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The Competitive Allocation Process Is Informationally Efficient Uniquely

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This paper establishes that the competitive allocation process is the only informationally decentralized mechanism for exchange environments which (i) achieves Pareto optimal allocations; (ii) gives each consumer an allocation which is, according to his preferences, at least as good as his endowment; (iii) satisfies certain regularity conditions; and (iv) has a message space of the smallest dimension necessary to satisfy (i–iii). *Journal of Economic Literature* Classification Numbers 021, 022, 024, 025, 026.

1. INTRODUCTION

Following the path-breaking work of Hurwicz [2] and Mount and Reiter [8], the competitive allocation process has been proved to be informationally efficient in the sense that any informationally decentralized allocation mechanism for exchange environments which achieves Pareto optimal allocations must use a message space which is dimensionally at least as large as the competitive message space (see Osana [9] and, in a stochastic context, Jordan [3]). For brevity, this result will be referred to as the Efficiency Theorem. The purpose of this paper is to establish that *only* the competitive process is informationally efficient. This result will be called the Uniqueness Theorem.

Intuitively, the Uniqueness Theorem is straightforward. The Second Fundamental Theorem of Welfare Economics states that every Pareto optimal allocation can be attained as a competitive allocation after a suitable redistribution of wealth. This implies that any process which achieves Pareto optimal allocations can be represented as the competitive process preceeded by a process which redistributes wealth. Since the redistribution of wealth presumably requires additional information, the Uniqueness Theorem should follow. Unfortunately this approach is inapplicable because the redistribution

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of wealth can be combined with the choice of allocation in such a way that no additional information is required. An example is given in 4.6 below. Although this method of proof cannot be salvaged, such counterexamples can be excluded by restricting attention to allocation processes which are "noncoercive" in that no trader's allocation is less preferred than his initial endowment. This restriction is not necessary for the Efficiency Theorem.

Another restriction which is necessary for the Uniqueness Theorem but not the Efficiency Theorem is topological in nature. Since informational efficiency is viewed here as a dimensional minimality property of message spaces, allocation processes must be subject to some regularity conditions. For the Efficiency Theorem, the relation between environments and equilibrium messages on the space of Cobb–Douglas environments must be similar to that of the competitive process in that on a neighborhood of each environment there exists a continuous function from environments to equilibrium messages. For the Uniqueness Theorem, it is necessary to strengthen this to the requirement that the relation between Cobb–Douglas environments and equilibrium messages be single-valued and continuous. Section 5 below presents an example to show that this restriction is essential, and contains a more thorough comparison of the two theorems.

Since this assumption is only imposed for Cobb–Douglas environments, it is not sufficient to yield the Uniqueness Theorem on any larger class. The single-valuedness requirement cannot be extended much beyond the Cobb–Douglas class without conflicting with the multiplicity of competitive equilibria. In order to extend the Uniqueness Theorem to more general classes of exhange environments, we will impose the additional regularity assumptions that the message space is connected and that the set of messages associated with Cobb–Douglas environments is a closed subset of the message space. The need for these assumptions is illustrated by examples in Section 5.

The model of allocation processes, which is essentially taken from [8], is defined in Section 2, and the analysis is motivated in Section 3 by an example of a social welfare maximizing allocation process of the type excluded by the Uniqueness Theorem.

It may be worthwhile to mention that the proof of the Uniqueness Theorem, specifically the proof of Lemma 4.3 below, relies on the local homology of manifolds. However, no knowledge of algebraic topology is required to read the statement of the Theorem or the statements of any of the supporting lemmas.

2. Allocation Mechanisms

We will begin by studying allocation mechanisms for Cobb–Douglas exchange environments. More general classes of exhange evironments need not be introduced until 4.10 below, after most of the analysis is finished.

2.1. *Cobb–Douglas Environments.* There are K agents, indexed by the superscript i, and L commodities, indexed by the subscript j. Each agent's consumption set is R^L_+.[1] For each i, let U^i denote the set of utility functions $u^i: R^L_+ \to R$ such that there is some $\alpha^i \in R^L_{++}$ with $u^i(x^i) = \prod_{j=1}^L (x^i_j)^{\alpha^i_j}$ for each $x^i \in R^L_+$. The space U^i is topologized as R^L_{++}, and a generic element of U^i will be denoted either u^i or α^i. The ith agent also has an endowment $\omega^i \in R^L_{++}$. For each i, let $E^i = R^L_{++} \times U^i$, and let $E = \prod_i E^i$. A generic element of E is denoted $e = (e^1,..., e^K) = (\omega^i, u^i)_i$.

2.2. *Allocation Mechanisms.* A *message process* is a pair (μ, M) where M is a set of abstract messages and μ is a (nonempty-valued) correspondence on E to M. Let $Y = \{y = (y^1,..., y^K) \in R^{LK}: \sum_i y^i = 0\}$. An *allocation mechanism* is a triple (μ, M, g), where (μ, M) is a message process and $g: M \to Y$. The function g is called the *outcome function.*

An allocation mechanism (μ, M, g) is *nonwasteful* if for each $e = (\omega^i, u^i)_i \in E$ and each $y \in g[\mu(e)]$

(i) $\omega^i + y^i \in R^L_+$ for each i; and

(ii) there is no $y' \in Y$ such that for each i, $\omega^i + y'^i \in R^L_+$ and $u^i(\omega^i + y'^i) \geqslant u^i(\omega^i + y^i)$, with strict inequality holding for some i.

An allocation mechanism (μ, M, g) is *noncoercive* if for each $e = (\omega^i, u^i)_i$ and each $y \in g[\mu(e)]$, $u^i(\omega^i + y^i) \geqslant u^i(\omega^i)$ for each i.[2]

A message process (μ, M) is *privacy-preserving* if there exists a correspondence $\mu^i: E^i \twoheadrightarrow M$ for each i such that for each $e \in E$, $\mu(e) = \bigcap_{i=1}^K \mu^i(e^i)$. For each i and each e, $e' \in E$, the environment $(e^1,..., e^{i-1}, e'^i, e^{i+1},..., e^K)$ is denoted $e' \otimes^i e$. An allocation process (μ, M, g) is *informationally decentralized* if (μ, M) is privacy-preserving. The following characterization of the privacy property is due to Mount and Reiter [8, Lemma 5, p. 171].

2.3. Lemma. *A message process (μ, M) is privacy preserving if and only if*

$$(\otimes) \quad \mu(e) \cap \mu(e') = \mu(e' \otimes^i e) \cap \mu(e \otimes^i e')$$

for each i and each e, $e' \in E$.

[1] $R^L_+ = \{x \in R^L: x_j \geqslant 0 \text{ for each } j\}$ and $R^L_{++} = \{x \in R^L: x_j > 0 \text{ for each } j\}$.

