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Adding Risks: Samuelson's Fallacy of Large Numbers Revisited

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Abstract

Samuelson called accepting a sequence of independent positive mean bets that are individually unacceptable a fallacy of large numbers, and subsequent researchers have extended Samuelson's condition on utility functions to assure that they would not allow this fallacy. By contrast, some behavioralists, arguing the merits of diversification, believe that it is simply wrong headed to refuse a long series of independent "good" bets out of a misguided faith in expected utility theory. Contrary to what one might infer from the literature, this paper shows that accepting sequences of good bets is both consistent with expected utility theory and quite usual.

I. Introduction

At the root of a contentious literature on economic behavior in the face of repeated random choices is the observation that maximizing the geometric growth rate of a portfolio will, with probability one, asymptotically outperform any other significantly different choice. This result has led some to suggest that any other choice would be irrational. Samuelson (1971), however, argued that convergence in probability is too weak to support such a strong behavioral conclusion. Indeed, since maximizing the geometric growth rate is equivalent to maximizing the expected log of wealth, the position of its advocates is tantamount to judging all other utility functions as irrational choices. In addition, as the horizon for a problem of maximizing terminal expected utility grows infinite, the natural "attractors" or turnpikes for all optimal policies are the optimal policies for constant relative risk aversion utility functions of which the log is just one example (see Ross (1974a) and Huberman and Ross (1983)).

This class of issues is a variant of an earlier theme in Samuelson's work on what he termed the fallacy of large numbers (see Samuelson (1963)). Samuelson offered a colleague a better than fair bet—a 50-50 chance of winning $200 or

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losing \$100. I use the adjective "good" to describe bets such as these with positive means. When the colleague rejected the bet, but said that he would be willing to accept a string of 100 such good bets, Samuelson demonstrated that if one bet were rejected and if that bet were rejected at all wealth levels, then any sequence of n such bets should also be rejected. The colleague's intuition was based on a loose application of the law of large numbers; he apparently felt that if the bet were repeated sufficiently often the probability of winning would converge to one and be a sure thing. Samuelson pointed out that the law of large numbers applies to *averages* and not to sums. As the gamble is repeated, the distribution of potential outcomes spreads—the variance goes as n—while the expected return from repeating good gambles increases as n.

Unfortunately, Samuelson's correct but purposely limited analysis has led to a more general perception that eventually accepting a sequence of good bets when a single one would be rejected is somehow truly a "fallacy of large numbers" and suspect behavior from the perspective of expected utility theory. Pratt and Zeckhauser (1987) and Kimball (1993) defined a von Neumann-Morgenstern utility function as "proper" if the sum of two independent undesirable gambles would be inferior to either of the gambles individually. These authors found sufficient and separate necessary conditions on utility functions for them to be proper. Kimball (1993) extended these results.[1] By contrast, in a delegated monitoring problem, Diamond (1984) found sufficient conditions on utility functions for "... the 'fallacy of large numbers' to be correct, rather than a fallacy...."

Rather than disagreeing with the view that preserving an initial rejection is an attribute of expected utility theory, Benartzi and Thaler (1996) have displayed it as an undesirable aspect of the theory itself. First, they argue that Samuelson misinterpreted his colleague's irrationality. They agree that the colleague was irrational, but they emphasize what they call "myopic loss aversion," i.e., excessive sensitivity to short-term losses for good bets. Naturally, they also believe it is implausible that someone would turn down 100 plays of the bet with an expected value of \$5000 and, what is for the example above, a less than 0.005 chance of losing money. Intuitively, as the gamble is repeated the mean return grows and the probability of any particular loss diminishes. Why, then, would any sensible individual reject such repeated gambles? Presumably, such shrinking violets grow irrationally risk averse as their wealth increases.

Benartzi and Thaler and other students of what is coming to be called "behavioral finance" offer as a substitute for von Neumann-Morgenstern utility a form of prospect theory (developed by Kahneman and Tversky (1979)) in which the outcomes of prior lotteries "frame" current choices by providing a reference point against which losses and gains are assessed. Whether this is of significance for finance remains to be determined.[2]

Contrary to the view that accepting a sequence of good bets while rejecting a single gamble is not easily reconciled with expected utility theory, I argue

[1]See Ross (1998) for a dissenting analysis of proper utility functions.

