ASYMPTOTIC INDEPENDENCE OF BIVARIATE EXTREMES

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1. Introdnction and Summary. Suppose a random sample of size n is drawn from a bivariate continuous population with probability density function (p. d. f.) $f(x, y)$. Suppose ξ and η are the random variables in the distribution of the population. Let X and X' be the minimum and the maximum observations on ξ, and Y and Y' on η. Suppose ξ', ξ'', η' and η'', are the random variables corresponding to X, X', Y and Y', respectively.

Sibuya (1960) has given a necessary and sufficient condition for asymptotic independence of two extremes for a sample from bivariate population. We shall obtain such a condition for asymptotic independence of all the four extremes X, X', Y and Y'. It assumes a very simple form when $f(x,y)$ is symmetrical in x and y, and the marginal p. d. f. of x and y have the same form. Under these conditions on the p. d. f., a modification is possible in the condition given by Sibuya (1960) which reduces to one given by Watson (1954) for other purpose. It is further shown that extremes for samples from bivariate normal population satisfy our condition if $|\rho| < 1$, where ρ is the population correlation coefficient. Geffroy (1958) and Sibuya (1960) have proved a particular result for asymptotic independence of only two extremes X and Y in the normal case.

2. Preliminaries for the condition. We can show that

$$P(X < \xi' < \xi'' < X') = \{G(X, X')\}^n, \quad \dots \quad (1)$$

$$P(Y < \eta' < \eta'' < Y') = \{H(Y, Y')\}^n, \quad \dots \quad (2)$$

and

$$P(X < \xi' < \xi'' < X', Y < \eta' < \eta'' < Y') = \{P(X < \xi < X', Y < \eta < Y')\}^n, \quad \dots \quad (3)$$

where

$$G(X, X') = P(X < \xi < X'), \; H(Y, Y') = P(Y < \eta < Y'). \quad \dots \quad (4)$$

For proof of (3), one may refer to Mardia (1964).

Our investigation is based on the concept of dependence function. Its detailed behaviour is studied in Sibuya (1960). Following him, we write

$$P(x_1 < \xi < x_2, y_1 < \eta < y_2) = \Omega\{G(x_1,x_2),\ H(y_1,y_2),\ G(x_1,\infty),\ H(y_1,\infty)\}\ G(x_1,x_2)\ H(y_1,y_2), \quad \ldots \quad (5)$$

where Ω denotes dependence function. Now

$$\Omega\{G(x_1,x_2),\ H(y_1,y_2),\ G(x_1,\infty),\ H(y_1,\infty)\} = 1,$$

if and only if ξ and η are independent. Also Ω is a single valued function, since $G(x_1, x_2) = G_0$, $G(x_1, \infty) = G_0'$, specify x_1 and x_2 for continuous distribution, and $H(y_1,y_2) = H_0$, $H(y_1, \infty) = H_0'$ specify y_1 and y_2.

In accordance with (5), we may write

$$P(X<\xi'<\xi''<X',\ Y<\eta'<\eta''<Y') = \Omega^*\{G^*(X,X'),\ H^*(Y,Y'),\ G^*(X,\infty),\ H^*(Y,\infty)\}\ G^*(X,X')\ H^*(Y,Y') \quad \ldots \quad (6)$$

where from (1) and (2)

$$G^*(X,X') = \{G(X,X')\}^n,\quad H^*(Y,Y') = \{H(Y,Y')\}^n. \quad \ldots \quad (7)$$

Again

$$\Omega^*\{G^*(X,X'),\ H^*(Y,Y'),\ G^*(X,\infty),\ H^*(Y,\infty)\} = 1, \quad \ldots \quad (8)$$

if and only if (ξ', ξ'') are independent of (η', η'') i.e. (X,X') are independent of (Y,Y').

