
Mathematics on a Distant Planet

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The purpose of this talk is to get you to think seriously about the extent to which mathematics is arbitrary and the extent to which it is fixed, and about what you think mathematics is.

This talk began in my mind long ago during the early SETI (search for extra terrestrial intelligence) when the giant radar dish at Arecibo was being used for the search. Now, many years later, we are searching with far more efficiency and still have no definite signs of life.

There are three aspects to mathematics. First, there are the postulates, axioms, or assumptions, however you may wish to call them. The Bourbaki approach has tended to focus on them. Alternate postulates that describe the same mathematics are not to be considered as different mathematics.

Second, there are the definitions; these determine a great deal of mathematics. For example, the current definition of a function allows a lot of peculiar things. If I have a continuous saw tooth function with alternating slopes of $\pm 45^\circ$ and rotate the coordinates 45° then the function is no longer continuous; it is not even a function! Neither Euler nor Fourier would have agreed with this! The idea of continuity arose from the concept of drawing the curve without lifting the (ideal) pen and had no relationship to the coordinates you happen to choose. To have distorted the primitive, intuitive idea of continuity so that it is now coordinate dependent seems to me to be foolish. Thus the idea of a function and the idea of a curve are now distinct concepts.

Third, there are the kinds of logical deductions that are permitted. The patterns of reasoning are seldom examined these days and will be looked at briefly, and in only one direction.

But before going farther I need to mention a few things in my life that have shaped my opinions. The first occurred at Los Alamos during WWII when we were designing atomic bombs. Shortly before the first field test (you realize that no small scale experiment can be done—either you have a critical mass or you do not), a man asked me to check some arithmetic he had done, and I agreed, thinking to fob it off on some subordinate. When I asked what it was, he said, “It is the probability that the test bomb will ignite the whole atmosphere.” I decided I would check it myself! The next day when he came for the answers I remarked to him, “The arithmetic was apparently correct but I do not know about the formulas for the capture cross sections for oxygen and nitrogen—after all, there could be no experiments at the needed energy levels.” He replied, like a physicist talking to a mathematician, that he wanted me to check the arithmetic not the physics, and left. I said to myself, “What have you done, Hamming, you are involved in risking all of life that is known in the Universe, and you do not know much of an essential part?” I was pacing up and down the corridor when a friend asked me what was bothering me. I told him. His reply was, “Never mind, Hamming, no one will ever blame you.” Yes, we risked all the life we knew of in the known universe on some mathematics. Mathematics is not merely an idle art form, it is an essential part of our society.

Since that day I had often, especially while at Bell Telephone Laboratories, made similar, less dramatic to be sure, predictions based on conventional mathematics; if you make the wing size $1/3$ as large, and give up the slant launch you will be much better off and can intercept the target much farther out; if you design a transistor this way instead of that you will see this much gain; this design of a central office will have less blocking than that design under the assigned load of traffic; etc. Many times I have made predictions about the physical world based on mathematics done at my desk. Surely Nature does not know nor care what I write, nor the mathematical postulates used, but the consequences can be serious. Therefore it is of significant importance to ask, "What kinds of mathematics can I depend on, and what kinds can I not?" That is the question!

Back to the main story; remember that the underlying hypothesis is that we are in two-way communication with a distant civilization via radio waves. We believe their physical and chemical world is similar to ours, so they also have gravity, inertia, heat, conservation of energy, thermodynamics, entropy, and other details of our world. They have the same physical problems that we do, and they must survive them. If you believe in creationism, and hence in God, then of course He could have created them to be very much different from us, but if you believe in evolution then they too must have evolved with the same physical laws, with upward compatibility at all stages of their evolution, from simple beginnings to their current complexity. Galileo once said, "Mathematics is the language of Science." Hence, facing the same laws of the physical world, their mathematics must have a good deal of similarity to ours.

The immediate question is, what kinds of communication could we expect to have with them—surely they will not speak English or any other Earthly natural language. I observe that on Earth while we have many different natural languages we seem to have essentially only one language for mathematics. Will they have essentially the same language?

I soon began, on my two week visits each summer to Los Alamos Scientific Laboratories as a consultant, to ask these questions of physicist friends. With the hypothesis as given above, they all agreed that the aliens would have essentially the same mathematics. But the word "essentially" needs clarification. Physicists all know that quantum mechanics has three different descriptions, the wave form, the matrix form, and there is also a group theory approach, hence a given body of experimental data, or Nature itself if you prefer, need not have a unique theory. The argument was that the aliens must have some equivalent form for Maxwell's equations. But these equations implied the calculus, more or less, as we have it, etc.