[2] The term "individually rational" is more frequently used for this property (e.g., Luce and Raiffa [5, p. 193]. The term noncoercive is taken from Miller [7].

2.4. *The Competitive Mechanism.* Let $\Delta = \{p \in R^L_{++} : \sum_{j=1}^L p_j = 1\}$ and let $M_c = \{(p, y) \in \Delta \times Y\colon py^i = 0$ for each $i\}$. For each i, define the correspondence $\mu_c^i\colon E^i \twoheadrightarrow M_c$ by $\mu_c^i(\omega^i, u^i) = \{(p, y)\colon u^i(\omega^i + y^i) \geqslant u^i(\omega^i + y'^i)$ for all y'^i with $py'^i \leqslant 0\}$, and define the correspondence $\mu_c\colon E \twoheadrightarrow M_c$ by $\mu_c(e) = \bigcap_{i=1}^K \mu_c^i(e^i)$. Then (μ_c, M_c) is the *competitive message process*. The *competitive allocation mechanism* is (μ_c, M_c, g_c), where $g_c\colon (p, y) \to y$ is the projection.

Using the conditions $\sum_i y^i = 0$ and $py^i = 0$ for each i, and the zero degree homogeneity of demand, the function $(p, y) \to (q, \tilde{y}) \in R^{L-1}_{++} \times R^{(L-1)(K-1)}$, where $(q_1,..., q_{L-1}) = (P_1/P_L,..., P_{L-1}/P_L)$ and $\tilde{y}^i = (y_1^i,..., y_{L-1}^i)$ for each $1 \leqslant i \leqslant K-1$ is a C^∞ diffeomorphism. Thus M_c is an $(L-1)K$-dimensional smooth manifold and, since preferences are Cobb–Douglas, μ_c is a smooth function.

2.5. *Remarks.* The competitive mechanism is of course nonwasteful and noncoercive. The competitive message process is privacy preseving by definition.

We have used utility functions rather than preference relations in defining individual characteristics in order to permit allocation mechanisms to be sensitive to intensities of preference. This makes the class of allocation mechanisms larger, and the Uniqueness Theorem stronger, than would otherwise be the case.

3. A Social Welfare Maximizing Mechanism

This section motivates the Uniqueness Theorem by describing a noncompetitive allocation mechanism which requires a larger message space.

3.1. Definitions. Define the choice function $h\colon E \to Y$ by letting $h[(\omega^i, \alpha^i)_{i=1}^K]$ maximize $\sum_i \ln[u^i(\omega^i + y^i)] = \sum_i [\sum_j \alpha_j^i \ln(\omega_j^i + y_j^i)]$ subject to $u^i(\omega^i + y^i) \geqslant u^i(\omega^i)$ for each i. The smoothness and strict concavity of the functions $\ln(u^i)$ for each i imply that for each environment e, there is a Lagrange multiplier $\lambda(e) \in R^L_{++}$ such that $\hat{y} = h(e)$ if and only if for each i, $\ln[u^i(\omega^i + \hat{y}^i)] - \lambda(e)\hat{y}^i \geqslant \ln[u^i(\omega^i + y^i)] - \lambda(e) y^i$ for all $y^i \in R^L$ with $u^i(\omega^i + y^i) \geqslant u^i(\omega^i)$. Therefore, let $M_s = R^L_{++} \times Y$, and let $g_s\colon M_s \to Y$ be the projection. For each i, define the correspondece $\mu_s^i\colon E^i \twoheadrightarrow M_s$ by $\mu_s^i(\omega^i, u^i) = \{(\lambda, y)\colon y^i$ maximizes $\ln[u^i(\omega^i + y'^i)] - \lambda y'^i$ subject to $u^i(\omega^i + y'^i) \geqslant u^i(\omega^i)\}$ and define $\mu_s\colon E \twoheadrightarrow M_s$ by $\mu_s(e) = \bigcap_{i=1}^K \mu_s^i(e^i)$. Then $g_s \cdot \mu_s = h$.

3.2. *Remarks.* Unfortunately μ_s is not a function because when $h(e) = 0$ (i.e., when $(\omega^i)_i$ is Pareto optimal) the Lagrange multiplier $\lambda(e)$ is not uniquely defined, so the multiplier component of $\mu_s(e)$ is an interval.[3] However, if we define the continuous function

$$f: M_s \to M_s \text{ by } f(\lambda, y) = (\lambda, y) \qquad \text{if } \|y\| \geqslant 1,$$
$$= ([\|y\| + (1 - \|y\|)\,\|\lambda\|^{-1}]\lambda, y) \qquad \text{if } \|y\| < 1;$$

define $\mu_s' = f \cdot \mu_s$ then μ_s' is a continuous function, $g_s \cdot \mu_s' = h$, and (μ_s', M_s) is privacy preserving by Lemma 2.3.

3.3. Definition. A topological space M is an *n-dimensional manifold* if it is locally homeomorphic to R^n.

3.4. Proposition. *Suppose that* (μ, M, g) *is an allocation mechanism with* $g[\mu(e)] = \{h(e)\}$ *for each* $e \in E$, *satisfying*:

(i) (μ, M) *is privacy preserving*;

(ii) M *is an n-dimensional manifold*; *and*

(iii) μ *is a continuous function.*

Then $n \geqslant KL = \dim M_s$.

Proof. The proof will parallel a proof of an analogous theorem for the competitive mechanism by Mount and Reiter [8, Theorem 35, p. 190]. Let $t: R^L_{++} \times Y \to E$ be any continuous function such that for each $(\lambda, y) \in R^L_{++} \times Y$, if $(\omega^i, \alpha^i)_i = t(\lambda, y)$ then for each i, j, $\alpha^i_j(\omega^i_j + y^i_j)^{-1} = \lambda_j$. Let $y^0 \in Y$ with $y^{0i} = (1,..., 1, -(L-1))$ for each $i < K$ and let $\lambda^0 = (1,..., 1)$. Let $e^0 = (\omega^{0i}, u^{0i})_i = t(\lambda^0, y^0)$. Then y^0 is the competitive equilibrium trade for e^0, so $u^{0i}(\omega^{0i} + y^{0i}) > u^{0i}(\omega^{0i})$ for each i. Since t is continuous there is a neighborhood A of (λ^0, y^0) in M_s such that $t(A)$ is contained in a neighborhood $\Pi_i B^i$ of e^0 in E with the property that for each $e \in \Pi_i B^i$, if $y = h(e)$ then $u^i(\omega^i + y^i) > u^i(\omega^i)$ for each i, It follows that $\mu_s \cdot t$ is the identity on A. We will use assumption (i) to prove that $\mu \cdot t$ is 1–1 on A. Suppose by way of contradiction that there exist distinct points (λ, y) and (λ', y') in A with $\mu[t(\lambda, y)] = \mu[t(\lambda', y')]$. Since $g(\mu[t(\lambda, y)]) = y$ and $g(\mu[t(\lambda', y')]) = y'$, we must have $y = y'$, so $\lambda \neq \lambda'$. Let $e = t(\lambda, y)$ and $e' = t(\lambda', y)$. By (i) and Lemma 2.3, $h(e' \otimes^1 e) = g(\mu(e)) = g(\mu(e')) = y$. However, $(\alpha_j'^1(\omega_j'^1 + y_j^1)^{-1})_{j=1}^L = \lambda' \neq \lambda = (\alpha_j^2(\omega_j^2 + y_j^2))_{j=1}^L$, so $h(e' \otimes^1 e) \neq y$. This contradiction proves that $\mu \cdot t$ is 1–1 on A. It follows from [1, Exercise 18.11, p. 82] that $n \geqslant KL$.