[2]One thing is for sure, it is very hard. In effect, all portfolio problems now become path dependent in the sense that what you do depends on how you got there. This is a high price to pay and a road to travel down only if we are sure that we will like what we find at the end. In research, unlike life, however satisfying the journey, we do not always try to climb the highest mountain.

that such behavior is quite normal. Interestingly, in a later unpublished paper, Samuelson (1984) has also taken the opposite side of the traditional fallacy argument and constructed an example of a utility function that will eventually accept a long enough sequence of good bets. This paper is not the first to further analyze Samuelson's fallacy. Nielsen (1985) found necessary and sufficient conditions for a concave utility function to eventually accept a sequence of bounded i.i.d. good bets. Lippman and Mamer (1988) generalized Nielsen's sufficiency result to sequences of i.i.d. good bets that could be unbounded, and Hellwig (1995) generalized Samuelson's problem to comparisons of sums of i.i.d. bounded random variables. Nielsen and Lippman and Mamer's principal condition is that the left tail of the utility function not decline exponentially as wealth declines, and Hellwig showed that, for his problems, a similar condition must also apply as wealth increases.

This paper finds necessary and sufficient conditions for the eventual acceptance of sequences of good bets that are independent, but not necessarily bounded or identically distributed. The results hold for monotone functions that are bounded by an affine function above but need not be concave, and they confirm the importance of the exponential lower bound on utility. An interesting way to think about the lower utility bound is that it requires that the agent not be as sensitive as the exponential to bad outcomes.[3] Since proper utility functions are defined by reference to pairs of independent but not necessarily identically distributed random variables, my results are directly in contradiction to the spirit of the research on "proper" utility functions.

Section II introduces the central problem and reviews Samuelson's theorem. Section III derives necessary and sufficient conditions for a utility function to eventually accept a long enough sequence of good bets. Section IV extends the results of previous sections to multiplicative independent gambles such as occur over time. Section V briefly summarizes and concludes the paper with some speculation on its possible uses.

II. Samuelson's Result

Throughout this paper, I consider individuals whose choices are governed by taking the expectation of a von Neumann-Morgenstern utility function. Unless specified to the contrary, this function will always be assumed to be nondecreasing.

Samuelson's basic theorem provides a sufficient condition for an individual to reject n repetitions of an undesirable gamble.

Theorem 1. (Samuelson (1963)) If a utility function, U, rejects a lottery x at all wealth levels, then it will reject any arbitrary partial sum,

$$S(n) \quad \equiv \quad x_1 + \cdots + x_n,$$

where the x_j are i.i.d. replications of x.

[3]The author is grateful to Stephen Brown for this observation.

Proof. Since at all wealth levels, w,

$$E\{U(w+x)\} \quad < \quad U(w),$$

conditional on any outcome of the first $n-1$ such gambles, the nth gamble would be rejected,

$$E\{U(S(n))|S(n-1)\} \quad = \quad E\{U(S(n-1)+x)|S(n-1)\} \quad < \quad U(S(n-1)),$$

hence,

$$E\{U(S(n))\} \quad < \quad E\{U(S(n-1))\},$$

and continuing,

$$E\{U(S(n))\} \quad < \quad U(0),$$

i.e., $S(n)$ is rejected for all n. □

Samuelson's basic theorem is appealing in its simplicity but, unfortunately, it does not tell very much about individual preferences. It is a result that applies jointly to specific pairs of utility functions and gambles, and one is left without a general sense of what there is about a utility function that will cause it to reject repeated gambles if it rejects one such gamble. One of the aims of this paper is to provide such general results.

The first step in exploring the classes of utility functions that preserve rejection or acceptance is to relate them back to Samuelson's criteria. Which utility functions have the property that if they reject a gamble at one wealth level they will reject it at all wealth levels? Unfortunately, as is well known in related contexts, the answer is disappointingly limited.

Theorem 2. The only utility functions that reject the same gambles at all wealth levels are the risk-neutral function and the exponential, i.e., the constant coefficient of absolute risk aversion utility functions,

$$U(x) \quad = \quad x, -e^{-Ax}.$$

Proof. See Appendix.

Theorem 2 confirms that if one tries to apply the strict Samuelson criterion, the linear and the exponential are the only utility functions that have the property that, if they reject a gamble at one wealth level, they will reject it at all levels. But, while these conditions are sufficient for rejecting a sequence any of whose members would be rejected, they are far from necessary and are overly strong. In the next section, Samuelson's problem will be studied in detail and, contrary to what might be thought from Theorems 1 and 2, I will show that there is a large and not at all unusual class of utility functions that accepts a sequence of gambles, any one of which considered individually would be rejected.

III. Eventual Acceptance Property (EAP)

I begin with a definition of the essential property being studied. This definition is more general than that originally posed by Samuelson and others (see, e.g., Nielsen (1985), Lippman and Mamer (1988), and Hellwig (1995)) in that it permits gambles to differ.

Definition 1. Eventual Acceptance Property (EAP) Let $x_1, x_2, \ldots,$ be a sequence of independent good random variables with means μ_i satisfying the following conditions,

$$\mu \quad = \quad \inf \mu_i \quad > \quad 0.$$

A utility function has the Eventual Acceptance Property (EAP) iff for each such sequence there exists a finite n such that the partial sum $S(n)$ is accepted.