3. The Condition. We shall prove the following theorem :

Theorem 1. *A necessary and sufficient condition for asymptotic independence of* (X, X') *and* (Y, Y') *is that the convergence of* $P\{G(x_1,\infty),\ G(-\infty, x_2),\ H(y_1,\infty),\ H(-\infty, y_2)\}$ *is of such order as*

$$P(1-s,\ 1-s,\ 1-s,\ 1-s) = O(s), \quad \ldots \quad (9)$$

where we have put

$$P[\xi\ \varepsilon\{(-\infty, x_1),(x_2,\infty)\},\ \eta\ \varepsilon\{(-\infty,y_1),(y_2,\infty)\}] = P\{G(x_1,\infty),\ G(-\infty,x_2),\ H(y_1,\infty),\ H(-\infty,y_2)\}. \quad \ldots \quad (10)$$

PROOF. On substituting (6) and (5) in (3), and utilizing (7), we get

$$\Omega^*\{G^*(X, X'),\ H^*(Y, Y'),\ G^*(X, \infty),\ H^*(Y, \infty)\} = (\Omega\{[G^*(X,X')]^{1/n},\ [H^*(Y,Y')]^{1/n},\ [G^*(X,\infty)]^{1/n},\ [H^*(Y,\infty)]^{1/n}\})^n\ ;$$

since, it is an identity, we have

$$\Omega^*\{G(X, X'),\ H(Y, Y'),\ G(X, \infty),\ H(Y, \infty)\}$$
$$=(\Omega\{[G(X,X')]^{1/n},\ [H(Y,Y')]^{1/n},\ [G(X,\infty)]^{1/n},\ [H(Y,\infty)]^{1/n}\})^n. \quad \ldots \quad (11)$$

Now by (8) and (11), (X,X') and (Y,Y') are asymptotically independent if and only if

$$\lim_{n\to\infty} (\Omega\{[G(X,X')]^{1/n},\ [H(Y,Y')]^{1/n},\ [G(X,\infty)]^{1/n},\ [H(Y,\infty)]^{1/n}\})^n = 1. \quad \ldots \quad (12)$$

Now from (5) and (10), we find

$$\Omega\{G(x_1, x_2),\ H(y_1, y_2),\ G(x_1, \infty),\ H(y_1, \infty)\}$$
$$=\frac{G(x_1,x_2)+H(y_1,y_2)-1+P\{G(x_1,\infty),\ G(-\infty,x_2),\ H(y_1,\infty),\ H(-\infty,y_2)\}}{G(x_1,x_2)\ H(y_1,y_2)},$$

so that the left member of (12) becomes

$$\lim_{n\to\infty} \frac{\{G(X,X')\}^{1/n}+\{H(Y,Y')\}^{1/n}-1+P[\{G(X,\infty)\}^{1/n},\ \{G(-\infty,X')\}^{1/n},\{H(Y,\infty)\}^{1/n}\{H(-\infty,Y')\}^{1/n}]}{G(X,X')\ H(Y,Y')}$$
$$=\exp\{\lim_{n\to\infty} n\, P[\{G(X,\infty)\}^{1/n},\ \{G(-\infty,X')\}^{1/n},\{H(Y,\infty)\}^{1/n},\ \{H(-\infty,Y')\}^{1/n}]\},\quad \ldots \quad (13)$$

where we have utilized

$$\{G(X,X')\}^{1/n}=\exp\left\{\frac{1}{n}\log G(X,X')\right\}=1+\frac{1}{n}\log\{G(X,X')\}+O\left(\frac{1}{n^2}\right),$$

and $\ldots$ (14)

$$\{H(Y,Y')\}^{1/n}=\exp\left\{\frac{1}{n}\log H(Y,Y')\right\}=1+\frac{1}{n}\log\{H(Y,Y')\}+O\left(\frac{1}{n^2}\right). \quad \ldots \quad (15)$$