3.5. *Remarks.* Thus the non-Walrasian choice function h requires more information, in this sense, than the Walrasian choice function which requires only a $K(L-1)$ dimensional message space.

[3] I would like to thank Kevin Cotter for this observation.

It may be noted that because of the constraint $u^i(\omega^i + y^i) \geqslant u^i(\omega^i)$ for all i, the range of μ_s is a proper subset of M_s, although of course it contains an open subset of M_s. It is easily seen that $\mu_s(E) = \{(\lambda, y) \in M_s$: for each i and each j, if $y^i_j < 0$ then there is some j' with $y^i_{j'} > 0\}$. If h and μ_s are redefined by dropping this constraint, we obtain a new allocation mechanism (μ''_s, M_s, g_s) with $h' = g_s \cdot \mu''_s$, which is no longer noncoercive. Then $\mu''_s(E) = M_s$. This example illustrates that since noncoerciveness is a property which can be checked for each agent independently, it will not necessarily increase the information requirement.

4. The Uniqueness Theorem

We will first obtain the Uniqueness Theorem for allocation mechanisms on the space of Cobb–Doublas environments E. If an allocation mechanism is nonwasteful, each allocation has an associated shadow price. If the mechanism is noncoercive, each agent's final consumption will be in R^L_{++} so the shadow price will equal the normalized derivative of each agent's utility function evaluated at his final consumption. The first step is to show, as in the proof of the Efficiency Theorem, that informational decentralization requires that the equilibrium message reveal the shadow price.

Lemma 4.1. *Suppose that (μ, M, g) is an allocation mechanism which is*

(i) *nonwasteful;*

(ii) *noncoercive; and*

(iii) *informationally decentralized.*

Then there is a function $h: \mu(E) \to \Delta \times Y$ defined by $h(m) = (p, y)$ where $y = g(m)$ and p is proportional to $Du^i(\omega^i + y^i)$ for each i and each $(\omega^i, u^i)_i$ with $m \in \mu[(\omega^i, u^i)_i]$.

Proof. Let $e = (\omega^i, u^i)_i \in E$, let $m \in \mu(e)$, and let $y = g(m)$. By assumption (ii), $\omega^i + y^i \in R^L_{++}$ for each i, so assumption (i) implies the existence of $p \in \Delta$ with $Du^i(\omega^i + y^i)$ proportional to p for each i. Let $e' = (\omega'^i, u'^i) \in E$ be any other environment with $m \in \mu(e')$. Since (μ, M) is privacy preserving, Lemma 2.3 implies that $m \in \mu(e' \otimes^i e)$ for each i. Therefore $y \in g[\mu(e' \otimes^i e)]$ for each i, so $Du'^i(\omega'^i + y^i)$ is proportional to p for each i. Thus h is well defined, which completes the proof.

4.2. *Remarks.* The next step is to show that if the message space M has minimal dimension and μ is a continuous function, then h is 1–1. In order words, the equilibrium message cannot reveal any more than the allocation and the shadow price. Since we have characterized individual agents in terms of utilities rather than preferences, it may be worthwhile to remark that the map from parameters to Cobb–Douglas preferences is continuous in the topology of closed convergence of preference relations [6]. Thus if agents were characterized by their preferences, the continuity of μ could be stated in terms of this topology on preferences.

4.3. LEMMA. *Suppose that* (μ, M, g) *is an allocation mechanism which is*

(i) *nonwasteful*;

(ii) *noncoercive*;

(iii) *informationally decentralized*; *and*

(iv) M *is a* $K(L-1)$ *dimensional manifold*; *and*

(v) μ *is a continuous function.*

Let $e^0 \in E$ *and let* $(p^0, y^0) = h[\mu(e^0)]$, *where h is defined in Lemma* 4.1. *If* $(\omega^{*i}, u^{*i})_i$ *is any environment such that* $\omega^{*i} + y^{0i} \gg 0$ *and* $Du^{*i}(\omega^{*i} + y^{0i})$ *is proportional to* p^0 *for each i, then* $\mu(e^*) = \mu(e^0)$. *In particular, h is* 1–1.

Proof. Let $e^0 = (\omega^{0i}, a^{0i})_i \in E$ and define the set $E^0 = \{(\omega^i, a^i)_i \in E$: $\omega^i = \omega^{0i}$ and $\sum_j a_j^i = \sum_j a_j^{0i}$ for each $i\}$. Note that for each $(\omega^{0i}, u^i)_i$ and $(\omega^{0i}, u'i)_i$ in E^0, if for some $y \in Y$, $Du^i(\omega^{0i} + y^i)$ is proportional to $Du'^i(\omega^{0i} + y^i)$ for each i then $u^i = u'^i$ for each i. Thus by Lemma 4.1, $h \cdot \mu$ is 1–1 on E^0 so μ is 1–1 on E^0.

The function $(\omega^{0i}, a^i)_i \to (a_1^i/a_L^i, \dots, a_{L-1}^i/a_L^i)_i$ is a homeomorphism on E^0 to $R_{++}^{K(L-1)}$, so we will consider E^0 as an open subset of $R^{K(L-1)}$. Let U be an open ball in E^0 containing e^0 and let $m^0 = \mu(e^0)$. Since M is a $K(L-1)$ dimensional manifold, $\mu(U)$ is an open neighborhood of m^0 and μ is a homeomorphism on U to $\mu(U)$ by [1, Exercise 18.10, p. 82]. Let $n = K(L-1)$, and let $H_n(M, M - m^0)$ denotes the nth singular homology module of M relative to $M - m^0$,[4] with integral coefficients. Then $H_n(M, M - m^0) = Z$ and, letting $V = \mu(U)$, the homomorphism i_*: $H_n(V, V - m^0) \to H_n(M, M - m^0)$ induced by the inclusion $i: (V, V - m^0) \subsetneq (M, M - m^0)$ is an isomorphism [1, p. 111]. By the hypothesis, for each i $D\ln[u^{*i}(\omega^{*i} + y^{0i})]$ is proportional to $D\ln[u^{0i}(\omega^{0i} + y^{0i})]$ so there is some $\lambda^{*i} > 0$ such that $a_j^{*i}/(\omega_j^{*i} + y_j^{0i}) = \lambda^{*i}[a_j^{0i}/(\omega_j^{0i} + y_j^{0i})]$ for each j. Define the function $G: U \times [0, 1] \to E$ by $G((\omega^{0i}, a^i)_i, t) = (\omega'^i, a'^i)_i$, where $\omega'^i = t\omega^{*i} + (1-t)\,\omega^{0i}$ and for each j,

$$a_j'^i = [ta_j^{0i} + (1-t)\,a_j^i][(\omega_j'^i + y_j^{0i})/(\omega_j^{0i} + y_j^{0i})][t\lambda^{*i} + (1-t)].$$

