OPTIMUM LOCATION IN SPATIAL COMPETITION

A. SMITHIES
University of Michigan

I. INTRODUCTORY

THE purpose of this paper is to take some further steps in the direction of generalizing the theory of spatial competition. The very fact that Professor Harold Hotelling's pioneer article[1] explained so successfully the close similarity of the Republican and Democratic platforms in 1928 indicates that something more was needed in 1936. It was probably true to say in 1928 that by moving to the center of electoral opinion neither party risked losing its peripheral support. The situation at the present time requires no elaboration; suffice it to say that neither party feels itself free to compete with the other for the undecided vote at the center, in full confidence that it will retain its support from the extremes of political opinion. Leaving the political analogy, Hotelling's assumption of completely inelastic demand means that neither competitor makes sacrifices at the ends of the market when he invades his rival's territory; thus there is no check on the two competitors' moving together. Actually, elastic demands do impose such a check and do account for the fact that equilibrium is frequently established, with the competitors free to move but spatially separated. I do not dispute Hotelling's conclusion that there is a tendency for two competitors to cluster nearer to the center than to the quartiles of a linear market; I suggest, however, that it is important to analyze not only the forces that tend to bring them together but also those that keep them apart. An important step in this direction was made by A. P. Lerner and H. W. Singer,[2] who modified Hotelling's assumption of complete inelasticity by postulating that demand was inelastic over a price range

[1] "Stability in Competition," *Economic Journal*, March, 1929, p. 41.

[2] "Some Notes on Duopoly and Spatial Competition," *Journal of Political Economy*, April, 1937, p. 145.

extending from zero to a finite upper limit. However, it seems desirable to go further and assume an elastic demand function at every point of the market. This I shall do in this paper, to the modest extent of assuming an identical linear demand function at every point of a linear market.

Hotelling and Lerner and Singer have confined themselves substantially[3] to the extreme competitive assumption that each competitor fixes his price and location, assuming that the price and location of his rival remain unaffected by his action. In this paper I shall consider, in addition, cases where each competitor makes his adjustments expecting reactions from his rival.

Finally, a considerable part of this paper will be concerned with the effects of the magnitude of the freight rates[4] and of changes in marginal costs for one or both producers.

The analysis of these problems can be carried out rigorously only by mathematical methods. Although the methods are elementary, their application is complicated.[5] For this reason and also for the reason that the mathematics do not bring out clearly the economic principles involved, I have attempted to present the whole argument in purely verbal form and to indicate in an appendix the general mathematical methods.[6]

Considerations of space suggest that the discussion should be

[3] Lerner and Singer do modify this assumption in cases where one competitor attempts to cut out his rival entirely. I shall deal with this point later.

[4] Hotelling's simple result (*op. cit.*, p. 50), that profits depend directly on freight rates, is true only in the special-demand situation he has examined. Lerner and Singer's treatment of freight rates (*op. cit.*) depend on the setup of their particular problem, and no general principles are developed. I know of no treatment of the problem of changes in marginal costs.

[5] G. H. Orcutt of the University of Michigan has constructed a mechanical model for solving this problem with a greater degree of generality than is possible by analytic methods. The principle of the machine is to represent, for each competitor, price, quantity per unit distance, and distance by voltage drops along linear resistance wires. These resistance wires are included in an electric circuit such that the product of these three voltages, i.e., total profits, can be read off a voltmeter. The machine is operated by varying price and distance for each competitor, in accordance with the assumptions of the problem, until a simultaneous maximum is achieved.

[6] The need to return to Marshallian orthodoxy in this problem has been impressed on me equally by Lerner and Singer's geometry and by my own algebra.

confined to cases where the producers are free to shift their locations at will. However, some indication of the solution of the fixed-location problem[7] in its relation to freight rates will be given in footnotes.

II. ASSUMPTIONS

We have now to formulate two sets of assumptions—the structural assumptions which limit the problem as a whole and the assumptions as to the character of the competition.