If a utility function has the EAP property, then it will eventually accept any sequence of good gambles even if each were rejected on its own. Notice that the EAP only requires eventual acceptance and not acceptance for all n larger than some specified n^*, which would, in general, depend on the specific sequence as well as the utility function. Notice, too, that the EAP is an additive property. Section IV extends the results to multiplied repeated gambles of the sort that arise in intertemporal portfolio theory.

The EAP is the property with which Samuelson was most concerned but, unfortunately, completely general necessary and sufficient conditions on utility functions for the EAP to hold are difficult to obtain. The problem is difficult because it turns on the behavior of random walks in the tails, and the usual sorts of local differential tricks employed to characterize utility functions in terms of coefficients of risk aversion are of little use. As a consequence, some additional properties will have to be imposed on the sequences, e.g., the variances might be required to be uniformly bounded, and the EAP will then be defined to hold for sequences satisfying the restriction.

The first result is a sufficient condition on a utility function for it to have the EAP. Later, a more general result will be obtained (Theorem 5), but the next theorem is of some independent interest because it generalizes a result in Samuelson (1984). Samuelson (1984) provided an example of a utility function that was bilinear with a single kink—a Domar-Musgrave utility function—and showed that the EAP held for the example. He then conjectured that the result would be true in general. The following theorem verifies this conjecture.

Theorem 3. A sufficient condition for a concave utility function, U, to have the EAP for sequences with uniformly bounded individual variances is that U be unbounded from above and that U' be bounded from above, i.e., as $x \downarrow -\infty$,

$$U' \uparrow m \quad < \quad \infty.$$

Proof. See Appendix.

Intuitively, as independent good gambles are added, the probability mass shifts to the right and also spreads. If the left tail of the utility function does not decline faster than the linear, then the law of large numbers will bound the negative contribution to expected utility from losses. Hence, if the utility function is unbounded from above, expected utility will be unbounded and the sequence eventually will be accepted. While it is nearly a special case of the above analysis, it is also worth separately considering the case where the utility function is no longer concave but, rather, is bounded from above and from below. Strictly speaking, to prevent extended versions of the St. Petersburg Paradox, this must be the case, although it is unclear what the negative domain means since presumably this entails losses that the individual cannot cover (see Ross (1974b) on the relation of this observation to bankruptcy). Conversely, presumably someone on the other side of these bets is making promises in the right-hand domain that they cannot keep. I will be able to finesse these issues when I interpret my results for multiplicative gambles in Section IV.

Theorem 4. If the utility function is bounded below and non-satiated above, then it has the EAP with respect to sequences with uniformly bounded individual variances.

Proof. See Appendix.

Theorems 3 and 4 expand understanding of which preferences have the property to eventually accept repeated good gambles, but I have not shown that they are necessary and, until I do so, it is difficult to get a more complete understanding of the EAP. In both theorems, what matters is the rather delicate behavior of the partial sums in the tails. Although the probability mass is shifting upward, it is also spreading and, particularly in the lower tail of the distribution of the partial sum, $S(n)$, if the utility function is approaching minus infinity fast enough, then the contribution of the negative portion of the expected utility integral may be unbounded from below. Theorem 3 shows that, in general, failure of the EAP requires that the marginal utility become unbounded as x declines. The following example illustrates the tension between the rapidly declining utility function in the left tail and the shifting mass of the distribution.

Example. Assume that each member of the sequence, $x_j \sim N(\mu, \sigma^2)$. Let the utility function be exponential,

$$U(x) \quad = \quad -e^{-ax},$$

on $(-\infty, 0)$ and arbitrary above with a smooth pasting at the origin so as to preserve overall concavity. Extending the exponential to the entire line, the expectation is minus the (real) moment-generating function for the exponential,

$$-e^{-\mu a n + \frac{1}{2}\sigma^2 a^2 n}.$$

Since the exponential is bounded from above by zero, by the Chebyshev inequality, the contribution of the portion of the expectation on $[0, \infty)$ approaches zero and the above term is the limiting form of the negative integral on $(-\infty, 0)$. If $\mu a < \frac{1}{2}\sigma^2 a^2$, then as $n \uparrow \infty$, this term approaches $-\infty$ exponentially.

By contrast, since the expected value of $S(n)$ is μn, it is straightforward to show that the positive integral is bounded above by a linear term in n. Hence, for any good gamble, the expected utility becomes arbitrarily low for large n violating the EAP.