Then $G(\cdot, 0)$ is the inclusion map and $G(\cdot, 1)$ is the constant map on U to e^*. Let $t \in [0, 1]$, $(\omega^{0i}, a^i)_i \in U$ with $(\omega^{0i}, a^i)_i \neq e^0$, and $(\omega'^i, a'^i)_i =$

[4] $M - m^0$ denotes the set $\{m \in M: m \neq m^0\}$.

$G((\omega^{0i}, \alpha^i)_i, t)$. Then for each i, $[\alpha_j'^i/(\omega_j'^i + y_j^{0i})]_{j=1}^L$ is proportional to $[(t\alpha_j^{0i} + (1-t)\,\alpha_j^i)/(\omega_j^{0i} + y_j^{0i}))]_{j=1}^L$. The environment $(\omega^{0i}, t\alpha^{0i} + (1-t)\,\alpha^i)_i \in E^0$, so by the argument in the first paragraph, for each $t < 1$, $\mu[G(e, t)] = m^0$ only if $e = e^0$.

Suppose by way of contradiction that $\mu(e^*) \neq m^0$. Then for each t, $\mu[G(e, t)] \neq m^0$ whenever $e \neq e^0$. Now define $G': (V, V - m^0) \times [0, 1] \to (M, M - m^0)$ by $G'(m, t) = \mu(G[\mu^{-1}(m), t])$. Then G' is a homotopy between the inclusion $i: (V, V - m^0) \subsetneq (M, M - m^0)$ and the constant map $j: (V, V - m^0) \to (m^*, M - m^0)$, where $m^* = \mu(e^*)$. Hence $i_* = j_*$ so i_* is the zero homomorphism, which completes the proof.

4.4. *Remarks.* So far we have used the noncoerciveness assumption only to ensure that allocations are interior so that shadow prices are normalized utility gradients. We will now use the full force of noncoerciveness to prove that h is a homeomorphism on $\mu(E)$ to M_c which identifies (μ, M, g) with the competitive mechanism.

4.5. PROPOSITION. *Suppose that* (μ, M, g) *is an allocation mechanism which is*

(i) *nonwasteful*;

(ii) *noncoercive*;

(iii) *informationally decentralized*; *and*

(iv) M *is a* $K(L-1)$ *dimensional manifold*; *and*

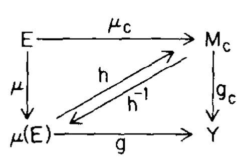
(v) μ *is a continuous function.*

Then there is a homeomorphism h *on* $\mu(E)$ *to* M_c *such that*

(a) $\mu_c = h \cdot \mu$; *and*

(b) $g_c \cdot h = g$.

The conclusion of the Proposition is summarized in the following commutative diagram:



Proof. Let $h: M \to \Delta \times Y$ be the function defined in Lemma 4.1. Suppose by way of contradiction that for some $e^0 \in E$, $h[\mu(e^0)] \neq \mu_c(e^0)$. Let $(p^0, y^0) = h[\mu(e^0)]$. Then y^0 is not the competitive trade for e^0, so for some i, say $i = 1$, $p^0 y^{01} < 0$. Let $x^0 = \omega^{01} + y^{01}$, where $(\omega^{0i}, u^{0i})_{i=1}^K = e^0$. Exploiting the linear homogeneity of preferences represented by u^{01}, we can find some

$\lambda > 1$ such that $u^{01}(\lambda x^0) < u^{01}(\lambda x^0 - y^{01})$. Let $\omega^{*1} = \lambda x^0 - y^{01}$ and $e^* = (\omega^{*1}, u^{01}; \omega^{02}, u^{02};..., \omega^{0K}, u^{0K})$. Then since (μ, M, g) is noncoercive, $\mu(e^*) \neq \mu(e^0)$. However, y^0 is a Pareto optimal trade for e^* and $Du^1(\omega^{*1} + y^{01})$ is proportional to p^0 so Lemma 4.3 implies that $\mu(e^*) = \mu(e^0)$. This contradiction proves that $h \cdot \mu = \mu_c$, and since g_c is the projection $(p, y) \to y$, it follows that $g_c \cdot h = g$.

By Lemma 4.3, h is 1–1; and since $h \cdot \mu = \mu_c$, the range of h is M_c. Hence it only remains to show that h and h^{-1} are continuous. To show that h^{-1} is continuous, let $\{m_c^n\}_{n=1}^{\infty}$ be a sequence in M_c converging to some $m_c^0 \in M_c$. It is straightforward to construct a sequence $\{e^n\}_{n=1}^{\infty}$ in E converging to some $e^0 \in E$ with $m_c^n = \mu_c(e^n)$ for each n and $m_c^0 = \mu_c(e^0)$. Since $h^{-1} \cdot \mu_c = \mu$, $h^{-1}(m_c^n) = \mu(e^n)$ for each n and $h^{-1}(m_c^0) = \mu(e^0)$. Since μ is continuous, $\mu(e^n)$ converges to $\mu(e^0)$, so h^{-1} is continuous. Since M_c and M are manifolds of the same dimension, h^{-1} is a homeomorphism on M_c to $h^{-1}(M_c) = \mu(E)$ [1, Exercise 18.10, p. 82], which completes the proof.

4.6. Example. Before extending this result beyond the Cobb–Douglas case we will describe an example which illustrates the role of noncoerciveness. Define the homeomorphism $f^i: R^L_{++} \times U^i \to R^L_{++} \times U^i$ by $f^i(\omega^i; \alpha^i) = (\omega^i; 2\alpha_1^i, \alpha_2^i, ..., \alpha_L^i)$ for each i, and define $f: E \to E$ by $f = (f^1, ..., f^K)$. First observe that an allocation y is Pareto optimal for an environment e if and only if it is Pareto optimal for $f(e)$. This implies that the allocation mechanism $(\mu_c \cdot f, M_c, g_c)$ is nonwasteful. Also, $(\mu_c \cdot f, M_c)$ is privacy preserving with $\mu^i(\omega^i, \alpha^i) = \mu_c^i[f^i(\omega^i, \alpha^i)]$ for each i. This mechanism therefore satisfies all the hypotheses of the Proposition except noncoerciveness, and $g_c \cdot \mu_c \cdot f \neq g_c \cdot \mu_c$. In this mechanism, each agent independently modifies his own preferences and the competitive mechanism is then applied to the modified environment. This type of mechanism can of course be constructed for more general classes of exchange environments.

