}x^n}{\underline{|n}}\,dx = -\epsilon^{-x}\left(\frac{x^n}{\underline{|n}} + \frac{x^{n-1}}{\underline{|n-1}} + \frac{x^{n-2}}{\underline{|n-2}} + \dots\right) = -w_2. \tag{61}$$

These are true for all values of n. Subtracting (61) from (60) we see that the function $w_1 + w_2$, or w say, must have the same value at any two finite limits we may choose for the integral. That is, the value of w is independent of the value of x.

Or we may proceed thus, and determine the value. Let n be greater than -1, and divide the integral (59) into two, one going from 0 to x, the other from x to ∞. For the first use (60), since $w_1 = 0$ when $x = 0$; and for the second use (61), since $w_2 = 0$ when $x = \infty$. We then get, by (59),

$$1 = (w_1 + w_2), \tag{62}$$

or, which is the same,

$$\epsilon^x = \frac{x^n}{\lfloor n} + \frac{x^{n+1}}{\lfloor n+1} + \frac{x^{n+2}}{\lfloor n+2} + \dots + \frac{x^{n-1}}{\lfloor n-1} + \frac{x^{n-2}}{\lfloor n-2} + \dots \,. \tag{63}$$

This is proved when n is greater than -1. But the change of n to $n-1$ in the series on the right of (63) makes no alteration. We therefore conclude that the series expresses ϵ^x for *all* values of n.

When $n = 0$ or any integer, positive or negative, we have the usual stopping series for ϵ^x. When n is fractional, we obtain semi-convergent series. Of course we obtain the whole series of forms by making n pass from 0 to 1. The most interesting case is that of $n = \frac{1}{2}$. This gives

$$\lfloor\tfrac{1}{2} = x^{\frac{1}{2}}\epsilon^{-x}\left(1 + \frac{2x}{3} + \frac{(2x)^2}{3.5} + \frac{(2x)^3}{3.5.7} + \dots \right.$$
$$\left. + \frac{1}{2x} - \frac{1}{(2x)^2} + \frac{1.3}{(2x)^3} - \frac{1.3.5}{(2x)^4} + \dots \right), \tag{64}$$

where the value of $\lfloor\frac{1}{2}$ we know to be $\frac{1}{2}\lfloor-\frac{1}{2} = \frac{1}{2}\pi^{\frac{1}{2}}$, by (45).

By means of this series we may pass from one to the other of the two forms of evaluation of Fresnel's integrals, due to Knochenhauer and to Cauchy respectively, which are given in works on Physical Optics.

21. The function called w_1 above we may obtain in a series of rising powers of x without the exponential factor in the following manner:—

$$w_1 = \nabla^{-1}\frac{\epsilon^{-x}x^n}{\lfloor n} = \nabla^{-1}\epsilon^{-x}\nabla^{-n} = \nabla^{-1}(\nabla+1)^{-n}\epsilon^{-x}$$
$$= \nabla^{-1}(\nabla+1)^{-n}\frac{\nabla}{\nabla+1} = (\nabla+1)^{-(n+1)}, \tag{65}$$

which is immediately integrable by the binomial expansion; thus

$$w_1 = \nabla^{-(n+1)} - (n+1)\nabla^{-(n+2)} + \frac{(n+1)(n+2)}{\lfloor 2}\nabla^{-(n+3)} + \dots$$
$$= \frac{x^{n+1}}{\lfloor n+1} - (n+1)\frac{x^{n+2}}{\lfloor n+2} + \frac{(n+1)(n+2)}{\lfloor 2}\frac{x^{n+3}}{\lfloor n+3} - \dots \tag{66}$$