The example violated the EAP because the exponential utility function diverged too rapidly in the lower tail relative to the speed with which the distribution spread upward. In fact, this is a feature of the exponential and it is a watershed case for a broad class of distributions. To make this precise, assume that the distributions possess moment-generating functions somewhere on the line, i.e., distributions will be assumed to have a real moment-generating function for some $\theta > 0$,

$$\phi_i(\theta) \quad = \quad E\left[e^{-\theta x_i}\right] \quad < \quad \infty,$$

where x_i is the ith good bet. As a practical matter, the restriction is not terribly confining since nearly all distributions of interest will satisfy it. The following lemma develops the principal tool that justifies our attention to generating functions.

Lemma 1. Let $x_1, x_2, \ldots,$ be a sequence of independent random variables with means μ_i satisfying the following conditions,

$$\begin{aligned} \mu &= \inf \mu_i > -\infty, \\ \theta_- &= \liminf \theta_i > 0, \end{aligned}$$

where $$\theta_i = \sup\{\theta | \phi_i(\theta) < \infty\} > 0.$$

It follows that, for $a < \mu$ and a given s,

$$P\left[S_n \leq s + an\right] \leq e^{\theta s} e^{n[a\theta + \kappa_n(\theta)]},$$

where $$\begin{aligned} S_n &= x_1 + \cdots + x_n, \\ \kappa_n(\theta) &= \frac{1}{n} \sum \log \phi_j(\theta), \end{aligned}$$

and, for $\theta > 0$ and sufficiently small,

$$\sup\left[a\theta + \kappa_n(\theta)\right] < 0.$$

Proof. See Appendix.

The main result can now be proved.

Theorem 5. Under the conditions on the sequence of random variables given in Lemma 1, a sufficient condition for the EAP to hold is that the utility function, U, be non-satiated and have the property that

$$U(x) e^{\gamma x} \quad \rightarrow \quad 0,$$

as $x \rightarrow -\infty$ for all $\gamma > 0$. If the utility function is bounded above by a linear function, e.g., if it is concave, then this condition is necessary as well.

Proof. See Appendix.

Theorem 5 is quite general and subsumes the previous two results. Perhaps counterintuitively, even though the mass is shifting upward, a utility function with decreasing absolute risk aversion on the whole line will not satisfy the integrability condition of Theorem 5 since it will decrease at least as rapidly as an exponential in the lower tail. The EAP requires that a utility function be less risk averse than the exponential in the lower tail. Notice that the central issue for necessity is the relative speed with which the left tail declines since, for a concave utility function, the upper tail is bounded above by a linear term in n. A parametric example is presented in the Appendix, which develops the arguments of Theorem 5 in detail for the exponential distribution. My condition is a variant of those found in Nielsen (1985), Lippman and Mamer (1988), and Hellwig (1995), but my theorem is somewhat stronger than theirs.

The next section applies these results to multiplicative gambles of the sort that arise in intertemporal portfolio problems.

IV. Multiplicative Gambles and Minimum Wealth Constraints

In many applications the gambles are multiplicative rather than additive. Fortunately, a number of my previous results carry over by a simple transformation to relative wealth gambles. Instead of working with a sequence of gambles whose outcomes add or decrement current wealth, consider a sequence of the form,

$$w \prod x_i,$$

which multiplies current wealth.

If the utility function is given by $U(\cdot)$ and defined on R^+, then an additive function on the line by the log transform is defined,

$$G(z) \quad = \quad U(e^z),$$

and

$$E(U(w \prod x_i) \quad = \quad E(G(\log w + S(n))),$$

where

$$S(n) \quad = \quad \log x_1 + \cdots + \log x_n.$$

If I now define a good gamble as one with a positive expected growth rate,

$$\mu_i \quad = \quad E[\log\, x_i] \quad > \quad 0,$$

then I can apply my previous results to G with the exception of those that relied on concavity. The concavity of U is not sufficient to assure that G is concave. Differentiating twice,

$$G'' \quad = \quad U'e^z[1-R],$$

where R is the coefficient of relative risk aversion, $R = -wU''/U' = wA$.

It follows that G is concave iff the coefficient of relative risk aversion is not greater than one. Since concavity serves primarily to bound the utility function by a linear function on the positive interval, Theorems 3 and 4 and the sufficiency part of Theorem 5 apply to G. However, for $R < 1$, G is no longer concave and while it may fail the lower tail integrability condition of Theorem 5, it may increase fast enough to permit the EAP to hold. Because the integral would then diverge on both the negative and the positive orthant, necessary as well as sufficient conditions for the EAP to hold for general functions are difficult to obtain, but weak sufficient conditions that will cover most useful cases are easy to obtain as extensions of my previous results.

Theorem 6. Assume that the conditions of Lemma 1 hold for the sequence of gambles, $\{\log x_I\}$ and that $G(z)$ satisfies the integrability condition of Theorem 5. The EAP holds for such multiplicative gambles if

$$\limsup_{w \to 0} R(w) \quad < \quad 1,$$

and U is non-satiated.