Highly imaginative people can imagine almost anything in this matter, and we have science fiction stories of gas clouds being sentient beings, but when you press them for probabilities and not mere idle conjectures a lot of this fancy imagination fades. Indeed, with regard to the Moon landings, some people claimed that the surface might be 17 feet of dust and the vehicles would be smothered in it when they landed. It turned out that the surface was pretty much as we had expected—our expectations allowed for more erosion from particles and solar wind, and less from water and wind erosion, all because of the very, very small amount of atmosphere on the Moon. Hence I tend to ignore the wild, far out, suggestions. Indeed, when they are all added up then the sum of all the probabilities assigned to the far out proposals was so small that we ignored them and assumed that the Moon was reasonably similar to the Earth.

You get nowhere with idle speculations; you need a base for starting. Recalling that Descartes, when he began to think about the world, once he realized that much of what he had been taught was not true, began with, "I think, therefore I am." I decided to begin with Kronecker's classic remark, "God made the integers, the rest is the work of man." You can pretend to believe in other than this, but the necessity of living, surviving, and of distinguishing one thing from another means to me that you will probably have a discrete counting system, and unlimited in extent—the integers are finite and linearly ordered, but the system of integers is unbounded.

The Peano postulates for the integers are amusing, but surely no one really thinks that these postulates gave rise to the integers; indeed if it turns out that they are inadequate then we will alter them to get what we want. Our ideas of the integers are independent of the Peano postulates!

Euclidean geometry is tied up with the continuous. The Greeks finally faced the difficulty of the two approaches, the continuous and the discrete, and in truth came out poorly, about as badly as we have. There were then two schools of philosophers—those who clung to no change (a thing that was itself could not change and still be itself), and those who believed in change being the basis of the universe (Heraclitus: all is in a state of flux, and you cannot step in the same river twice). The Zeno paradoxes were, I believe, a dramatic attempt to show the inherent troubles that arise between the discrete and continuous points of view. Take only one, the flying arrow paradox. If time is composed of instants, much as the real line is said to be composed of points, then at any instant the arrow is in some position and hence it is never moving. We say the same sort of thing about the line; it is composed of points that have no dimension but the unit line has dimension! Is that reasonable? On the other hand those who believed in the change had to say, finally, that there is a limit to the ability to subdivide, and hence the world was composed of indivisible atoms. The discrete position was stuck with the logic that no change was possible; change was all illusion, since otherwise nothing would be definite and fixed. The continuous and discrete do not mix, they are like oil and water.

I am inclined to believe that on the imagined distant planet they too would have found the same sort of logic we use even if their life were based on silicon rather than carbon compounds; after all they apparently have to deal with the same sort of physical universe that we have locally, and hence they would have the same troubles that Zeno dramatized. Again, as Galileo observed, mathematics is the language of Science, and since the physical world they face has the same laws as ours, they cannot deviate too far from the mathematics we have. Or can they?

From the integers the Greeks came to the fractions, the rational numbers, the ratio numbers, and I believe the distant planet would too. But the Greeks, when recognizing that the diagonal of a unit square is not a rational number, decided it is not a number at all, at best it is a magnitude, and we have a lot of Euclid's *Elements*, beginning in Book 5, trying to deal with the Eudoxean theory of magnitudes. Somewhere in the Middle Ages, with the rise of the decimal system of notation, we seem to have decided that our irrational numbers are numbers. Need the aliens have done so? Probably, or else, like Euclid, they carried on with two parallel theories. The transcendentals such as π , γ , and e are probably not avoidable, and they on their planet will have had to deal with them in some manner.

In 1957 Turing introduced into logic the (paper, not physical) Turing machine to meet Hilbert's desire for mechanical proofs, which Hilbert supposed would be

perfect proofs valid for all time. In so doing Turing changed the definition of numbers! Before that time numbers were, apparently, regarded as being their representations (almost always in their decimal form as if God had five fingers!—rarely in binary form). Turing concentrated on what a machine could produce using programs, which were supposed to be a finite string of instructions (binary digits for convenience), and which when run on the corresponding computing machine would halt so you knew when you had the answer. Thus a *computable number* is a number for which there is some program to compute it on some Turing machine to as many (necessarily finite) digits as you specify. Thus Turing, in trying to get (ideal to be sure) mechanical proofs, changed the concept of a number from its *representation* to the corresponding *process* of getting as many digits as you please. The number π is now a program, say the program that generated some 6 billion digits of it, and is no longer the original infinite representation. Thus Turing introduced another modest step away from the actual infinite and back to the finite, but unbounded: the potential infinite of Aristotle as opposed to the actual infinite of Cantor. Most mathematicians seem to be unaware of this change, but those in computing find it natural. This change in the definition will ruin some old proofs and results, and at the same time make possible other proofs of other things.