The structural assumptions are as follows:

1. There is a linear market bounded at both ends.

2. At every point of the market there can be only one price, and there are identical linear demand functions relating price to quantity sold per unit of time at that point. Thus, the total amount sold at any point is supplied by the competitor charging the lower delivered price at that point.

3. There are two competitors, A and B, having single locations. We can, without loss of generality, assume that A is located to the left of B.

4. The competitors are subject to constant marginal costs. Except where we are considering the effects of (small) changes in the costs of one or both competitors, marginal costs for both competitors will be assumed equal and zero. Fixed costs will be ignored throughout.

5. There is a uniform freight rate per unit of distance for both competitors, which is independent of distance and of the price and quantity of the goods transported.

6. Each competitor will sell on an f.o.b. mill basis.[8] That is, he

[7] For a discussion of equilibrium in the fixed-location problem see Erich Schneider, "Bermerkungen zu einer Theorie der Raümwirtschaft," *Econometrica*, January 1935, p. 79. Schneider does not deal with the freight-rate problem.

[8] I am restricting this paper to f.o.b. mill selling entirely for reasons of space. The analysis of the case where each producer attempts to maximize his expected profits at every point of his sections of the market is very similar to the f.o.b. case. Whereas in the f.o.b. case delivered price, by definition, exceeds mill-price by the full amount of freight costs from the mill to the point of delivery, in this case, given the linear demand conditions we have assumed, delivered price will be equal to mill-price *plus* half the freight costs from the mill to the point of delivery. Thus, since in both cases we have a linear relation between mill-prices and delivered prices, the qualitative results of the argument will be the same.

will fix a mill-price to prevail at the point where he is located, and his delivered price will be computed by adding to the mill-price the freight cost from his mill to the point of delivery.

7. Each competitor is free to move his location instantaneously and without cost.[9]

8. Each competitor will attempt to fix his mill-price and his location so as to maximize his instantaneous rate of profits in respect of his total sales.

9. The relation of freight rates to demand conditions is such that, in all the cases under examination, there are sales at every point of the market.[10]

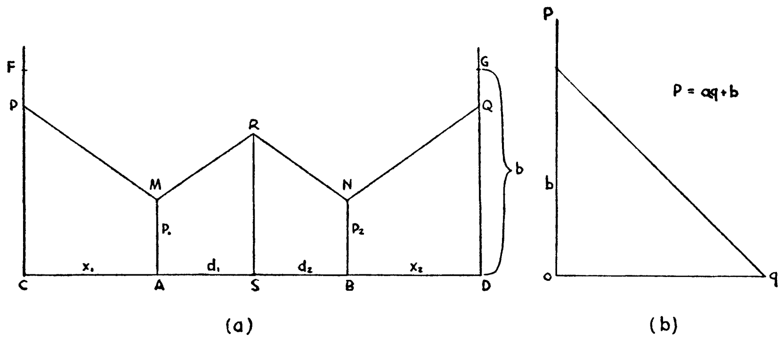

FIG. 1

The setup described by the above assumptions can be illustrated by Figure 1. *CD* represents the linear market of length l, and *A* and *B* the positions of producers A and B, respectively. *AM* and *BN* are their respective mill-prices, denoted by p_1 and p_2. The lines *MP* and *MR* show delivered prices for A, while *NR* and *NQ* show delivered prices for B. The slopes of these delivered-price

[9] Although these assumptions are quite unrealistic in many cases, I feel—and I take it that Lerner and Singer also feel—that they are useful for demonstrating clearly certain fundamental economic tendencies that have a wider application than merely to the problem at present in hand. I need only refer once more to Hotelling's brilliant analogies and to the use made by Chamberlin (*Theory of Monopolistic Competition* [Harvard University Press, 1933]) of the theory of spatial competition to illustrate the theory of product differentiation.