We will now specify a more general class of utility functions. In order to ensure that prices are strictly positive, strict monotonicity will be required. However, this requirement will be stated in such a way that preferences, such as Cobb–Douglas preferences, which do not satisfy strict monotonicity on the boundary of R^L_+ are permitted.

4.7. Definitions. For each i, with each function u^i on R^L_+ to $R \cup \{-\infty\}$ we associate a set C^i defined by

$$C^i = \begin{cases} R^L_{++} \text{ if the closure in } R^L_+ \text{ of the set} \\ \{x: u^i(x) \geqslant u^i(\omega^i)\} \text{ is contained in } R^L_{++} \\ \text{for each } \omega^i \in R^L_{++}; \text{ and } R^L_+ \text{ otherwise.} \end{cases}$$

Let $U^{\#i}$ denote the set of utility functions u^i on R^L_+ to $R \cup \{-\infty\}$ such that

(i) u^i is continuous and real-valued on C^i;

(ii) u^i is strictly monotone on C^i; that is, if $x \in C^i$ and $x' \geqslant x$ and $x' \neq x$ then $u^i(x') > u^i(x)$; and

(iii) u^i is quasi-concave on C^i; that is, if $x, x' \in C^i$ and $u^i(x) \geqslant u^i(x')$ then $u^i(\lambda x + (1-\lambda)x') \geqslant u^i(x')$ for all $0 < \lambda < 1$.

Let $E^{\#} = \prod_i (R^L_{++} \times U^{\#i})$. Then $E \subset E^{\#}$. The definitions in sections 2.2 and 2.4 extend to $E^{\#}$ in the obvious way, as does Lemma 2.3.

4.8. Proposition. *Suppose that (μ, M, g) is an allocation mechanism on $E^{\#}$ which is*

(i) *nonwasteful*;

(ii) *noncoercive*;

(iii) *informationally decentralized*; *and*

(iv) *M is a connected $K(L-1)$ dimensional manifold*;

(v) *the restriction of μ to E is a continuous function*; *and*

(vi) *$\mu(E)$ is closed in M.*

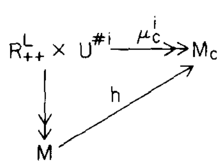
Then there is a homeomorphism h on M to M_c such that

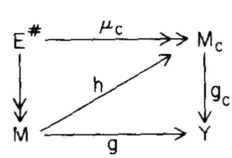
(a) $h[\mu^i(\omega^i, u^i)] \subset \mu^i_c(\omega^i, u^i)$ *for each* $(\omega^i, u^i) \in R^L_{++} \times U^{\#i}$;

(b) $h[\mu(e)] \subset \mu_c(e)$ *for each* $e \in E^{\#}$; *and*

(c) $g_c \cdot h = g$.

Diagramatically:





Proof. It was proved at the beginning of the second paragraph of the proof of Lemma 4.3 that $\mu(E)$ is an open subset of M. Since M is connected and $\mu(E)$ is closed in M, $\mu(E) = M$. Let h be the homeomorphism given by

Proposition 4.5, which satisfies (c). To prove (a), let $(\omega^i, u^i) \in R^L_{++} \times U^{\#i}$, let $m \in \mu^i(\omega^i, u^i)$, and let $(p, y) = h(m)$. Since $(p, y) \in M_c$, $py^i = 0$. We have to show that y^i maximizes $u^i(\omega^i + z)$ subject to $pz \leqslant 0$. By assumption (vi), there is some $e^0 \in E$ with $m \in \mu(e^0)$. Then by Proposition 4.5, $(p, y) = \mu_c(e^0)$. Since $m \in \mu^i(w^i, u^i)$, $m \in \mu(e^*)$, where $e^* = (\omega^{01}, u^{01}; ...; \omega^{0i-1} u^{0i-1}; \omega^i, u^i; \omega^{0i+1}, u^{0i+1}; ..., \omega^{0K}, u^{0K})$, so y must be Pareto optimal for e^* by assumption (i). This implies that y^i maximizes $u^i(\omega^i + z)$ subject to $pz \leqslant 0$. This proves (a), and (b) follows directly.

4.9. *Remarks.* Unfortunately, it is not possible to conclude that $\mu_c(e) \subset h[\mu(e)]$ for all e. In particular, there exist allocation mechanisms satisfying the hypotheses of Proposition 4.8 such that for some environments, not all Walrasian allocations are achieved by the mechanism. For example, suppose that $K = L = 2$, and that agent 1 has the endowment $\omega^{01} = (3, 1)$ and the preferences depicted in Fig. 1. Figure 2 is the graph of the agent's excess demand for commodity 1 as a function of its price, where $p_1 + p_2 = 1$. Let u^{01} represent the preferences depicted in Fig. 1 and consider the allocation mechanism (μ^0, M_c, g_c), where $\mu^0[(\omega^i, u^i)^2_{i=1}] = \mu^{01}(\omega^1, u^1) \cap \mu^{02}(\omega^2, \mu^2)$, and $\mu^{02} = \mu^2_c$, and μ^{01} is defined by

$$\mu^{01}(\omega^1, u^1) = \begin{cases} \mu^1_c(\omega^1, u^1) \text{ if } (\omega^1, u^1) \neq (\omega^{01}, u^{01}); \text{ and} \\ \{(p, y) \in \mu^1_c(\omega^{01}, u^{01}) \colon y^1 = (-1, 1) \text{ if} \\ p = (1/2, 1/2)\} \text{ if } (\omega^1, u^1) = (\omega^{01}, u^{01}). \end{cases}$$

Thus $\mu^{01}(\omega^{01}, u^{01})$ is obtained by deleting the "tail" on the excess demand correspondence in Fig. 2. The mechanism (μ^0, M_c, g_c) clearly satisfies all of the hypotheses of 4.8.