This change from the infinite to the finite is in the direction of many other parts of mathematics; for example the delta-epsilon process. Again, I do not see how they on their planet could have avoided facing the problems of the discrete and continuous, and the consequent troubles with infinity. In one form or another, they probably had their Zeno with his paradoxes, as well as other troubles. After all we apparently live in a finite universe and the old idea of $1/3 = 0.3333\dots$ out to infinity, flies in the face of our own beliefs and probably their beliefs too!

Without attempting a proof, and I will get to this matter of mathematical rigor and proof in a little while, you can see, since a program is defined to be a finite sequence of instructions, that the possible computer programs are necessarily each finite but the set of all programs is unbounded, and hence would be countable, indeed by a suitable ordering even enumerable, hence the computable numbers, being a subset, are also countable. Note that this subset of programs of the enumerable set of strings of instructions is not well defined in the sense, to be shown later, that you cannot tell if an arbitrary string is a program or not. But most of you believe, because you were told so, and shown some kind of proof of it, that the numbers from 0 to 1 are not countable.

Now there is the well known Lowenheim-Skolem theorem in logic, that any finite set of postulates has a countable realization—and again with no proof offered by me you can still sort of see why that would be true; we simply do not accept infinitely long proofs as yet. But if so, how from the finite number of postulates could you prove the non-countability of the numbers defined by the postulates, since surely what you can prove from the postulates alone must apply to every realization of them? So you recall the proof Cantor gave, and his diagonalization process, which used the *representations* of numbers. Which do you prefer, the intuitively obvious proof that any finite number of postulates has a countable realization or the diagonalization process used to prove the non-countability? What would they, on their planet, have done since sooner or later it would come up? I am inclined to believe that they would have confined Cantor *in his old age* to an insane asylum. I will return to his diagonalization process later.

It will be noticed that I have avoided going deeply into conventional logic. As a graduate student I found, and studied, Boole's *Laws of Thought*, and I found it

interesting, relevant, and believable. But when I examine even introductions to mathematical logic I find the hair splitting that goes on unbelievable; I cannot believe that any one, or even all the fine distinctions together, could ever change a prime number into a composite number, undo Cauchy's contour theorem, or cause a missile, instead of hitting the target, to miss by a mile. The logician's activities seem to be irrelevant to mathematics as I understand it, but rather logic is an ideal game played by pure mathematicians for their own amusement.

So long as you confine yourself to ordered sequences and ordered sets then the natural way of computing the density of, say, the even integers, is to take the limiting ratio of those having the property to the total number examined, and of course you get $1/2$ for the density of the even integers, and you avoid the paradoxical statements that at first confuse people until it becomes clear to them that you have chosen one particular, rather unnatural, way of comparing the sizes of ordered sets. Even Galileo noted the paradox that arises when you try the method of 1 to 1 pairing off with the integers; in particular he observed that the pairing off way gave an equal number of perfect squares and positive integers. You get no such strange results when you stick with ordered sets and densities in intervals; it was because Cantor wanted to deal with infinite *unordered* sets that he used the 1 to 1 definition of equal size of sets. I am by no means sure that on the distant planet they would have so chosen, and this has serious consequences for Lebesgue integration, which assigns a measure 0 to each countable set, hence to all the computable numbers, hence in my opinion, all of reality that you can ever name or talk about! Indeed, for more than 40 years I have claimed that if whether an airplane would fly or not depended on whether some function that arose in its design was Lebesgue *but not Riemann* integrable, then I would not fly in it. Would you? Does Nature recognize the difference? I doubt it! You may, of course, choose as you please in this matter, but I have noticed that year by year the Lebesgue integration, and indeed all of measure theory, seems to be playing a smaller and smaller role in other fields of mathematics, and none at all in fields that merely use mathematics. Recently people have shown that the Henstock integral, which is a simple, reasonably natural extension of Riemann integration, is more general than Lebesgue integration with all its peculiar properties.