To corroborate the method of getting (65) we may use (60). For this gives

$$w_1 = \epsilon^{-x}\nabla^{-(n+1)}(1+\nabla^{-1}+\nabla^{-2}+\ldots) = \epsilon^{-x}\nabla^{-(n+1)}\epsilon^{x} = (\nabla+1)^{-(n+1)}, \quad (67)$$

again, as in (65). Now if we endeavour to express w_2 in a similar manner, we find that it will not work. But direct multiplication of the series in the brackets in (61) by the usual expansion of ϵ^{-x} gives

$$\begin{aligned} w_2 = {} & \frac{x^n}{\lfloor n}\left\{1-n+\frac{n(n-1)}{\lfloor 2}-\frac{n(n-1)(n-2)}{\lfloor 3}+\ldots\right\} \\ & +\frac{x^{n-1}}{\lfloor n-1}\left\{1-(n-1)+\frac{(n-1)(n-2)}{\lfloor 2}-\frac{(n-1)(n-2)(n-3)}{\lfloor 3}+\ldots\right\} \\ & \qquad +\ldots \\ & +\frac{x^{n+1}}{\lfloor n+1}\left\{-(n+1)+\frac{(n+1)\,.\,n}{\lfloor 2}-\frac{(n+1)\,.\,n(n-1)}{\lfloor 3}+\ldots\right\} \\ & +\frac{x^{n+2}}{\lfloor n+2}\left\{\frac{(n+2)(n+1)}{\lfloor 2}-\frac{(n+2)(n+1)n}{\lfloor 3}+\ldots\right\}+\ldots, \end{aligned} \quad (68)$$

or, which is the same,

$$\begin{aligned} w_2 = {} & \frac{x^n}{\lfloor n}(1-1)^n+\frac{x^{n-1}}{\lfloor n-1}(1-1)^{n-1}+\ldots+\frac{x^{n+1}}{\lfloor n+1}(1-1)^{n+1}+\ldots \\ & -\frac{x^{n+1}}{\lfloor n+1}+(n+1)\frac{x^{n+2}}{\lfloor n+2}-\frac{(n+1)(n+2)}{\lfloor 2}\frac{x^{n+3}}{\lfloor n+3}+\ldots\,. \end{aligned} \quad (69)$$

Now the last line we know to express $-w_1$. Therefore, by (62), we get

$$1 = \frac{x^n}{\lfloor n}(1-1)^n+\frac{x^{n-1}}{\lfloor n-1}(1-1)^{n-1}+\ldots+\frac{x^{n+1}}{\lfloor n+1}(1-1)^{n+1}+\ldots, \quad (70)$$

and this is the result we shall obtain by multiplying the series on the right of (63) by the usual expansion of ϵ^{-x}. But (70) is only a special form of a more general formula that will appear later. We may use (44) to convert (70) to circular functions.

A Bessel Function Generalized.

22. The generalized expansion of ϵ^x may be at once applied to generalize other formulæ. Thus, we know that the solution of

$$\left(\nabla^2+\frac{1}{x}\nabla\right)u = u \quad (71)$$

in rising powers of x is

$$I_0(x) = 1+\frac{x^2}{2^2}+\frac{x^4}{2^2 4^2}+\frac{x^6}{2^2 4^2 6^2}+\dots\,. \tag{72}$$

Put $x^2/4 = y$, and it becomes

$$I_0(x) = 1+\frac{y}{(\underline{|1})^2}+\frac{y^2}{(\underline{|2})^2}+\frac{y^3}{(\underline{|3})^2}+\dots\,. \tag{73}$$

Let the differentiator d/dy be called Δ. Then (73) may be at once written

$$I_0(x) = 1+\frac{\Delta^{-1}}{\underline{|1}}+\frac{\Delta^{-2}}{\underline{|2}}+\frac{\Delta^{-3}}{\underline{|3}}+\dots = \epsilon^{\Delta^{-1}}. \tag{74}$$

There is nothing hypothetical about this. What is, however, entirely speculative in the absence of trial is whether it is legitimate to substitute the generalized exponential for the ordinary, and therefore generalized for complete differentiations. But on trial it will be found to work. Thus, using (63) with Δ^{-1} in place of x, we convert (74) to

$$u = \frac{\Delta^{-m}}{\underline{|m}}+\frac{\Delta^{-(m+1)}}{\underline{|m+1}}+\dots+\frac{\Delta^{-(m-1)}}{\underline{|m-1}}+\dots, \tag{75}$$

writing u for the result; or, which means the same,

$$u = \frac{y^m}{(\underline{|m})^2}+\frac{y^{m+1}}{(\underline{|m+1})^2}+\dots+\frac{y^{m-1}}{(\underline{|m-1})^2}+\dots, \tag{76}$$

where, as before, $y = x^2/4$ in connexion with the Bessel function. Or

$$u = \frac{\left(\frac{x}{2}\right)^n}{\left(\underline{|\tfrac{n}{2}}\right)^2}+\frac{\left(\frac{x}{2}\right)^{n+2}}{\left(\underline{|\tfrac{n+2}{2}}\right)^2}+\dots+\frac{\left(\frac{x}{2}\right)^{n-2}}{\left(\underline{|\tfrac{n-2}{2}}\right)^2}+\dots, \tag{77}$$

if $n = 2m$. We have $u_m - u_{m-1}$ for all values of m, and $u_0 = I_0(x)$.