Therefore the equation (12) is equivalent to

$$P[\{G(X,\infty)\}^{1/n},\{G(-\infty,X')\}^{1/n},\ \{H(Y,\infty)\}^{1/n},\{H(-\infty,Y')\}^{1/n}]$$
$$=P\left[1-\frac{G(X,\infty)}{n}+O\left(\frac{1}{n^2}\right),\ 1-\frac{G(-\infty,X')}{n}+O\left(\frac{1}{n^2}\right),\right.$$
$$\left.1-\frac{H(Y,\infty)}{n}+O\left(\frac{1}{n^2}\right),\ 1-\frac{H(-\infty,Y')}{n}+O\left(\frac{1}{n^2}\right)\right]=o\left(\frac{1}{n}\right). \quad \ldots \quad (16)$$

where we have used (14) and (15) in (13) for the first equality while for the second, (12) and (13) are used. Since we have

$$P(l,l,l,l) < P(l_1,l_2,l_3,l_4) < P(m,m,m,m), \quad \ldots \quad (17)$$

where

$$l_1 = G(X, \infty),\ l_2 = G(-\infty, X'),\ l_3 = H(Y, \infty),\ l_4 = H(-\infty, Y'),$$

$$l = \text{maximum}\ \{G(X, \infty),\ G(-\infty, X'),\ H(Y, \infty),\ H(-\infty, Y')\},$$

and

$$m = \text{minimum}\ \{G(X, \infty),\ G(-\infty, X'),\ H(Y, \infty),\ H(-\infty, Y')\},$$

we find by (16) and (17) that (9) implies (16) and vice-versa.

Theorem 2. *The necessary and sufficient condition for asymptotic independence of the four extremes is given by* (9).

PROOF. We know asymptotic independence of X and X' from Gumbel (1958), section 3.2.7, as well as of Y and Y'. On using Theorem 1, the proof is complete.

Theorem 3. *If (i) $f(x,y)$ is symmetrical in x and y, (ii) the marginal p. d. f. of x and y have the same forms, and (iii) the variates x and y are unlimited, then the condition* (9) *can be replaced by the condition*

$$\lim_{l \to \infty} \frac{P(|x| > l,\ |y| > l)}{P(|x| > l)} = 0. \qquad \dots \quad (18)$$

PROOF. In this case, we have by symmetry $G(-\infty, -x_1) = G(-\infty, x_2)$, so that we may take $x_2 = -x_1 = l$ (say).

Since the marginal p d. f. of x and y have the same form, we have $y_2 = -y_1 = l$. Hence (10) becomes

$$\begin{aligned} &P(1-s, 1-s, 1-s, 1-s) \\ &\quad = P[\ \xi\ \varepsilon\{(-\infty, -l), (l, \infty)\},\ \eta\ \varepsilon\{(-\infty, -l), (l, \infty)\}] \\ &\quad = P(|x| > l, |y| > l). \end{aligned} \qquad \dots \quad (19)$$

Now $1 - s = G(x_1, \infty) = G(-\infty, x_2)$ so that $s \to 0$ if and only if $G(-\infty, l)$ and $G(-l, \infty) \to 1$ simultaneously, i. e., if and only if $P(|x| > l) \to 0$ as $l \to \infty$, because the variate is unlimited. Thus we can take

$$s = P(|x| > l), \qquad \dots \quad (20)$$

where $s \to 0$ should be read as $l \to \infty$.

On using (19) and (20) in (9), we obtain (18).

Remark. Under the above-mentioned conditions of Theorem 3, we can similarly replace the condition for asymptotic independence of maximas X' and Y', given in Sibuya (1960), by

$$\lim_{l\to\infty} \frac{P(x > l, y > l)}{P(x > l)} = 0. \qquad \ldots \quad (21)$$

For normal distribution, the above condition is also obtained and proved by Watson (1954) for extreme values in samples from m-independent stochastic process.

4. Independence of extremes in normal population. Suppose $h(x,y,\rho)$ is p. d. f. of the bivariate normal population with zero means, unit standard deviations and correlation coefficient ρ. Let $h(x)$ and $h(y)$ be the marginal p. d. f. of x and y.