I know that the great Hilbert said, "We will not be driven out of the paradise Cantor has created for us," and I reply, "I see no reason for walking in!" Indeed, in time, as more and more people get used to computers, I am inclined to believe that we here on this Earth will decide that the computable numbers are enough. Apparently you never need a non-computable number! Take, for example, the classic real line from 0 to 1, and remove the computable numbers. You have a non-countable number of numbers left, no one of which you can ever describe (how can you describe a number adequately if you cannot give, at least implicitly, a way of finding it)! Yet the axiom of choice says you can select one! Can you? Which one, if you can never describe it so another person knows what you are talking about? Is the axiom of choice reasonable? Is it safe to depend on this axiom in this real world? Just as the physicists finally decided, after years of arguing about properties of the ether that it turned out could not be measured, I too believe it is better to ignore entirely what you cannot talk about or measure! Some things do not arise naturally!

Many years ago, as I was picking up a paper to read on the non-computable numbers, I suddenly realized that no one could ever come into my office and ask for a non-computable number. If they never can occur, why bother? So I threw the article in the waste basket unread!

At this point it should become apparent that I do not think that theorems are really proved. As G. H. Hardy said long ago, we emit some symbols, another person reads them, and they are either convinced or not by them. To simple people who believe whatever they read and do not question things for themselves, a proof is a proof is a proof, but to others a proof merely supplies a way of thinking about the theorem, and it is up to the individual to form an opinion. Formal proofs, where there is deliberately no meaning, can convince only formalists, and of the results obtained they themselves seem to deny any meaning. Is that to be the mathematics we are to use in understanding the world we live in?

When Shannon first published his *Information Theory* (1947), most people recognized that the theorems were true but the proofs were inadequate. Professor Doob at Illinois took the strict view of proofs, and openly doubted, in his Math Review, Shannon's mathematical integrity! Thus, again, there are two quite different views of what mathematics really is, and I have no way of deciding if one or the other won out on the distant planet, or perhaps, as on Earth, both flourish side by side with little mutual understanding between them. For me theorems are true or false pretty much independent of their corresponding proofs; my internal beliefs must be the final arbiter of whether I accept or reject the mathematics I see. But the purists believe that the postulates, definitions, and the accepted logic determine such things!

Coming back from fancy, mystical theories, which seem uncheckable in reality, to Euclidean geometry and its postulates, there is a standard proof, using accepted methods, that all triangles are isosceles, and the corollary follows that all triangles are equilateral. The proof rests, as you no doubt know, on an improperly drawn figure. Hilbert recognized that Euclid had assumed, but had not proved, things about intersections and betweenness, and to cope with such proofs Hilbert added many more postulates than Euclid had started with! I first read up on this as a graduate student, and I discovered the remarkable fact that not one of the 400-odd theorems in Euclid was thereby shown to be false! After a lot of thought I realized that Hilbert made the added postulates so this would be true—meaning that the theorems were being viewed as true independent of the inadequate proofs (beginning with Theorem 1)—and from there I soon realized that Euclid had been in the same position; he had a lot of theorems he “knew were true,” including the Pythagorean theorem, and he had to find postulates that would support them. Mathematics is not simply laying down some arbitrary postulates and then making deductions, it is much more; you start with some of the things you want and you try to find the postulates to support them! Bourbaki to the contrary, notwithstanding!

We now get to the questions of proofs and rigor. I have long argued that given a rising standard of rigor we cannot now be sure of any proofs. Surely, most of our current proofs will have to be fixed up, much as in my time I have found that I had to fix up proofs of some of the greatest mathematicians. Thus Gauss, in his doctoral thesis, after showing how previous proofs of the fundamental theorem of algebra were not reliable, gave a proof himself—indeed he gave several different proofs over his lifetime—but probably all of them would be found to have holes in them by a modern topologist! Did Gauss ever prove the fundamental theorem of algebra? In what sense do you mean he proved it? In an argument with a very good mathematician in the Mathematics Department's common room at Bell Telephone Laboratories, I sprung the rising level of proof problem on him. He was driven to asserting that we had at that time (1960's) reached the ultimate of rigor in proofs and there would be no more need for patching up old proofs! Of course later, with time to calm down and collect his thoughts, he might well have changed his mind!

Perhaps it is not true that a theorem can ever either be definitively proved or disproved!