Let (ω^{02}, u^{02}) be characteristic for agent 2 which give rise to the excess demand correspondence in Fig. 3. Let $e^0 = (\omega^{01}, u^{01}; \omega^{02}, u^{02})$. Then $\mu^0(e^0)$ is the singleton $(p^0, y^0) = (1/2, 1/2; -1, 1; 1, -1)$ whereas $\mu_c(e^0)$ is the set

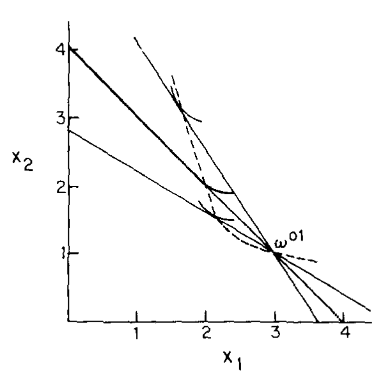


FIGURE 1

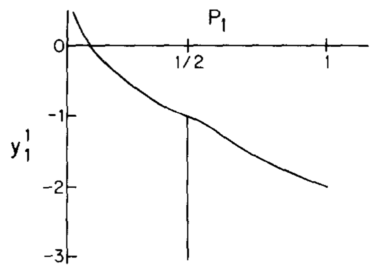


FIGURE 2

$\{(p^0; -\lambda, \lambda; \lambda, -\lambda): 1 \leqslant \lambda \leqslant 3\}$. In this example, agent 1 modifies his true excess demand correspondence by taking a continuous single-valued selection from it. Since this modification is independent of agent 2's characteristics, the resulting reduction in the set of Walrasian allocations is consistent with the Proposition. Of course if preferences are strictly convex then excess demand is single-valued to begin with, so no such reduction is possible.

Surprisingly, the desired result can also be obtained if preferences can be represented by a concave utility function. More specifically, if u^i is either strictly quasi-concave or concave, for any $(p, y) \in \mu_c^i(\omega^i, u^i)$ there exist endowments and utility functions for the other agents such that (p, y) is the unique competitive equilibrium of the resulting environment. This is stated in Proposition 4.11.

4.10. DEFINITIONS. For each i, let U^{*i} denote the set of utility functions u^i in $U^{\#i}$ satisfying either

(i) u^i is strictly quasi-concave on C^i; that is, for each $x, x' \in C^i$, if $x \neq x'$ and $u^i(x) \geqslant u^i(x')$ then $u^i(\lambda x + (1-\lambda)x') > u^i(x')$ for all $0 < \lambda < 1$; or

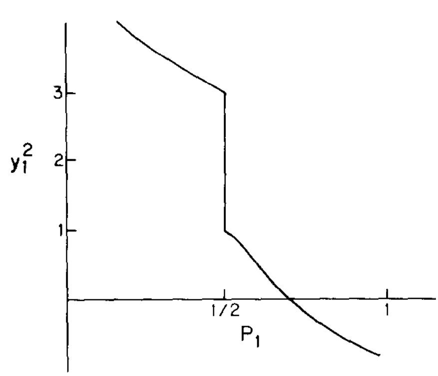


FIGURE 3

(ii) u^i is concave on C^i; that is, for each $x, x' \in C^i$, $u^i(\lambda x + (1-\lambda)x') \geqslant \lambda u^i(x) + (1-\lambda)u^i(x')$ for each $0 < \lambda < 1$.

Let $E^* = \prod_i (R^L_{++} \times U^{*i})$. Then $E \subset E^* \subset E^{\#}$. The definitions in 2.2 and 2.4 extend to E^* in the obvious way, as does Lemma 2.3.

4.11. PROPOSITION. *For any i, let $(\omega^{*i}, u^{*i}) \in R^L_{++} \times U^{*i}$ and let $(p^*, y^*) \in \mu^i_c(\omega^{*i}, u^{*i})$. Then for each $i' \neq i$ there exists $(\omega^{i'}, u^{i'}) \in R^L_{++} \times U^{*i'}$ such that $\mu_c(\omega^1, u^1; ...; \omega^{i-1}, u^{i-1}; \omega^{*i}, u^{*i}; \omega^{i+1}, u^{i+1}; ...; \omega^K, u^K)$ is the singleton (p^*, y^*).*

Proof. The proof will be symmetric in i so let $(\omega^{*1}, u^{*1}) \in R^L_{++} \times U^{*1}$ and let $(p^*, y^*) \in \mu^1_c(\omega^{*1}, u^{*1})$. First suppose that u^{*1} is strictly quasi-concave on C^1. For each $1 < i < K$, if any, let $(\omega^i, u^i) \in R^L_{++} \times U^i$ (that is, u^i is Cobb–Douglas) with $(p^*, y^*) \in \mu^i_c(\omega^i, u^i)$. Define the utility function $u^K: R^L_+ \to R$ by $u^K(x) = p^*x$, and choose ω^K large enough so that p^* is the unique competitive equilibrium price for the environment $(\omega^{*1}, u^{*1}; \omega^2, u^2; ...; \omega^K, u^K)$. (Informally, if $p \neq p^*$ agent K's excess demand is a corner solution which, if ω^K is large enough relative to $\omega^{*1} + \sum_{i=2}^{K-1} \omega^i$, floods the market). Since p^* is the unique competitive equilibrium price, and u^{*1} and u^i for each $1 < i < K$ are strictly quasi-concave, it follows that $\mu_c[(\omega^{*1}, u^{*1}; \omega^2, u^2; ...; \omega^K, u^K)]$ is the singleton (p^*, y^*). In the more difficult case in which u^{*1} is concave on C^1, the result is given by [4, Corollary 3.2].

4.12. THE UNIQUENESS THEOREM. *Suppose that (μ, M, g) is an allocation mechanism on E^* which is*

(U.i) *nonwasteful*;

(U.ii) *noncoercive*;

(U.iii) *informationally decentralized*; *and*

(U.iv) *M is a connected $K(L-1)$ dimensional manifold*;

(U.v) *the restriction of μ to E is a continuous function*; *and*

(U.vi) *$\mu(E)$ is closed in M.*

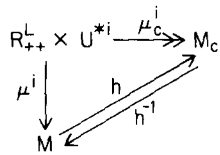
Then there is a homeomorphism h on M to M_c such that

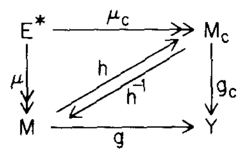
(a) *$h[\mu^i(\omega^i, u^i)] = \mu^i_c(\omega^i, u^i)$ for each i and each $(\omega^i, u^i) \in R^L_{++} \times U^{*i}$*;

(b) *$h[\mu(e)] = \mu_c(e)$ for each $e \in E^*$*; *and*

(c) *$g_c \cdot h = g$.*

Diagramatically:





Proof. As in the proof of 4.8, $\mu(E)=M$, so let h be the homeomorphism given by Proposition 4.5, which satisfies (c). For any i, let $(\omega^i, u^i) \in R^L_{++} \times U^{*i}$. One proves that $h[\mu^i(\omega^i, u^i)] \subset \mu^i_c(\omega^i, u^i)$ exactly as in the proof of 4.8(a). Let $(p, y) \in \mu^i_c(\omega^i, u^i)$ and, using Proposition 4.11, let $e = (\omega^1, u^1;...; \omega^{i-1}, u^{i-1}; \omega^i, u^i; \omega^{i+1}, u^{i+1};...; \omega^K, u^K) \in E^*$ such that $\mu_c(e)$ is the singleton (p, y). Since $\mu(e) \subset \mu^i(\omega^i, u^i)$ and $h[\mu(e)] \subset \mu_c(e)$, it follows that $(p, y) \in h[\mu^i(\omega^i, \mu^i)]$. This proves (a), and since h is 1–1, (b) follows directly.