Proof. The condition implies that there exists δ and r, $0 < \delta, r < 1$, such that, for $w \in (0, \delta)$, $R(w) < r$. Integrating this inequality twice and scaling the utility function, I obtain for $w \in (0, \delta)$,

$$U(w) \quad \geq \quad cw^{1-r},$$

where $c > 0$ is a constant of integration. Hence,

$$G(z) \quad = \quad U(e^z) \quad \geq \quad ce^{(1-r)z},$$

for $z \leq \log \delta$. Since $r < 1$, G satisfies the conditions of Theorem 5 and the EAP holds. □

Theorem 6 verifies that if $G(x)e^{\gamma x}$ is integrable over the negative line for $\gamma > 0$, and G is non-satiated, then the EAP holds. Converting this back into conditions on the original utility function U, $U(e^x)e^{\gamma x}$ must be integrable and U must be non-satiated.

As a rule of thumb, for multiplicative gambles, the constant relative risk aversion utility functions play the role that the exponential does for additive gambles. For instance, Theorem 2 generalizes immediately to this class when applied to multiplicative gambles. As a consequence, for any sequence of independent gambles, if each is rejected then the sequence will be rejected, violating the EAP.

In the special case of the class of constant relative risk aversion utility functions defined on $(0, \infty)$,

$$U(w) \quad = \quad \left[\frac{1}{(1-R)}\right] w^{1-R} \quad \text{for } R \geq 0, \text{ and } R \neq 1,$$

and $$U(w) \quad = \quad \log w \quad \text{for } R = 1.$$

Applying the exponential transform,

$$\begin{aligned} G(z) \quad &= \quad U(e^z) \\ &= \quad \left[\frac{1}{(1-R)}\right] e^{(1-R)z} \quad \text{for } R \geq 0, \text{ and } R \neq 1, \end{aligned}$$

and $$G(z) \quad = \quad z \quad \text{for } R = 1.$$

For $R \leq 1$, the conditions of Theorem 6 are satisfied and the EAP holds, but it fails for $R > 1$. If one is willing to indulge in a discussion about the reasonable properties of utility functions, though, a case can be made that $R > 1$ is unreasonable for low values of wealth, which is precisely the region where the conditions for the EAP are violated.

As a simple example, with $R = 2$ a 50-50 gamble of size a will require an insurance premium of π, where

$$\frac{\pi}{w} = \left(\frac{a}{w}\right)^2.$$

To put this in perspective, imagine an individual with a total wealth of \$25,000 facing a 50-50 gamble of losing 90% of his wealth, \$22,500, or winning \$22,500. The premium to insure against this bet is 81% of total wealth, or \$20,250. In other words, this individual is so risk averse that, despite having a current wealth level of only \$25,000, he would pay up fully \$20,250 of it, leaving him with \$4,750 for sure to avoid having a 50% chance of having \$2,500 and a 50% chance of having \$47,500. Such behavior is certainly possible and one can imagine unusual circumstances where it would occur, but it does not seem to be a reasonable description of the behavior of economic everyman to pay 80% of his wealth to avoid a 50% chance of losing 90% and pass up a 50% chance of nearly doubling.

Whatever the worth of such speculations about what is reasonable behavior, at a minimum they have the plausible implication of the eventual acceptance of a long enough string of independent proportional good bets. To put the matter another way, if the utility function is not too risk averse, i.e., $R \leq 1$, then diversification over time will assure that a string of independent proportional good bets eventually will be accepted.

V. Conclusion

I have argued that accepting a long enough sequence of independent good bets, even though the individual bets would be rejected, is not only consistent with expected utility theory, it is quite usual. Whatever the appeal of this line of reasoning, it directly contradicts the work of Pratt and Zeckhauser (1987) and Kimball (1993) who argue that two independent individually undesirable independent gambles are jointly undesirable. Contrary, too, to the assertions of Benartzi and Thaler (1996), expected utility theory is rich enough to encompass acceptance as well as rejection of sequences of good bets.

These matters are not solely questions about rational individual choice behavior. Samuelson's main point was that the law of large numbers applies to risks that are cut up among shareholders and not to sequences of gambles that are added. Risk is controlled and lessened by being cut up into smaller pieces among the shareholders of a large insurance or investment portfolio. Adding n identically distributed uncorrelated gambles raises the total variance as n, but cutting a given risk into n independent risks lowers the total variance as n. But what exactly is being "cut up" and how this is done is not so obvious. And the insurance company analogy is not particularly compelling. After all, when an insurance company or a "swaps shop" opens its doors, it attracts n independent risks, it does not cut up some larger existing risk. The presumption is that the race between a financial market that cuts up risks and a business that adds them is won by the market but, whether or not companies behave as though they have utility functions, this tension nevertheless puts a greater burden on understanding what is rational choice across repeated gambles.