Well, I face the same problem with respect to the distant planet. Could they have found a certain, sure mathematics that does not need the continual re-proving of theorems that we have? Can such mathematics exist? I wish I knew!

Back to geometry again. On this Earth we have chosen carefully to ignore anti-symmetry; two plane triangles are congruent even if we have to flip one over in three dimensions to get the match. It seems probable to me that the distant planet might have preferred to admit orientation in the first place and not force most users of geometry to tack it on later. In our classical Euclidean geometry we cannot have the important theorem that in three dimensions there are only two orientations, those of left- and right-handed threads.

Now, consider one of the most widely cited theorems of Euclidean geometry, that you cannot trisect an arbitrary angle with straight edge and compass. It is true, in a sense, but if you allow two marks on the straight edge then it is false (that was known to Archimedes)! In practical use it is a trivial difference! Would they have had a Plato who was so obsessed with the ideal that he wanted no physical instruments in geometry outside the straight edge and compass? Not even two marks on the straight edge! That mathematicians should so often cite the theorem whose truth or falsity hangs on such a trivial difference in definition is unreasonable! It is not a theorem relevant to the real world.

Along these lines I leave it to you to consider the matter of how we conveniently count multiple zeros of a function as being multiple (deliberately confusing factors and zeros!), so you can then extend the reasoning to dual graphs, justify counting self-dual graphs twice, and thus obtain the theorem that there are six regular solids in three space and not five. After all, six is a perfect number! Mere words to be sure; nothing has changed, the world is still as it was, but the statement of the theorem is quite different!

Well, much of our mathematics is like that, and hence we cannot assume that the aliens would have followed the same narrow path as we did. Without defining it, I will claim that the “robust” parts of mathematics can *generally* be depended on when care is taken in the identification of the parts of reality and the corresponding mathematics (along with a careful inspection of the underlying assumptions, general structure, and its usefulness in other places), and that the “non-robust” parts are useless to us, as useless as the idea of the ether was to the physicists, best forgotten!

By now you are probably wanting me to say what I believe so you can attack me for all the outrageous things I have pointed out about your mathematics. I will begin with Hermite who said, “We are not the master of Mathematics, we are the servant.” I have often said the opposite, “We are the master of Mathematics, not the servant; it shall do as we want it to do.” In truth, I seem to believe in a blend of the two remarks; at times we are driven and at times we are in control of mathematics. So too, the aliens will find themselves, and because they live in the same kind of physical world and have established radio contact with us, their “robust,” useful mathematics will have a reasonable analogy with ours, but the “non-robust” parts could be very different. Would they even know or care about all of our trivial theorems? Although it is widely assumed to have been proved with our assumptions, definitions, and methods of reasoning, is the famous Fermat theorem true in their mathematics? Is it not necessary to consider what they regard as a proof and as no proof, or even what are meaningful statements to them and to us?

Before you think that I am excluding much of higher mathematics let me observe another thing I have long said. If you come into my office and show me that Cauchy's contour theorem is false, I will be very interested, but in the end I will say to you that you should go back and find other assumptions so that it is true because I "know" it is "true"; it is too needed in some form, perhaps only Green's theorem, not to have it be true. It gives the potential function behind vector fields, among other things, and it is the basis of our complex variables work, though we all know that there are at least three distinct approaches to complex variables: (1) Cauchy's contour theorem, (2) the power series approach of Lagrange and Weierstrass, and (3) the approach via harmonic functions, which never uses $i = \sqrt{-1}$. This again, illustrates a point I have made several times; the theory need not be unique to provide the support for actions that we are postulating the aliens have been able to carry out; there could be very different mathematical foundations, as well as surface details, so long as these support the results needed for the real world.

Finally, it is not only the postulates and definitions that need to be examined but also the logic used in mathematics. I cannot in a short time discuss most of the troubles in logic, hence we now come to consider just the role of self-reference in mathematics. You all know the statement, "This statement is false." It is grammatically correct, and there is nothing wrong with it *until* you apply it to itself, when if it is true then it is false, and if it is false then it is true! There are many other examples, such as the classic, "Moderation in all things." But the word "all" means an extreme, again a self-contradiction when you work it out. A friend of mine kept a notebook of such common statements.

Russell's paradox is the usual example used in logic to illustrate the dangers of using self-reference in definitions, and Russell developed an hierarchical theory of *types* as an escape from such problems, but later seemed to have rejected it! So far as I can see, his theory of types has not been widely accepted, and we are still in the quandary of contradicting ourselves in our assumptions via self-reference, possibly fairly remote.