[10] This assumption avoids the necessity of considering the possibility of locational indeterminateness. This question has been adequately dealt with by Lerner and Singer (*op. cit.*, p. 150).

lines will depend on the freight rate. It is evident that A will not be able to sell to the right of S, or B to the left of it. The vertical lines CF and DG are drawn for reference, of height b—b being the price intercept of the demand curve—shown in Figure 1 (b), which is assumed to be of the form $p = aq + b$,[11] where p denotes the delivered price charged at any point in the market and q the quantity sold at that point, and $a < 0$, $b > 0$. The distances AC and BD of the competitors from their respective ends of the market are denoted by x_1 and x_2, respectively, while their distances from S are denoted by d_1 and d_2. It is worth emphasizing that x_1 and x_2 are within the complete control of the competitors, whereas d_1 and d_2 depend on the mutual interactions of their price policy and their location policy. We shall term the regions AC and BD the "hinterlands" of A and B, while AB will be called their "competitive region."

Our next problem is to consider the conjectural hypotheses made by the competitors as to each other's behavior. In contrast to the nonspatial problem of imperfect competition, each competitor's policy will depend on his estimate of his rival's reactions in respect both of price and of location. We shall be concerned with the following three cases.

1. Each competitor in making an adjustment assumes that his rival will set a price equal to his own and will adopt a location symmetrical with his own. By this is meant that A in fixing his location will assume that B will fix his location so as to make $BD = AC$ in Figure 1. Our analysis will show that if the competitors behave in this way the equilibrium position they will achieve will be the same as if they had acted jointly as a monopolist. Thus, for want of a better term, I shall describe this situation as "full quasi-co-operation."

2. Each competitor assumes that his rival will have the same price reactions as above but will keep his location unchanged. This situation may be termed "quasi-co-operative as to prices and competitive as to locations."

[11] Although in our present problem price is the independent variable, the demand function is written in this form for the sake of convenience in the mathematical analysis.

3. Lastly,[12] we have the case which is substantially that examined by Hotelling and Lerner and Singer, where each competitor assumed that both the price and the location of his rival will be fixed independently of his own. That is, there is competition both as to prices and as to location. This we may term "full competition." However, I shall not assume, as Lerner and Singer do originally,[13] that this assumption holds where one competitor adopts a price and location policy designed to cut his rival out of the market entirely; such credulity as this implies is fantastic—as Lerner and Singer realize. Nor shall I consider their amendment[14]—that a competitor adopts such a price and location, not with the hopes of cutting his rival out entirely, but with the strategic purpose of forcing him into a disadvantageous location. It seems to me that such action is virtually a declaration of economic war which is likely to be reciprocated and that the competitors will try to achieve *Lebensräume* satisfactory to both before resorting to policies of extermination. However, we shall see that in some conditions no such equilibrium is possible—and in those cases the possibilities of economic warfare must be considered.

The three cases proposed for examination are, of course, extreme, and in general the situation would be somewhere between quasi-co-operation and full competition. I have elsewhere[15] used generalized methods of formulating the problem so as to cover such intermediate cases, but in the present problem the complexity involved is too great to make that procedure worth while. Examination of the extreme cases indicates, at any rate, the qualitative results for the intermediate cases.[16]

[12] Considerations of symmetry would seem to suggest that we should consider also competition as to prices and quasi-co-operation as to location. However, since such a situation seems of little importance, it is omitted for the sake of brevity.

[13] *Op. cit.*, p. 151.

[14] *Ibid.*, p. 161.

[15] A. Smithies, "Equilibrium in Monopolistic Competition," *Quarterly Journal of Economics*, November, 1940, p. 95; and A. Smithies and L. J. Savage, "A Dynamic Problem in Duopoly," *Econometrica*, April, 1940, p. 130.

[16] The foregoing statement of the problem has ignored (for the sake of simplicity) the possibility that a competitor expects his rival to react to a price or location change by adjusting, respectively, his location or price.