To test the validity when m is fractional, take $m = \frac{1}{2}$ or $n = 1$, then, by (77), we obtain

$$u = \frac{2}{\pi}\left(x+\frac{x^3}{3^2}+\frac{x^5}{3^2 5^2}+\frac{x^7}{3^2 5^2 7^2}+\dots+\frac{1}{x}+\frac{1^2}{x^3}+\frac{1^2 3^2}{x^5}+\frac{1^2 3^2 5^2}{x^7}+\dots\right). \tag{78}$$

This will by numerical calculation be found to give the same value as the series (72) for $I_0(x)$. When x is as large as 10 the values agree to the fourth place by the convergent series in (78) alone, the semi-convergent series in (78) having a relatively small value. The value

I make to be 2815·71 by the ordinary formula, and 2815·75 by (78), including the semi-convergent part. But the last two figures are probably wrong, as there is a good deal of figuring involved in the calculation of both (72) and the convergent series in (78).

When smaller values of x are taken, the numerical agreement persists as far as the initial convergency of the descending series permits, as in the case of the series (31), for example. Later on I will co-ordinate (78) and (72) with the descending formula (31).

The companion formula to (78) is

$$v = \frac{2}{\pi}\left(\frac{1}{x}-\frac{1}{x^3}+\frac{1^2 3^2}{x^5}-\frac{1^2 3^2 5^2}{x^7}+\ldots\right)-\frac{2}{\pi}\left(x-\frac{x^3}{3^2}+\frac{x^5}{3^2 5^2}-\ldots\right). \quad (79)$$

We might expect this to be a form of the oscillatory function $J_0(x)$. But it is not. It represents the oscillatory companion to $J_0(x)$, say $G_0(x)$, which may be exhibited in an ascending series of the whole powers of x^2 together with a logarithm, so standardized as to vanish at infinity. This function will appear later. The double series (79) occurs in Lord Rayleigh's 'Sound.'* The series (78) I have not come across.

The Binomial Theorem Generalized.

23. Let us next generalize the binomial theorem in a similar manner. We have

$$\frac{(1+x)^n}{\underline{|n}} = \frac{1}{\underline{|n}}+\frac{x}{\underline{|1}\,\underline{|n-1}}+\frac{x^2}{\underline{|2}\,\underline{|n-2}}+\ldots \quad (80)$$

in ascending powers of x. Or

$$\frac{(1+x)^n}{\underline{|n}} = \frac{1}{\underline{|n}}+\frac{\nabla^{-1}}{\underline{|n-1}}+\frac{\nabla^{-2}}{\underline{|n-2}}+\ldots = \frac{1}{\nabla^n}\left(\frac{\nabla^n}{\underline{|n}}+\frac{\nabla^{n-1}}{\underline{|n-1}}+\ldots\right). \quad (81)$$

But in descending powers of x, which is the only other form generally known, we have

$$\frac{(x+1)^n}{\underline{|n}} = \frac{x^n}{\underline{|n}}+\frac{x^{n-1}}{\underline{|n-1}}+\frac{x^{n-2}}{\underline{|2}\,\underline{|n-2}}+\ldots$$

$$= \frac{1}{\nabla^n}\left(1+\nabla+\frac{\nabla^2}{\underline{|2}}+\frac{\nabla^3}{\underline{|3}}+\ldots\right) = \frac{\epsilon^\nabla}{\nabla^n}. \quad (82)$$

That we may use the generalized exponential we might infer from the two forms (80) and (82) being equivalent, combined with previous

* See p. 154, vol. 2. That (79) is $G_0(x)$ is not explicitly shown there, but it may be readily deduced.