4.13. *Remarks.* Statement (a) of the Uniqueness Theorem, which is summarized in the first diagram, states that (μ, M, g) is the competitive mechanism even at the individual agent level.

5. Comparison with the Efficiency Theorem

The section discusses the stronger assumptions of the Uniqueness Theorem in comparison with the Efficiency Theorem. There are several not quite equivalent statements of the Efficiency Theorem. The one stated and proved below is essentially due to Hurwicz [2] and Osana [9, Remark, p. 73].

5.1. Definitions. Let (μ, M, g) be an allocation mechanism on E^* and let $e^0 \in E$. Then μ is *locally threaded at* e^0 if there is a neighborhood U of e^0 in E and a continuous function f: $U \to M$ with $f(e) \in \mu(e)$ for each $e \in U$. An allocation mechanism (μ, M, g) is *interior on E* if for each $(\omega^i, a^i)_i \in E$ and each $y \in g(\mu[(\omega^i, a^i)_i])$, $\omega^i_j + y^i_j > 0$ for each i, j.

5.2. THE EFFICIENCY THEOREM. *Suppose that* (μ, M, g) *is an allocation mechanism on* $E^{\#}$ *which is*

(E.i) *nonwasteful*;

(E.ii) *interior on* E;

(E.iii) *informationally decentralized*; *and*

(E.iv) M *is a manifold*; *and*

(E.v) μ *is locally threaded at some* $e^0 \in E$.

Then $\dim M \geqslant \dim M_c$.

Proof. Let $(\omega^{0i}, \alpha^{0i})_i = e^0$ and define the set $E^0 = \{(\omega^i, \alpha^i)_i \in E\colon \omega^i = \omega^{0i}$ and $\sum_j \alpha^i_j = \sum_j \alpha^{0i}_j$ for each $i\}$. Then one proves exactly as in the proof of Lemma 4.3 that for each e, $e' \in E^0$ and $m, m' \in M$ with $m \in \mu(e)$ and $m' \in \mu(e')$ we must have $m \neq m'$. Using assumption (v), let U be an open neighborhood of e^0 in E and let $f\colon U \to M$ be a continuous function with $f(e) \in \mu(e)$ for each $e \in U$. Let $U^0 = U \cap E^0$ and let f^0 denote the restriction of f to U^0. Then f^0 is 1–1 so $\dim M \geqslant \dim U^0 = K(L-1) = \dim M_c$ [1, Exercise 18.11, p. 82].

5.3. *Remarks.* It is clear from the proof of the Efficiency Theorem that the conclusion would remain unchanged if $E^{\#}$ were replaced by the smaller class E^*. Since the hypothesis is strictly weaker than that of the Uniqueness Theorem, it follows that assumption (U.iv) of the Uniqueness Theorem could be replaced by

(U.iv′) M is a connected manifold of dimension $\leqslant K(L-1)$, without affecting the conclusion.

5.4. *The Information Axioms.* Assumption (E.v), like assumptions (U.v) and (U.vi), is used to ensure that the dimension of the message space adequately measures the amount of information used by the allocation mechanism. For the Efficiency Theorem it is only necessary to rule out message processes which compress a $K(L-1)$ dimensional manifold, such as E^0, into a lower dimensional manifold in a 1–1 fashion. Since 1–1 dimension reducing functions are necessarily not continuous on any open set, (E.v) is all that is needed. For the Uniqueness Theorem, it is necessary to assume that M is connected (U.iv), that μ is single-valued and continuous on E (U.v) and that $\mu(E)$ is closed in M (U.vi). The need for these much stronger assumptions can be seen examining the informational requirements of lump-sum income transfers.

For simplicity, suppose that $K = L = 2$. For any real number r, define $M_r = \{(p, y) \in \Delta \times Y\colon py^1 = r\}$. Consider the allocation process obtained by associating with each environment the competitive equilibria subject to a transfer of r units of income from agent 2 to agent 1. Then M_r is the message space for this process and $\dim M_r = \dim M_c = 2$. However, if $r > 0$ and agent 2's endowment is sufficiently small, such allocations may fail to be noncoercive, and may even fail to exist. For this reason the transfer r will have to be adjusted in response to changes in agents' characteristics. Intuitively, these adjustments require information not needed by the competitive mechanism, and the purpose of the additional assumptions is to ensure that his information is reflected in the dimension of the message space.

To see that additional assumptions are needed, for each real number r, let $\mu_r^1: R_{++}^2 \times U^{*1} \twoheadrightarrow M_r$ be defined by $\mu_r^1(\omega^1, u^1) = \{(p, y) \in M_r: y^1$ maximizes $u^1(\omega^1 + z)$ subject to $pz \leqslant r$ and $u^1(\omega^1 + z) \geqslant u^1(\omega^1)\}$, and define $\mu_r^2: R_{++}^2 \times U^{*2} \twoheadrightarrow M_r$ by $\mu_r^2(\omega^2, u^2) = \{(p, y) \in M_r: y^2$ maximizes $u^2(\omega^2 + z)$ subject to $pz \leqslant -r$ and $u^2(\omega^2 + z) \geqslant u^2(\omega^2)\}$. Let $M = \bigcup_{r \in R} M_r$, topologized as a disjoint union,[5] and for each $r \in R$, define $\mu_r: E^* \twoheadrightarrow M_r$ by $\mu_r(e) = \mu_r^1(\omega^1, u^1) \cap \mu_r^2(\omega^2, u^2)$. For each i, let $\mu^i = \bigcup_{r \in R} \mu_r^i$, and define $\mu: E^* \twoheadrightarrow M$ by $\mu(e) = \bigcup_{r \in R} \mu_r(e) = \mu^1(\omega^1, u^1) \cap \mu^2(\omega^2, u^2)$. Let $g: M \to Y$ be defined by $g(p, y) = y$ for each $(p, y) \in M$. Then (μ, M, g) satisfies all of the assumptions of the Efficiency Theorem (to verify (E.v), note that $\mu_0 = \mu_c$, so μ_0 is a continuous selection from μ on E). Also, for each $e \in E^*$, $g[\mu(e)]$ is the entire contract curve. Thus, under the hypothesis of the Efficiency Theorem, the adjustments of lump sum income transfers needed to preserve noncoerciveness require no increase in the dimension of the message space. This mechanism and the mechanism defined in 4.6 above are the two main "contrary cases." We now show that slight modifications of the present example establish the need for each of the additional regularity assumptions of the Uniqueness Theorem.