III. VERBAL ARGUMENT

The attempt to solve our problem by purely verbal argument will involve little more than an appeal to informed common sense, but it seems worth while to make it in order to establish some general principles which will help to indicate the solution to problems that are too complex to be treated by rigorous methods. Our first problem is to determine the equilibrium position under our various competitive hypotheses. However, it seems pedagogically helpful to begin with a discussion of the (well-known) equilibrium position of a monopolist in the type of market and subject to the cost and f.o.b. selling conditions we have postulated for competitors. Second, we shall consider the dependence of equilibrium on the level of freight rates and, third, determine the effects of changes in marginal costs for one or both competitors.

1. CONDITIONS FOR EQUILIBRIUM

A. *Monopolist.*—A monopolist will be free from the asymmetry of having a hinterland on one side of him and a competitive region on the other. His markets on both sides of his mill will be equally exploitable. Assuming the monopolist has one plant, then, if he had no freight to consider, it would be indifferent to him where he located, and he would charge a price of $b/2$ at every point in the market in order to maximize his profits. Now, how will the existence of a freight rate affect him? It will mean, first, that he is unable to charge a uniform price at every point and, second, that he must decide how much of the freight to absorb himself and how much to "pass on" to consumers. Both of these exigencies will adversely affect his profits, so that his third problem is to locate himself at the point where their burden is minimized. It requires no argument to answer the third question; obviously, the monopolist will locate at the center of the market. Turning to the second question, the imposition of a freight rate is precisely analogous to the imposition of an excise tax per unit of commodity, uniform at every point of the market but, for any one point, linearly dependent on its distance from the mill of the producer. Now, for the cost and demand conditions assumed it is well known that a monopolist will absorb part of the tax himself and will pass some on to consumers in the form of a higher price. These considerations apply to the present case, and when the requirement of f.o.b. selling is introduced it can be adduced that at his mill (located at the center) he will charge a price which is less than $b/2$ by a determinate amount which is less than the cost of transporting a unit of commodity over half the length of the market, while at the ends of the market he will charge a price which is greater than $b/2$ by another such amount.

If a monopolist should have two plants instead of one,[17] the considerations of the foregoing paragraph make it evident that in order to maximize profits he will locate one plant at each quartile of the market, and in respect of each plant he will behave, in respect of its own half of the market, in the same way as the monopolist with the single plant behaved in respect of the whole market.

B. *Full quasi-co-operation.*—The essential difference between two competitors and a two-plant monopolist is that each competitor strives to occupy more than half the market, whereas the monopolist aims at maximizing profits in each half of the market. Successful invasion on the part of one competitor involves both enlarging his hinterland and occupying a greater fraction of the competitive region. The essential limitation on such incursions is that every move to add new territory is accompanied by less successful exploitation of the original hinterland (because of the higher freight charges involved).

Now, in this case of full quasi-co-operation we are, in effect, postulating that there are no profits to be gained from invasion—for each competitor assumes that any price he sets and any location he adopts will be identically met by his rival. Then, no matter what he does, neither competitor can expect to occupy more than half the market. Hence, he will not be prepared to make any sacrifices in his hinterland, and his efforts will be directed to exploiting half the market so as to maximize profits. It is evident that in this case the mutual actions of both competitors will re-

[17] I am not here concerned with the interesting theoretical problem of the optimum number of plants for a monopolist. I merely assume that the optimum number is two.

sult in an equilibrium position that is identical with that of the two-plant monopolist.

C. *Quasi-co-operation as to prices; competition as to locations.*—In this case each competitor believes to an equal degree that he has one effective strategy for increasing his territory—namely, moving closer to the center while expecting that his rival will not change his location but will meet price competition. We can thus expect (except in the limiting cases to be dealt with) that equilibrium will be achieved with each competitor at an equal distance from the center of the market and closer to it than the quartiles. Although each competitor will be disappointed in his hopes for territorial expansion, neither will retire toward the quartile position because he believes that any gains he may make in his hinterland will be more than offset by losses to his rival, whom he does not expect to retreat.