experience relating to the Bessel function. Using it in (82) we find

$$\frac{(1+x)^n}{\lfloor n} = \frac{1}{\nabla^n}\left(\frac{\nabla^m}{\lfloor m} + \frac{\nabla^{m+1}}{\lfloor m+1} + \dots + \frac{\nabla^{m-1}}{\lfloor m-1} + \dots\right), \tag{83}$$

which by immediate integration gives

$$\frac{(1+x)^n}{\lfloor n} = \frac{x^{n-m}}{\lfloor m \lfloor n-m} + \frac{x^{n-m+1}}{\lfloor m-1 \lfloor n-m+1} + \dots + \frac{x^{n-m-1}}{\lfloor m+1 \lfloor n-m-1} + \dots, \tag{84}$$

for all values of m.

The case $m=0$ or any integer is that of (82), and $m=n$ is that of (80). The whole series of forms ranges between $m=0$ and $m=1$, because they recur. When $m=\frac{1}{2}$ we have

$$\frac{(1+x)^n}{\lfloor n} = \frac{x^{n-\frac{1}{2}}}{\lfloor \frac{1}{2} \lfloor n-\frac{1}{2}} + \frac{x^{n+\frac{1}{2}}}{\lfloor -\frac{1}{2} \lfloor n+\frac{1}{2}} + \dots + \frac{x^{n-\frac{3}{2}}}{\lfloor \frac{3}{2} \lfloor n-\frac{3}{2}} + \dots \,. \tag{85}$$

If in this we take $n=1$, we find that the terms can be arranged in pairs, thus

$$1+x = \frac{x^{\frac{1}{2}}}{\lfloor \frac{1}{2} \lfloor \frac{1}{2}} + \frac{x^{\frac{3}{2}}+x^{-\frac{3}{2}}}{\lfloor -\frac{1}{2} \lfloor \frac{3}{2}} + \frac{x^{\frac{5}{2}}+x^{-\frac{5}{2}}}{\lfloor -1\frac{1}{2} \lfloor 2\frac{1}{2}} + \dots, \tag{86}$$

or, which is the same,

$$\frac{\pi}{4}(1+x) = x^{\frac{1}{2}}\left(1 + \frac{x+x^{-1}}{1.3} - \frac{x^2+x^{-2}}{3.5} + \frac{x^3+x^{-3}}{5.7} - \dots\right). \tag{87}$$

The best value of x is obviously 1.

When two variables x and y are used in the binomial theorem, we have, using ∇ for d/dx and Δ for d/dy,

$$\frac{(y+x)^n}{\lfloor n} = \frac{x^n}{\lfloor n} + \frac{x^{n-1}}{\lfloor n-1} y + \frac{x^{n-2}}{\lfloor n-2} \frac{y^2}{\lfloor 2} + \dots = \nabla^{-n} + \nabla^{-(n-1)}\Delta + \dots$$

$$= \frac{\nabla^{-n}}{1+\nabla/\Delta} = \frac{\Delta^{-n}}{1+\Delta/\nabla} = \nabla^{-n}\epsilon^{y\nabla} = \Delta^{-n}\epsilon^{x\Delta} \tag{88}$$

We may use any of these forms. Selecting the last but one, and using the generalized exponential, we have

$$\epsilon^{y\nabla}\nabla^{-n} = \nabla^{-n}\left\{\frac{(y\nabla)^m}{\lfloor m} + \frac{(y\nabla)^{m-1}}{\lfloor m-1} + \dots + \frac{(y\nabla)^{m+1}}{\lfloor m+1} + \dots\right\}; \tag{89}$$

therefore

$$\frac{(x+y)^n}{\underline{|n}} = \frac{y^m x^{n-m}}{\underline{|m}\ \underline{|n-m}} + \frac{y^{m-1}x^{n-m+1}}{\underline{|m-1}\ \underline{|n-m+1}} + \dots + \frac{y^{m+1}}{\underline{|m+1}}\ \frac{x^{n-m-1}}{\underline{|n-m-1}} + \dots, \quad (90)$$

where m may have any value, and x, y may be exchanged.

Taylor's Theorem Generalized.