Lemma 1. *For any function $V(x, y)$,*

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} V(x, y)\, dxdy = \int_0^{\infty}\int_0^{\infty} [V(x,y) + V(-x,y) + V(x, -y) + V(-x, -y)dxdy.$$

Theorem 4. *For a sample from a bivariate normal population with $|\rho| < 1$, X, X', Y and Y' are asymptotically independent.*

PROOF. Since the p. d. f. $h(x, y, \rho)$ satisfies the conditions of Theorem 3, we shall require to prove (21). Its numerator is

$$P(|x| > l, |y| > l) = 1 - P(|x| < l) - P(|y| < l) + P(|x| < l, |y| < l)$$

$$= 1 - \int_{-l}^{+l} h(x)\, dx - \int_{-l}^{+l} h(y) dy + \int_{-l}^{+l}\int_{-l}^{+l} h(x, y) dxdy$$

$$= 1 - 4\int_0^l h(x) dx + 2\int_0^l\int_0^l \{h(x,y,\rho) + h(x,y,-\rho)\} dxdy, \qquad \ldots \quad (22)$$

where in the last equation, we have utilized Lemma 1 and symmetry of $h(x,y,\rho)$. While denominator of (21) is

$$P(|x| > l) = 1 - 2\int_0^l h(x) dx. \qquad \ldots \quad (23)$$

Since as $l \to \infty$, (22) and (23) tend to zero, we may apply L'Hospital's rule to calculate the limit of (21). For this purpose, we obtain from (22) and (23)

$$\frac{d}{dl}\{ P(|x|>l, |y|>l)\} = -4h(l) + 2\int_0^l h(x, l, \rho)\, dx + 2\int_0^l h(x, l, -\rho) dx,$$

and

$$\frac{d}{dl}\{ P(|x|>l)\} = -2h(l).$$

Hence on the application of L'Hospital rule, we find

$$\lim_{l\to\infty} \frac{P(|x|>l, |y|>l)}{P(|x|>l)} = 2 - 2(2\pi)^{-1/2}\{\lim_{l\to\infty} g(l)\}, \quad \ldots \quad (24)$$

where

$$g(l) = (1-\rho^2)^{-1/2}\left[\int_0^l \exp\left\{-\frac{(x-\rho l)^2}{2(1-\rho^2)}\right\} dx + \int_0^l \exp\left\{-\frac{(x+\rho l)^2}{2(1-\rho^2)}\right\} dx\right]. \quad \ldots \quad (25)$$

On using suitable transformations in (25), we can show that

$$g(l) = \int_{-\rho l(1-\rho^2)^{-1/2}}^{l(1+\rho)^{1/2}} e^{-\frac{1}{2}z^2} dz + \int_{\rho l(1-\rho^2)^{-1/2}}^{l(1-\rho)^{1/2}} e^{-\frac{1}{2}z^2} dz,$$

and therefore

$$\lim_{l\to\infty} g(l) = (2\pi)^{1/2} + 0 \quad \text{if } \rho \geqslant 0 \text{ and } \rho \neq 1$$

$$= 0 + (2\pi)^{1/2} \quad \text{if } \rho \leqslant 0 \text{ and } \rho \neq -1,$$

that is

$$\lim_{l\to\infty} g(l) = (2\pi)^{1/2}, \text{ if } |\rho|<1. \quad \ldots \quad (26)$$

On using (26) in (24), we get the desired result.

Corollary 1. The ranges $R_1(=X'-X)$ and $R_2(=Y'-Y)$ of sample drawn from bivariate normal population are asymptotically independent for $|\rho|<1$.

In quality control, if R-charts are prepared for two measurable characteristics of same product then the range of one in general would be dependent on the other. Further each sample could be regarded as drawn from bivariate normal population for large sample size. However if n is large enough, then by the above theorem we should not worry about their dependence.

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