As another kind of self-reference that may be dubious, consider the self-reference in the logic used in a proof. I will first choose Turing's main result for the halting problem, which states that there can be no program that can determine if an arbitrary program will halt or not. To prove it Turing starts by assuming that he has this program, which of course he does not believe exists! Since he can know nothing about the insides of the program all he can do is tinker with the input and output, and he simply reverses the outputs, "halt" and "does not halt." Next he applies one program, the unaltered, to altered one (or has it really been?), and comes out with a contradiction! Hence he claims the nonexistence of the program. Is that an acceptable form of proof? Does it convince you?

In impossibility proofs we often use the initial assumption that what we are going to prove does not exist, does exist! For example, in the classic proof that the square root of 2 is not a fraction, we assume we have a fraction in its lowest terms, square both sides, and get a contradiction involving the divisibility by 2. On the distant planet they might well prefer the broader theorem that no fraction when raised to any integer power can be an integer. On Earth, at present, both the Turing theorem and the irrationality of the square root of 2 are usually considered as proven, though one of them is rather self-referential.

Next we look at the Cantor diagonalization process, which seems to fall between the two examples. Again, he opens by *assuming* that if the numbers were countable then you could list them in some order, and he then makes a change in the

first digit of the first number, the second in the second, etc. down the infinite list to the end! Then he claims that the altered digits form a number that is not in the list. Hence there could be no such list, and the numbers between 0 and 1 are not countable (enumerable). Again, in the proof he deals with the actual infinite and not the potential infinite of Aristotle. After all he had no conception of computable numbers, so it is hard to blame him too much; he was using the traditional number representation of his time and we (at least some of us) have changed the ground under him!

It is a question of how much self-reference is acceptable in a self-reference proof. We know that some simple self-references can give self-contradictory results. We tend to feel that the simple square root proof, which algebraically rewrites the expression without other alterations, is a safe argument. I, at least, have my doubts of the validity of the Turing proof as it is usually given, and I am saying nothing about the truth of the theorem itself. The Cantor diagonalization, which does some alteration in the objects being discussed, supposes that we can cope with the actual infinity, and uses a much weaker degree of self-reference than Turing, hence falls in between the two, and I am ambiguous as to my belief in the safety of relying on the result. The Lowenheim-Skolem paradox does seem to undercut the non countability argument Cantor gave. Indeed, how would you test it? Remember we seemed to have shifted the definition of a number from its possibly infinite representation (using Dedekind cuts for example) to a finite process that will generate another digit. Is the Cantor diagonalization process still valid? Turing's theorem, quoted earlier (if you believe it), shows that there can exist no definite way to choose the programs that lead to computable numbers, so how could Cantor's initial listing be made for even the computable numbers when there can be no mechanical method of recognizing a program? Even if you do not believe Turing's theorem just how is the listing to be done? Would you risk your life on its truth in the real world, or is it merely an artifact of contemporary mathematics having nothing to do with reality? If the latter then it seems to me impossible to make a serious pronouncement on whether or not the aliens would have such mathematics. Any argument about the agreement of our mathematics with theirs must be based on having the same physical world to explain, and not on wild imaginations.

It seems to me that there are degrees of self-reference in such proofs, since some of the proofs convince me and some do not. Could the aliens on the distant planet have found some objective method of discrimination in self-reference so that they would not put me in the ambiguous position I find myself, believing some and not others with no sharp line between them? When you return to the contradiction I offered about the noncountability of the numbers on the real line, the classic proof you were raised on is the Cantor diagonal process. Suppose I offer to put the binary numbers in the order, all the one-digit numbers, then the two-digit numbers, then the three, etc., thus: 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, You give me any string of binary digits, and in this Universe it must be a finite string since the Universe is apparently finite, and I can name where it occurs! Alter any digit of any entry in any finite string and I can tell you where the altered one occurs! Of course this is not what Cantor did! He seized the infinite immediately, and supposed he could do what no program can do, whereas I am sticking with the attitude that a number is a process and you can give me only another digit of the number representation, not the string of all of the digits in its infinite representation. How would the aliens have chosen? Indeed, how do you wish to choose when perhaps your life is going to depend on the result?