24. We may also apply the generalized exponential to Taylor's theorem for the expansion of a function in powers of the variable. For this theorem is expressed by

$$f(x+h) = \epsilon^{h\nabla} f(x), \quad (91)$$

and, if this be true generally, irrespective of the wholeness of the differentiations, we must have

$$f(x+h) = \left\{ \frac{h^n}{\underline{|n}} \nabla^n + \frac{h^{n+1}}{\underline{|n+1}} \nabla^{n+1} + \dots + \frac{h^{n-1}}{\underline{|n-1}} \nabla^{n-1} + \dots \right\} f(x). \quad (92)$$

Whether this is true for any function $f(x)$, with the usual limitations, I cannot say. There are probably other necessary limitations.

As examples, take $f(x) = 1$. Then we obtain

$$1 = \frac{h^n x^{-n}}{\underline{|n}\ \underline{|-n}} + \frac{h^{n-1}x^{1-n}}{\underline{|n-1}\ \underline{|1-n}} + \dots + \frac{h^{n+1}x^{-n-1}}{\underline{|n+1}\ \underline{|-n-1}} + \dots. \quad (93)$$

Here put $c = h/x$; then, by using (44), we have the result

$$1 = c^n \frac{\sin n\pi}{n\pi} + c^{n+1} \frac{\sin (n+1)\pi}{(n+1)\pi} + \dots + c^{n-1} \frac{\sin (n-1)\pi}{(n-1)\pi} + \dots, \quad (94)$$

where c is to be positive. When $n = \frac{1}{2}$, this reduces to

$$\frac{\pi}{2} = a + \frac{1}{a} - \frac{1}{3}\left(a^3 + \frac{1}{a^3}\right) + \frac{1}{5}\left(a^5 + \frac{1}{a^5}\right) - \dots, \quad (95)$$

where a is written for $c^{\frac{1}{2}}$. It is obviously right when $a = 1$.

The formula (70) may be derived from (94) by the use of (44).

Special Formulæ for Factorials.

25. The binomial generalization before given is, of course, a special case of (92), namely, $f(x) = x^n/\underline{|n}$. It will be observed that the series it gives may be convergent. Thus we may obtain convergent special formulæ for $\underline{|n}$. Thus, take $m = \frac{1}{4}$, $n = \frac{1}{2}$ in (84). We obtain

$$\frac{(1+x)^{\frac{1}{2}}}{\underline{|\tfrac{1}{2}}} = \frac{x^{\frac{1}{4}}}{(\underline{|\tfrac{1}{4}})^2}\left\{1+\frac{x+x^{-1}}{5}-\frac{3(x^2+x^{-2})}{5.9}+\frac{3\,.\,7(x^3+x^{-3})}{5\,.\,9\,.\,13}-\dots\right\}, \tag{96}$$

and when $x = 1$, we have the series

$$\left(\frac{8}{\pi}\right)^{\frac{1}{2}}(\underline{|\tfrac{1}{4}})^2 = 1+\tfrac{2}{5}(1-\tfrac{3}{9}(1-\tfrac{7}{13}(1-\tfrac{11}{17}(1-\dots\;. \tag{97}$$

Similarly, $m = \frac{2}{3}$, $n = \frac{1}{3}$, $x = 1$, gives

$$\frac{8\pi^2}{81\,.\,4^{\frac{1}{3}}(\underline{|\tfrac{1}{3}})^3} = 1-\tfrac{1}{5}(1-\tfrac{4}{8}(1-\tfrac{7}{11}(1-\tfrac{10}{14}(1-\tfrac{13}{17}(1-\dots; \tag{98}$$

and so on.

Property of the Generalized Exponential.

26. Notice that the operation ∇^m performed upon the generalized ϵ^x reproduces it when m is integral, but gives an equivalent series when m is fractional. If, then, we take the special form of the ordinary stopping series for ∇^m to work upon, we require to imagine that the zero terms are in their places, thus,

$$\epsilon^x = \dots + \frac{x^{-2}}{\underline{|-2}}+\frac{x^{-1}}{\underline{|-1}}+1+x+\frac{x^2}{\underline{|2}}+\dots\;. \tag{99}$$

All terms before the 1 are zero, but not their rates of variation with x in the generalized sense, if we are to have harmony with the behaviour of the general form of ϵ^x. This is transcendental: and there is much that is transcendental in mathematics.

The above generalizations are somewhat on one side of our subject of the treatment of operators, though suggested thereby. I propose to continue the main subject in a second paper.