5.5. *The Connectedness of M.* The allocation mechanism (μ, M, g) violates the connectedness of M (U.iv) and the single-valuedness of μ on E (U.v). If μ is redefined by

$$\begin{aligned} \mu'^i(\omega^i, u^i) &= \mu_0^i(\omega^i, u^i) && \text{if} \quad u^i \in U^i \\ &= \mu^i(\omega^i, u^i) && \text{otherwise,} \end{aligned}$$

for each i, and $\mu'(e) = \mu'^1(\omega^1, u^1) \cap \mu'^2(\omega^2, u^2)$ for each $e \in E^*$, then μ' restricted to E is equal to μ_c, so (U.v) is satisfied, and (U.vi) is also satisfied. However, $g[\mu'(e)]$ is the contract curve unless u^1 or u^2 is Cobb–Douglas. All hypotheses of the Uniqueness Theorem are satisfied except the connectedness of M.

[5] Definition 3.3 does not require manifolds to be second countable. If this property is added to the definition, an example in the same spirit is obtained by confining r to the integers.

5.6. *The Single-Valuedness of μ on E.* Suppose we require M to be connected and $\mu(E)$ to be closed in M but drop the single-valuedness of μ on E. Then let $M = \{(p_1, y_1^1) \in (0, 1) \times R\}$; define $\mu^1(\omega^1, u^1) = \{(p_1, y_1^1) \in M$: if $p_1 \in (0, 1/2)$, then y_1^1 maximizes $u^1[\omega^1 + (z, -2p_1 z/(1-2p_1))]$ for $z \in R$, and if $p_1 \in [1/2, 1)$, y_1^1 maximizes $u^1[\omega^1 + (z, -[(p_1 z - 1)/(1-p_1)])]\}$ for each $(\omega^1, u^1) \in R^2_{++} \times U^{*1}$; and define $\mu^2(\omega^2, u^2) = \{(p_1, y_1^1) \in M$: if $p_1 \in (0, 1/2)$, then y_1^1 maximizes $u^2(\omega^2 - (z, -2p_1 z/(1-2p_1))]$ for $z \in R$, and if $p_1 \in [1/2, 1)$, y_1^1 maximizes $u^2(\omega^2 - (z, -[(p_1 z - 1)/(1-p_1)])]$ subject to this utility level being not less than $u^2(\omega^2)\}$ for each $(\omega^2, u^2) \in R^2_{++} \times U^{*2}$. Define $\mu: E^* \twoheadrightarrow M$ by $\mu(e) = \mu^1(\omega^1, u^1) \cap \mu^2(\omega^2, u^2)$. Let $g: M \to Y$ be defined by

$$g(p_1, y_1^1) = (y_1^1, -2p_1 y_1^1/(1-2p_1); -y_1^1, 2p_1 y_1^1/(1-2p_1)) \quad \text{if} \quad p_1 \in (0, 1/2);$$
$$= (y_1^1, -(p_1 y_1^1 - 1)/(1-p_1); -y_1^1, (p_1 y_1^1 - 1)/(1-p_1)) \quad \text{if} \quad p_1 \in [1/2, 1).$$

Then (μ, M, g) satisfies all the hypotheses of the Uniqueness Theorem except (U.v), and $g[\mu(e)]$ contains a nonWalrasian allocation for many environments in E.

5.7. *The Closedness of $\mu(E)$ in M.* Suppose we require (U.v) but drop (U.vi). Note that the function $f: M_c \to M$ defined by $f(p, y) = (p_1/2, y_1^1)$ is a homeomorphism on M_c to $\{(p_1, y_1^1) \in M: p_1 \in (0, 1/2)\}$. Then modify the definition of μ in 5.6 by

$$\mu'^i(\omega^i, u^i) = f \cdot \mu_c^i(\omega^i, u^i) \quad \text{if} \quad u^i \in U^i; \text{ and}$$
$$= \mu^i(\omega^i, u^i) \quad \text{otherwise};$$

and $\mu'(e) = \mu'^1(\omega^1, u^1) \cap \mu'^2(\omega^2, u^2)$. Then the restriction of μ' to E is equal to $f \cdot \mu_c$, so (U.v) is satisfied, but $\mu'(E) = f(M_c)$ is not closed in M. All other hypotheses of the Uniqueness Theorem are satisfied, but $g[\mu(e)]$ contains nonWalrasian equilibria for many non-Cobb–Douglas environments.

5.8. *The Continuity of μ on E.* Finally, suppose we require that M be connected, that $\mu(E)$ be closed in M, that μ be single-valued on E, but drop the requirement that μ is continuous on E. Let $E_a^2 = \{(\omega^2, u^2) \in R^L_{++} \times U^{*i}$: there is some $p_1 \in [1/2, 1)$ such that there is no $z \in R$ with $u^2[\omega^2 - (z, -(p_1 z - 1)/(1-p_1)] \geqslant u^2(\omega^2)\}$. Given μ^1 and μ^2 as defined in 5.6, define $\mu''^2: R^L_{++} \times U^{*i} \twoheadrightarrow M$ by

$$\mu''^2(\omega^2, u^2) = \{(p_1, y_1) \in \mu^2(\omega^2, u^2): p_1 \in (0, 1/2)\} \quad \text{if} \quad (\omega^2, u^2) \in E_a^2;$$
$$= \{(p_1, y_1) \in \mu^2(\omega^2, u^2): p_1 \in [1/2, 1)\} \quad \text{otherwise};$$

and define $\mu'': E^* \twoheadrightarrow M$ by $\mu''(e) = \mu^1(\omega^1, u^1) \cap \mu''^2(\omega^2, u^2)$. Then μ'' is single-valued but not continuous on E, and satisfies all other hypotheses of the Uniqueness Theorem, but $g \cdot \mu''(e)$ is non-Walrasian for many environments in E.

5.9. *Concepts of Informational Size.* Much of the literature on the Efficiency Theorem has been devoted to extending the comparison of informational size to topological spaces which are not manifolds. The following comparison was introduced by Walker [10]. Given topological spaces M and M^0, we will say that $M \geqslant_F M^0$ (M is Frechet as large as M^0) if M^0 can be embedded in M. Since the proof of the Uniqueness Theorem only requires that M be (homeomorphic to) a subset of $K(L-1)$ dimensional manifold, assumption (U.iv) can be replaced by "M is connected and $M_c \geqslant_F M$" without affecting the proof. For the Efficiency Theorem, if assumption (E.iv) is replaced by "M is a Hausdorff space," the conclusion can be replaced by the statement "$M \geqslant_F M_c$." The last step of the proof is replaced by a straightforward argument based on the fact that a continuous 1–1 function on a compact set to a Hausdorff space is an embedding. A more general comparison, which also generalizes that of Mount and Reiter [8, Definition 9, p. 174], is introduced by Osana [9, pp. 71–72]. We will not discuss the Mount and Reiter or Osana definitions here, since they cannot be applied to either theorem without considerable modification of the information axioms.

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