As a final note, there is now a growing interest in risk control for both the internal management and the external regulation of financial firms, and there is an increasing use of measures such as value at risk (VAR). Whether or not firms behave as if they have utility functions, the choice of a VAR level is akin to the choice of a utility function with certain tail properties of the sort analyzed here.

Appendix

Proof of Theorem 2. For any local (compact) gamble,

$$E\{U(w+x)\} \quad \le \quad U(w) \quad \text{iff} \quad \mu U'(w) + \frac{1}{2}\sigma^2 U''(w) \quad \le \quad 0,$$

where μ is the mean of the gamble and σ is the standard deviation. As a consequence, acceptance or rejection is determined by the coefficient of absolute risk aversion,

$$A(w) \quad \equiv \quad -\frac{U''(w)}{U'(w)}.$$

It follows immediately that only the constant coefficient of absolute risk aversion utility functions, i.e., the exponential and the linear, have the same rejection regions at all levels of wealth. □

In what follows, the well-known Chebyshev inequality, which is repeated here for completeness, is employed.

Chebyshev's Inequality. Let x be a mean zero random variable and let $\epsilon > 0$. Denoting the probability of an event by the symbol $P(\cdot)$,

$$P(|x| \ge \epsilon) \quad \le \quad \frac{\sigma^2}{\epsilon^2},$$

where σ^2 is the variance of x.

Proof of Theorem 3. Let $S(n)$ denote the nth partial sum of a sequence of good gambles, x_j, each of which is distributed independently with mean $\mu_j > \mu > 0$, and variance $\sigma_j^2 < \sigma^2 < 0$. Let F_n denote the distribution function of the nth partial sum.

Without loss of generality, the utility function will be normed so that $U(0) = 0$. The criterion for acceptance then becomes

$$E\{U(S(n))\} \quad > \quad U(0) \quad = \quad 0.$$

To study this, the integral is broken into its component positive and negative parts,

$$E[U(S(n))] \quad = \quad \int_{(-\infty,\infty)} U(x)dF_n(x)$$

$$= \int_{(-\infty,0]} U(x)dF_n(x) + \int_{(0,\infty)} U(x)dF_n(x).$$

Since U is unbounded from above, for any $\delta > 0$, there exists w such that for all $x > w$, $U(w) > \delta$. Using the Chebyshev inequality,

$$\begin{aligned}
\int_{(0,\infty)} U(x)dF_n &\geq \int_{(w,\infty)} U(x)dF_n \\
&\geq U(w)P(S(n) \geq w) \\
&\geq \delta P(\mu n + \epsilon(n) \geq w) \\
&= \delta[1 - P(\epsilon(n) \leq w - \mu n)] \\
&\geq \delta\left[1 - \frac{n\sigma^2}{(\mu n - w)^2}\right] \\
&\rightarrow \delta,
\end{aligned}$$

where $n > w/\mu$. Hence, the positive integral is unbounded in n.

From concavity and the assumption that the marginal utility is bounded above, it follows that the utility function is bounded from below on $(-\infty, 0]$ by the linear function mx. This implies that the negative integral,

$$\begin{aligned}
\int_{(-\infty,0]} U(x)dF_n &\geq m\int_{(-\infty,0]} U(x)dF_n \\
&= mxF_n\,\big|_{-\infty}^{0} - m\int_{(-\infty,0]} F_n(x)dx.
\end{aligned}$$

Letting $\mu(n) > \mu$ denote the mean of the nth partial sum, as $x \downarrow -\infty$, the Chebyshev inequality implies that for any n,

$$0 \geq xF_n(x) \geq x\frac{n\sigma^2}{(\mu(n)n - x)^2} \rightarrow 0,$$

hence the limit in the first term is 0. Again employing the Chebyshev inequality, the integral,

$$\int_{(-\infty,0]} F_n(x)dx \leq \int_{(-\infty,0]} \frac{n\sigma^2}{(\mu(n)n - x)^2}dx \leq \frac{n\sigma^2}{\mu n} = \frac{\sigma^2}{\mu}.$$

Combining these results,

$$\int_{(-\infty,0]} U(x)dF_n \geq -m\frac{\sigma^2}{\mu}.$$

Choosing $\delta > m\sigma^2/\mu$ completes the proof. □

Proof of Theorem 4. The proof is a simple modification of the proof of Theorem 3. From the Chebyshev inequality, for any x,

$$F_n(x) \equiv P(S(n) \leq x) = P(\epsilon(n) \leq x - \mu(n)n) \leq \frac{n\sigma^2}{(\mu n - x)^2} \rightarrow 0,$$

as $n \uparrow \infty$.