Although we have excluded competitive price-cutting from this case, the (equal) equilibrium mill-prices for the competitors will be lower than in Case B. This is due to the fact that average freight charges to the hinterland of each producer will be higher than if he were situated at his quartile, and in order to maximize his profits he will charge a price lower than $b/2$ by an amount greater than that in Case A.[18]

D. *Full competition.*—Here each producer thinks he can increase his territory both by moving toward the center and by price-cutting. And, what is more, he believes that these strategies

[18] In discussing the process of adjustment in this and the following case, I am implicitly assuming that the starting-point of each competitor is farther from the center of the market than his ultimate equilibrium position. This is not necessary; the equilibrium position is independent of the starting-points. Suppose (in Fig. 1) B is located as a monopolist at the center. Then A, assuming B will remain there, will locate at the optimum position between *C* and *B*. Producer B will then find it profitable, to some extent, to sacrifice some of the competitive region in order the more effectively to exploit his hinterland. This retirement of B will encourage further advance of A, and so on. The process of adjustment will involve successive advances by A and successive strategic retirements by B, until each reaches his equilibrium position. It is also evident from this example that if B is originally located at the center we have lost no generality by assuming that A locates to the left of B. Also, once A has located to the left of B, B will move to his right; and A will have no inducement to move to the right of B, since the greater segment of the market lies to the left of B, where A already is.

are not independent of each other in their effectiveness. Price-cutting increases the advantages of territorial advance and vice versa.[19] And, as before, he has to make hinterland sacrifices in respect of both strategies. Again equilibrium will be achieved, with equal prices and equal territories, closer to the center than to the quartiles. The prospects of gain from price-cutting and of loss from price-raising will result in a lower equilibrium price than in Case B, while the complementary relationship of price-cutting and locational advance will mean that the latter policy will be carried further than in Case C and equilibrium will be established closer to the center of the market.

In the light of our analysis we can now see the implications of Hotelling's assumption of zero elasticity of demand. This means that a producer by altering his position does not affect his position in the hinterland, since he can always pass on to the consumer his entire freight charges without affecting profits. Thus, there are no restraints to territorial advance, so that the competitors, if both are free to move, will inevitably both move to the center of the market.

2. DEPENDENCE ON FREIGHT RATES

The argument of the preceding paragraphs has clearly indicated that the relation of freight charges to demand conditions is of critical importance in the quantitative determination of equilibrium. In our present problem our special assumptions make it possible to say that the critical relation is the ratio of the cost of transporting a unit of commodity the whole length of the market to the price intercept of the demand curve, i.e., b. For brevity, I shall denote this ratio by s. Our present purpose, then, is to supplement the description of the four equilibrium positions we have determined by considering specifically their dependence on s.

Cases A and B (two-plant monopoly and full quasi-co-operation).—In these cases we have seen that the equilibrium location is uniquely determined—at the quartiles. The magnitude of s is therefore relevant only in so far as it affects the equilibrium price

[19] This proposition can be proved mathematically. It is to be noted that this complementary relationship also obtains in Case C, but since in that case price-cutting is not undertaken, the relation in that case will be inoperative in inducing the competitors to move nearer the center than they otherwise would.

and profits. We have already seen that if the freight rate approaches zero the equilibrium price will be $b/2$, and as freight rate increases it will be profitable for the producers to absorb some of the increase in the form of a lower mill-price. Hence, the greater the value of s, the smaller will be the equilibrium mill-price. Also, it follows from general principles that profits will decrease as s increases.

It remains to determine the conditions under which the whole market will be supplied. Clearly, s can have a value in excess of which it will be unprofitable for each producer to supply the outlying parts of his market. This critical value of s is determined by finding the value for s for which the delivered price both at the ends and at the center of the market is b; and sales at these points, consequently, are zero. A simple calculation shows that this is the case if $s = 8/3$, and at that point the equilibrium mill-price $p = p_1 = p_2 = b/3$.