I would not have you believe that I think of mathematics solely as being useful, that the artistic aspect is not useful—especially in teaching where beauty can add to the understanding. But with the doubling of mathematical results every 17 years or so, we are now at well over 100,000 new theorems each year, extrapolating from an estimate made long ago by Ulam (see *The Mathematical Experience*, Davis and Hersh, pp. 20–21). And there is the remark made by Ralph P. Boas when he was an editor of *Math Reviews*, that of the new results in the papers reviewed most are true but the corresponding proofs are perhaps half the time plain wrong! Some people claim that the great increase in the volume of published results is due to republication, in some disguised form that is hard to recognize in the new jargon, and is not new mathematics—still we have to cope with it. Again, try explaining this on the basis of the concept of the mathematics of Platonic idealism.

In the Platonic world that most mathematicians seem to think mathematics is, one “discovers” theorems that apparently were already there immediately after the big bang occurred. Opposed to that view is the view that I “create” the result when I find it. When I try to examine my own beliefs, without any prior conventional beliefs, I find that if the result seems to be important then I found it, but if it seems to be rather trivial then I created it! Their view on their distant planet? Mathematicians on this Earth generally realize that the Platonic view is not defensible logically, but stick to it anyway, except when pressed as to what mathematics is when they shift to a defensible position and claim that mathematics is an idle game of symbol manipulation with no inherent meaning. To paraphrase Hilbert, “When rigor enters, meaning departs.” After all, they on their planet will, like us, want outside financial support to continue, indeed to increase. I fear that they will have similar logical troubles about defining mathematics and what it is. Both here and there, it must be more than meaningless symbol manipulations if it is to result in communication via the radio waves predicted by Maxwell’s equations, but what it is in a more positive sense is hard to say.

In writing one book (*Methods of Mathematics Applied to Calculus, Probability, and Statistics*) I was forced to say that the essence of mathematics is extension, generalization, and abstraction. These are three similar, but distinct, attributes of mathematics, and I think they are the essence of mathematics, both here on Earth and there on their planet. But perhaps mathematics is “merely clear thinking,” and nothing more.

When we teach mathematics we should at all times be aware of this dual nature; its abstract beauty and its practicality, which is necessary to help us to cope with the Universe we find ourselves in. But we need to avoid the old Greek belief that mathematics is sure, certain knowledge. The aliens must also face the similar problem on their planet.

The purpose of this talk was to explore and sensitize you to the arbitrary elements in our accepted mathematics, as well as to the extent that we are forced to have the mathematics we have, since it is of use to explain the real world.

I also want to suggest strongly that if in the future you want government grants and support, then the latter kind of mathematics should get a good deal of your attention, but not to the complete neglect of elegance. The argument that in the past, pure, as opposed to directed, research has led to much useful mathematics is true, but until you can also seriously estimate the mathematics that was not done, because they did the pure mathematics instead of useful mathematics, then the argument has no validity, and mathematicians should be ashamed to use it. Furthermore, with the enormous growth of results, at well over 100,000 new (?) theorems every year (look at all the volumes of relevance to your own field alone

that come out each month), then the chance of a new piece of pure mathematics being spotted by you and also being at hand when you need it, and not have to be recreated when needed, is increasingly small; instead we will more often create the mathematics we need as we need it. Regeneration is increasingly easier than retrieval. In fact, even now it is often easier to reinvent some sort of concept than it is to find its description in the literature—which helps in the proliferation of “new results”!

You should not believe everything I have said; the talk is only a stimulus and a guide for you to examine your thoughts on what mathematics is and should be in near the future, as opposed to what the books and experts tell you it is. I found the search very illuminating and worthwhile. As I regularly tell my students, “In science and mathematics we do not appeal to authority, but rather *you are responsible for what you believe.*” I leave you to consider what you believe about mathematics.

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R. W. HAMMING worked at Los Alamos during WWII in the computing group for the design of the atom bomb. He then went to Bell Telephone Laboratories for 30 years, retired, and went to the Naval Postgraduate School to teach various courses for 21 years. As a consequence he is acutely aware of the need to have our predictions based on our mathematics agree, as best we can, with the real world we live in. This paper is based on a luncheon talk given by Professor Hamming at a Northern California Section Meeting in San Francisco, February 22, 1997. He was able to prepare the paper for publication before his death on January 7, 1998—*Editor*

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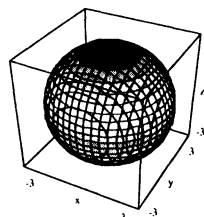
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