If b is the lower bound of U, it follows that

$$\begin{aligned} E[U(S(n))] &= \int_{(-\infty,\infty)} U(x)dF_n \\ &= \int_{(-\infty,a)} U(x)dF_n + \int_{[a,\infty)} U(x)dF_n \\ &\geq U(a)[1 - F_n(a)] + bF_n(a) \\ &\to U(a), \end{aligned}$$

which, by non-satiation, exceeds $U(0)$ for "a" sufficiently large. □

Proof of Lemma 1. The proof is an adaptation of an argument in Durrett (1996). Since $\theta_- > 0$, there exists N such that for all $n > N$, $\phi_i(\theta) < \infty$. Let $\theta_N = \min(\theta_1, \ldots, \theta_N, \theta_-) > 0$. For $\theta \in (0, \theta_N)$, the exponential function is a positive and decreasing function and for any n,

$$\begin{aligned} (\Pi\phi_j)\, n &= E\left[e^{-\theta S_n}\right] \\ &\geq P[S_n \leq s + na]e^{-\theta s}e^{-na\theta}, \end{aligned}$$

or

$$P[S_n \leq s + an] \leq e^{s\theta}e^{n[a\theta+\kappa_n(\theta)]}.$$

Since $a < \mu$, it suffices to show that $\kappa_n(\theta) \leq -\mu$ as $\theta \to 0$. This can be shown rigorously by an application of the dominated convergence theorem and the monotone convergence theorem (see Durrett (1996)), but the argument only serves to verify the formal analysis that follows from observing that

$$\kappa_n(0^+) = (1/n)\sum \phi_j'(0)/\phi_j(0) = -(1/n)\sum \mu_j \leq -\mu. \;\square$$

Proof of Theorem 5. First I show sufficiency. Without loss of generality, I will set $U(0) = 0$. From non-satiation, the contribution to expected utility from the mass above zero will be positive. From the limit condition, for any $\gamma > 0$, there exists x_γ such that for all $x < x_\gamma$,

$$U(x) > -e^{-\gamma x}.$$

Hence,

$$\int_{(-\infty,0]} U(s)dF_n \geq -\int_{(-\infty,x_\gamma)} e^{-\gamma s}dF_n + \int_{[x_\gamma,0]} U(s)dF_n.$$

For an appropriate choice of $\gamma > 0$, it is shown that this converges to 0 as $n \to \infty$.

From Lemma 1, for any fixed s,

$$F_n(s) \leq e^{\theta s}e^{nk_n(\theta)},$$

where for $\theta > 0$ and sufficiently small, $k_n(\theta) < 0$. Hence, as $n \to \infty$,

$$F_n(s) \to 0,$$

assuring that the integral of $U(s)$ on $[x_\gamma, 0]$ converges to 0 as n grows large.

Integrating by parts,

$$-\int_{(-\infty,x_\gamma)} e^{-\gamma s} dF_n = -e^{-\gamma s} F_n(s) \mid_{\infty}^{x_\gamma} + \int_{(-\infty,x_\gamma)} e^{-\gamma s} F_n(s) ds.$$

Choosing $\gamma < \theta$, as $s \to -\infty$,

$$e^{-\gamma s} F_n(s) \leq e^{-\gamma s} e^{\theta s} e^{nk_n(\theta)} \to 0.$$

Hence, $$e^{-\gamma s} F_n(s) \big|_{-\infty}^{x_\gamma} = e^{-\gamma x_\gamma} F_n(x_\gamma) \to 0,$$

as $n \to \infty$. Similarly, as $n \to \infty$,

$$\begin{aligned} \int_{(-\infty,x_\gamma)} e^{-\gamma s} F_n(s) ds &\leq \int_{(-\infty,x_\gamma)} e^{-\gamma s} e^{\theta s} e^{nk_n(\theta)} ds \\ &= e^{nk_n(\theta)} \int_{(-\infty,x_\gamma)} e^{(\theta-\gamma)s} ds \\ &\to 0. \end{aligned}$$

To verify necessity, observe first that if U is satiated at some wealth level, w^*, then any bet with some mass below w^* will be rejected. Hence, for EAP, we must not have satiation. To conclude the necessity argument, consider a sequence of i.i.d. good bets that have $\mu > 0$ and that assign an atom of mass $e^{-\lambda z}$ at $-z$. For any n,

$$\int_{[-\infty,0]} U(s) dF_n(s) ds \leq e^{-\lambda nz} U(-nz).$$

Assume, to the contrary, that $U(s)e^{\gamma s}$ is not integrable on $[-\infty, 0)$ for some $\gamma > 0$. Letting $\lambda < \gamma$ implies that the above right-hand side bound $\to -\infty$ of exponential order, $e^{(\gamma-\lambda)zn}$ as $n \to \infty$. Since $U(s)$ is bounded above by a linear function, $a + bs$, the right-hand integral,

$$\int_{[0,\infty]} U(s) dF_n(s) ds \leq a + b\mu n,$$

which only grows linearly in n. It follows that choosing z large enough assures that the sequence of bets will be rejected for any n. □