Cases C and D.—In both these cases it follows from our previous argument that hinterland sacrifices will be greater and prospective gains from territorial expansion toward the center will be less, the greater the value of s. This suggests, first, that s may be sufficiently high to force the competitors to establish equilibrium at the quartiles and, second, that s may be sufficiently low for the hinterland deterrents to be inoperative, so that the competitors will both move to the center.

Such is the case. If we determine the conditions under which the competitors are charging a price b and selling zero quantity at the ends of the market, we find that they will be located at the quartiles and will also be charging price b and selling zero quantity at the center. The necessary and sufficient conditions are again $s = 8/3$, and again we have $p = b/3$. Hence we can say that in the cases examined, for $s = 8/3$, the equilibrium price and location of the competitors is independent of the nature of the competition.

Let us next consider the effect of low values of s. We have seen that the tendency to move toward the center of the market is stronger in Case D than in Case C, so that we should expect the minimum value of s necessary to keep the competitors apart to be greater in the latter case than in the former. Our calculations confirm this inference; we find that d_1 and d_2 in Figure 1 will be zero if $s = 4/7$ in Case C, and $8/11$ in Case D. The corresponding prices are $3b/7$ and $3b/11$, respectively. If s is less than $4/7$ in the one case and $8/11$ in the other, the competitors will still move to the center and remain there in their efforts to maximize profits, although a maximum in the mathematical sense will not have been attained.

Hotelling[20] recognizes the instability of this situation; and this implies that in his case the stability in competition depends on the difficulties of shifting location, which may be overcome in the long run. Equilibrium at the center would be stable only if one assumes that each competitor sells only in his own hinterland and does not attempt to invade the hinterland of his rival. I prefer to say that the forces of competition that eliminate the competitive region also destroy the inviolability of the hinterlands, and that once the competitors have come together they compete as duopolists in the entire market; and that the whole question must then be reopened and examined from the point of view of the theory of duopoly in a nonspatial market, which theory can be applied to the present case with but trivial modifications and upon which I shall not attempt to embark here.

Our next problem is the somewhat more complicated one of determining the general relations between s and the equilibrium price and profits. We have seen that forces of competition drive the competitors nearer to the center of the market than in the case of full quasi-co-operation, in futile endeavors to increase their territory; and the deterrent to these activities is the magnitude of s with which they are faced. In this case it is by no means clear that maximum profits are associated with the lowest possible freight rates or that, as freight rates increase, equilibrium mill-prices decrease. In fact, for Case D, we have already seen that where $s = 8/3$ the mill-price is greater than where $s = 8/11$. One is led to believe that there are values of s between these extremes which will maximize prices and profits, respectively. In other words, up to a certain level freight rates will serve to protect the competitors from their own self-destructive instincts. (This point may be readily apprehended by imagining the extreme case of an

[20] *Op. cit.*, p. 52.

insuperable wall erected at the center of the market. This would undoubtedly increase profits for both producers, since they would then be forced to act as monopolists.) Our previous argument has shown that in Case C the competitors need less protection against themselves than in Case D. This suggests that the optimum value of *s* is lower in Case C than in Case D. The effect of higher freight rates is to make the behavior of the competitors more monopolis-

TABLE 1

Situation	*s*	Prices	Profits	Location $x_1=x_2$
		b Multiplied by	b^2l Multiplied by	*l* Multiplied by
Case C:				
Equilibrium at the center	4/7 (0.57)	3/7 (0.43)	9/98 (0.092)	1/2 (0.50)
Maximum profits	0.72	0.44	0.094	0.43
Maximum price	0.72	0.44	0.094	0.43
....	2.00	0.37	0.072	0.27
Equilibrium at the quartiles	8/3 (2.67)	1/3 (0.33)	1/18 (0.056)	1/4 (0.25)
Case D:				
Equilibrium at the center	8/11 (0.73)	3/11 (0.26)	9/121 (0.074)	1/2 (0.50)
Maximum profits	1.00	0.33	0.084	0.39
Maximum price	1.70	0.36	0.077	0.29
....	2.00	0.36	0.070	0.27
Equilibrium at the quartiles	8/3 (2.67)	1/3 (0.33)	1/18 (0.056)	1/4 (0.25)

tic; this has the effect initially of increasing both prices and profits. But in the cases under examination the rise of prices under the influence of increasing freight rates will persist longer than the rise of profits—appreciably longer in Case D and inappreciably longer in Case C. Eventually, however, the competitors will find it profitable to absorb part of the increase of freight rates by charging lower mill-prices.

The results of this argument can now be summarized by giving the results of the numerical calculations in Table 1.[21]

[21] It is worth recording that the analysis of the section applies also to the fixed-location problem dealt with by Hotelling in the first part of his paper (pp. 45–47), where locations are fixed and each competitor assumes his rival will keep his price

3. CHANGES IN MARGINAL COSTS

In this section we shall consider the effects on the equilibrium situation of (*a*) a small change in marginal costs equal for both producers (i.e., we shall consider the effects of marginal costs rising above the zero level that we have hitherto assumed) and (*b*) a small increase in the level of marginal costs for one producer alone. We shall also, as before, consider the case of the monopolist for the sake of comparison.

a) This case offers no difficulty; the general reasoning from nonspatial markets indicates that in all the cases examined, including the monopolist, it will be profitable to pass on part, but only part, of the increased costs to the consumer in the shape of higher prices. In Cases A and B, where equilibrium is established at the quartiles, this rise of price will have no effect on location. In Cases C and D, where the equilibrium positions of the competitors are, in general, closer to the center than to the quartiles, they will find it profitable to move back toward the quartiles in order to reduce the incidence of higher prices on sales in their hinterlands, which constitute the greater part of their respective markets. In fact, if marginal costs rise sufficiently we shall have a case analogous to that examined in the last section, where the

---

unchanged. If the freight rate is zero, it is obvious that the competitors will cut prices to zero on these assumptions. Also, if we assume that the fixed locations are at the quartiles, the situation where the whole market is only just being supplied will be identical with the cases already examined; namely, we shall have $s = 8/3$ and $p = b/3$. In the same way as before, freight rates up to a certain level will protect competitors against the destructive effects of price competition; up to that level a rise of freight rates will increase profits, while prices will continue to rise somewhat longer. Table 2, analogous to Table 1, summarizes the numerical results.

TABLE 2

SITUATION	s	PRICES: b Multiplied by	PROFITS: b^2l Multiplied by
Zero freight rates	0.00	0.00	0.00
Maximum profits	0.50	0.375	0.110
Maximum prices	1.00	0.40	0.095
Zero quantities at the ends and center of the market	8/3 (2.66)	1/3 (0.33)	1/18 (0.056)

competitors will not move from the quartiles, no matter what competitive assumptions are made.

b) This case is more complicated, and we must consider the various cases separately.

Case A: If the costs rise for one plant of a monopolist, it will be profitable for him to raise the price charged in that plant, but it will also be profitable for him to use his low-cost plant to supply more than half the market. Further, as we have seen, it is profitable for him to locate each plant at the center of the part of the market that it supplies. Hence, the monopolist will move his high-cost plant nearer to its end of the market than to the quartile and his low-cost plant nearer the center. This readjustment will also involve reducing the mill-price charged by the latter plant.