Parametric Example Illustrating Theorem 5. Let $x_1, x_2, \ldots,$ be i.i.d., with $x_i = a + \xi_i$, where $a > 1$ and ξ_i is distributed as a negative exponential,

$$f_i(x) = e^x \quad \text{if } x \leq 0, \quad 0 \text{ if } x > 0.$$

Since $E\{x_i\} = a - 1 > 0$, these are good bets. The moment-generating function of ξ_i is given by

$$\phi(\theta) = \int_{[-\infty,0]} e^{-\theta x} e^x dx = \frac{1}{1-\theta},$$

which converges for $\theta < 1$. The moment-generating function for $\sum \xi_i$, is

$$\phi^n(\theta) = \frac{1}{(1-\theta)^n}.$$

Integrating the Laplace transform gives the resulting density function for $\sum \xi_i$,

$$f_n(x) = \frac{1}{(n-1)!}(-x)^{n-1}e^x, \quad x \leq 0.$$

Hence, for $s \leq 0$,

$$\begin{aligned} P[S_n \leq s] &= P\left[\mu n + \sum \xi_i \leq s\right] \\ &= P\left[\sum \xi_i \leq s - \mu n\right] \\ &= \frac{1}{(n-1)!}\int_{[-\infty, s-\mu n]} (-x)^{n-1}e^x dx \\ &= e^{(s-\mu n)}\left[1 + (\mu n - s) + \cdots + \frac{1}{(n-1)!}(\mu n - s)^{n-1}\right]. \end{aligned}$$

From the previous analysis,

$$P[S_n \leq s] = e^{\theta s}e^{n\kappa(\theta)},$$

where $\kappa(\theta) < 0$ for θ sufficiently small.

Notice that, as observed in the proof of Theorem 5, this is not sufficient for the exponential utility function to converge,

$$\begin{aligned} E[U(w)] &= \int_{[-\infty, -\mu n]} -e^{-(-\mu n + x)}\frac{1}{(n-1)!}(-x)^{n-1}e^x dx \\ &= -\frac{e^{-\mu n}}{(n-1)!}\int_{[-\infty, -\mu n]} (-x)^{n-1}dx. \end{aligned}$$

Interestingly, consider the function,

$$U(x) = -(-x)^{-m}e^{-x},$$

where m is a positive integer. In the region, $x < -m$, the function is monotone and concave. Patching it to preserve monotonicity and concavity above $-m - a$ for some $a > 0$, then the resulting utility function has the interesting property that it converges for $n < m+1$ and fails to converge for $n \geq m+1$. In other words, even though the distribution is shifting to the right, the mass in the left tail is growing sufficiently rapidly that, for more than $m + 1$ bets, the integral diverges.

The case of exponential divergence can be explored by amending the example to permit ξ_i to be distributed as an exponential with parameter $\lambda > 0$,

$$f(\xi) = \lambda e^{\lambda \xi}, \quad \xi \leq 0,$$

and where $\mu > 1/\lambda$ to assure that $E[x_i] = \mu - 1/\lambda > 0$.

The moment-generating function of the sequence,

$$\phi^n(\theta) = \left[\frac{\lambda}{\lambda - \theta}\right]^n,$$

which applies for $\lambda < \theta$, can be Laplace inverted to produce the distribution function for the sum

$$\xi(n) = \xi_1 + \cdots + \xi_n,$$

and, hence, for $s \leq \mu n$,

$$\begin{aligned} F_n(s) &= P[S_n \leq s] = P[\xi(n) \leq s - \mu n] \\ &= \int_{[-\infty, s-\mu n]} \frac{1}{(n-1)!}(-\lambda x)^{n-1} e^{\lambda x} dx, \end{aligned}$$

which implies that

$$f_n(s) = \frac{1}{(n-1)!}(-\lambda[s - \mu n])^{n-1} e^{\lambda[s-\mu n]}.$$

If $U(x)e^{\gamma x}$ is not integrable for some $\gamma > 0$, i.e., if $U(x) \to -\infty$ more rapidly than the exponential, then for $\lambda \leq \gamma$ expected utility will not converge.

This, in turn implies, that for $x > 0$,

$$\begin{aligned} \int_{[-x,0]} U(s) f_n(s) ds &= \frac{1}{(n-1)!} \int_{[-x,0]} U(s)(-\lambda[s - \mu n])^{n-1} e^{\lambda[s-\mu n]} ds \\ &\to -\infty \end{aligned}$$

of $O(x^n)$ as $x \to \infty$. □

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