Case B: The case of full quasi-co-operation is now somewhat different from that of the monopolist. For the small rise of marginal costs for one competitor is assumed not to alter either producer's expectation that he will continue to supply half the market. The competitors will continue to locate at the quartiles. Producer A, whose marginal costs have risen, will raise his price in order to charge the monopoly price appropriate to his new cost situation for half the market, while producer B, whose costs have not changed, will continue to charge his old price. Although A will be disappointed with the results and B will be pleasantly surprised, we are still entitled to regard the situation as one of equilibrium. However, this equilibrium will contain the germs of instability. Depending on the size of the rise of A's marginal costs, B's joy and A's consternation will tend to make them revise their assumptions in the direction of B's expecting A to charge a higher price than his own and vice versa. Our analysis must, therefore, be confined strictly to small changes of marginal costs.

Cases C and D: In both these cases A, whose marginal costs have risen, will charge a higher mill-price and move back toward his quartile. A's retreat will improve B's position in the competitive region. This opens up to B the possibilities of charging higher prices and of moving toward the center, except that the more he moves toward the center, the less profitable will it be for him to charge a higher price. In fact, it may be profitable for him to charge a lower price, depending on the extent of his move. What he actually does depends on the freight rate or, more accurately, on s. The smaller the value of s, the greater will be the tendency for him to move toward the center and charge lower prices, while for larger values of s he will move less toward the center and will raise his price. The critical values of s are approximately 0.65 in Case C and 1.70 in Case D.

This concludes the verbal argument. It should be pointed out that in this type of argument it is impossible to do justice to the essential character of the adjustment as one of mutual determination, and in this respect the arguments of this part sin more than once.

IV. MATHEMATICAL APPENDIX

In this appendix I shall merely indicate the general methods used to reach the conclusions arrived at by verbal argument in Part III.[22]

We shall make use of the following symbols. Geometrical references are to Figure 1. Let $l =$ the length of the market; r the freight rate per unit of distance; x_1 and x_2 the distances of A from D and B from C, respectively; d_1 and d_2 the distances from S of A and B, respectively; p_1 and p_2 their mill-prices; and c_1 and c_2 their marginal costs.

The variables within the control of the competitors are p_1 and x_1 for A and p_2 and x_2 for B. The quantities d_1 and d_2 are dependent variables and may be expressed in terms of the four independent variables. Then letting π_1 and π_2 be the profits of A and B, respectively, we may write

$$\pi_1 = \pi_1 (p_1, x_1, p_2, x_2, r, c_1, l) ,$$
$$\pi_2 = \pi_2 (p_2, x_2, p_1, x_1, r, c_2, l) .$$

The producers aim at maximizing not their actual profits but their expected profits. Expected profits for A may be obtained from these functions by substituting for p_2 and x_2 the values that A

[22] A detailed presentation of the mathematical argument is available in mimeographed form on application to the author.

expects these variables to take on as the result of his own action. Thus, in Case A he will expect $p_1 = p_2$ and $x_1 = x_2$, while in Case B he will expect $p_1 = p_2$ and x_2 to remain unchanged by his action. In Case C he will expect both p_2 and x_2 to remain unaffected by his action.

Now, for A and B simultaneously to maximize their expected profits, the following conditions are necessary:

$$\frac{\partial \pi_1}{\partial x_1} = \frac{\partial \pi_2}{\partial p_1} = \frac{\partial \pi_2}{\partial x_2} = \frac{\partial \pi_2}{\partial p_2} = 0 \, .$$

In order to obtain manageable solutions to these four equations, it was necessary to assume $c_1 = c_2$ and to assume a linear demand function at every point of the market. The solutions then express the optimum values of p_1, p_2, x_1, x_2; π_1 and π_2 as functions of r, l, and c.

By investigating the behavior of these functions with respect to r the results given in Table 1 can be obtained. The results given in Table 2 are obtained by imposing the restriction $x_1 = x_2 = l/4$.

The effects of a change in marginal costs for both producers are found by determining the effects of a small change in c, while the effects of a change in the marginal costs of one producer are found by taking as our starting-point $c_1 = c_2$ and determining the effects of a small change in c_1, c_2 remaining unchanged.