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On the tumbling toast problem

Riccardo Borghi

Dipartimento di Elettronica Applicata, Università degli Studi 'Roma Tre', Via della vasca navale 84, I-00144 Rome, Italy

E-mail: borghi@uniroma3.it

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Abstract
A didactical revisitation of the so-called tumbling toast problem is presented here. The numerical solution of the related Newton's equations has been found in the space domain, without resorting to the complete time-based law of motion, with a considerable reduction of the mathematical complexity of the problem. This could allow the effect of the different physical mechanisms ruling the overall dynamics to be appreciated in a more transparent way, even by undergraduates. Moreover, the availability from the literature of experimental investigations carried out on tumbling toast allows us to propose different theoretical models of growing complexity in order to show the corresponding improvement of the agreement between theory and observation.

1. Introduction

The study of the so-called tumbling toast (TT) problem was first proposed in 1995 in a paper by Matthews [1], which raised some questions about the interaction between Newton's and Murphy's laws [2–5]. Six years after [1] appeared, Bacon *et al* [6] proposed an experimental investigation of the TT problem in which the reproducibility features were guaranteed by the use, in place of real bread-made toast, of a plywood board of comparable size. Moreover, the use of software packages aimed at facilitating the analysis of video recordings and at numerically solving complex (nonlinear) differential equations allowed the authors of [6] (i) to experimentally determine the free-fall angular velocity of the board tumbling off the edge of a table for different values of the overhang of the toast's centre of mass (CM henceforth) and (ii) to quantitatively compare their experiment with the time-domain numerical solution of Newton's equations. In particular, an important outcome of their investigation was that, in order to explain in a correct way the butter-side up or down behaviour of the TT, it is mandatory to take into account, in the theoretical model, the *slipping* of the toast surface with respect to the edge of the table before leaving it. To this end, a theoretical model of the TT, in which both the presence of static and kinetic frictions and the thickness of the board were taken into account, was used by numerically solving the corresponding Newton's equations in the time domain [6].

It is not so usual to find nontrivial problems of physics to allow undergraduate students to grasp how subsequent refinements of a theoretical model improve agreement with observation. Our opinion is that the TT problem, in the form presented in [6], could be one of them. In fact, on one hand, there is the availability of accurate experimental measurements, and on the other hand, there is the presence of several different physical mechanisms, all of which contribute to the complex overall dynamics of the tumbling. This, in turn, could allow models of growing complexity to be built up simply by sequentially including the effects of each of the above-mentioned phenomena.

This paper aims to give a didactical presentation of this. In doing so, we keep the analysis of the toast dynamics as simple as possible, in order to be grasped even by first-year undergraduates. Moreover, we also wish to emphasize how quite often a complete solution of Newton's equations in the time domain is not required, but rather the overall mathematical complexity (even of a nontrivial problem) can be considerably reduced by using some tricks that should become part of the 'box of tools' [7] of any physics student. We start with the analysis of the toast motion within the hypothesis that any kind of friction between the board surface and the table edge could be neglected. Such an assumption, although rather unrealistic, leads to a highly simplified theoretical model. At the same time, the extreme simplicity of the model serves to introduce in a transparent way to students the mathematical techniques employed to avoid resorting to the time-domain solution. We shall see that, despite the several approximations, such a model is nevertheless able to provide theoretical predictions that, although far from showing an acceptable quantitative agreement with the experimental results, should allow the student to grasp the key point in the TT dynamics, i.e. the slipping of the board over the table edge [6]. Equipped with the basic tools, such a simplified model will then be subsequently refined, by adding sequentially the static and the dynamic friction at the table edge, and showing from time to time the corresponding improvement of the agreement with the experimental results, up to the most complete model which also includes the nonzero thickness of the board. We believe that the approach pursued in this paper could also be used to analyse other similar physical problems, for instance that addressed by Bacon regarding balls rolling out from edges [8].

2. Theoretical analysis

2.1. The simplest model: dynamics in the absence of friction

We start our study with the simplest situation depicted in figure 1, in which the only forces acting on the board are the weight $M\boldsymbol{g}$ and the normal reaction at the table edge $\boldsymbol{N}$. The toast will be assumed to be an infinitely thin square of mass M and size L.

Newton's law for the CM of the board reads

$$M\boldsymbol{g} + \boldsymbol{N} = M\boldsymbol{a}_{\mathrm{CM}}, \tag{1}$$

where $\boldsymbol{a}_{\mathrm{CM}}$ denotes the CM acceleration. On introducing the polar coordinates of CM (r, φ), equation (1) splits into the system of two scalar equations

$$\begin{aligned} g\sin\varphi &= \ddot{r} - r\dot{\varphi}^2, \\ g\cos\varphi - n &= r\ddot{\varphi} + 2\dot{r}\dot{\varphi}, \end{aligned} \tag{2}$$

where dots denote temporal derivatives and $n = N/M$ denotes the modulus of the edge reaction normalized to the board mass[1]. The torque–angular momentum equation written with respect

[1] This is equivalent to set $M = 1$ in suitable units.

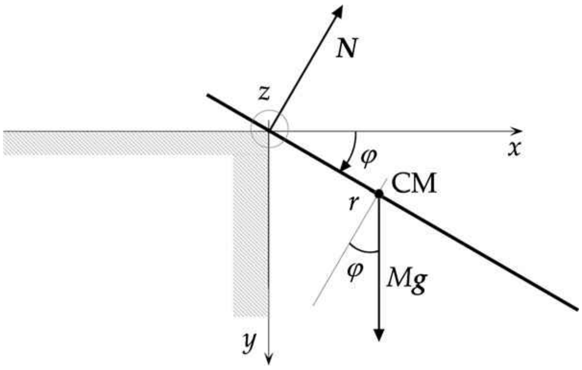

Figure 1. Geometry for the simplest model.

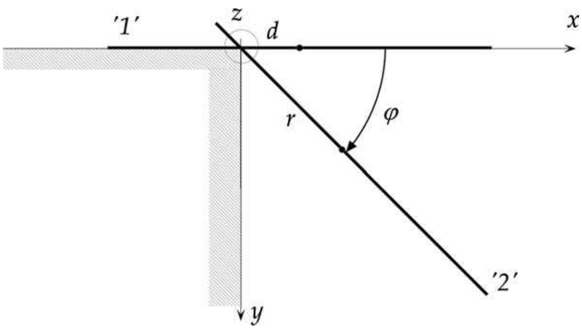

Figure 2. Energy conservation law.

to the axis passing through the CM and parallel to the table edge gives

$$N\,g\,r\cos\varphi \;=\; \dot{\mathcal{L}}_{\rm CM}. \tag{3}$$

Here, $\mathcal{L}_{\rm CM}$ denotes the z-component of the angular momentum with respect to the CM, which, in the limit of an infinitely thin board, can be written as

$$\mathcal{L}_{\rm CM} \;=\; M\,a^2\,\dot{\varphi}, \tag{4}$$

where $a = L/2\sqrt{3}$. On substituting equation (4) into equation (3), we have

$$n \;=\; \frac{a^2}{r}\,\ddot{\varphi}. \tag{5}$$

Equations (2) and (5) constitute a system with respect to the three unknowns of the problem, namely $r(t)$, $\varphi(t)$ and the modulus of the reaction $n(t)$, all of them thought of as functions of the time t. In particular, on imposing the initial conditions $r(0) = d$, $\dot{r}(0) = 0$, $\varphi(0) = 0$, and $\dot{\varphi}(0) = 0$, the above differential system can be solved in the time domain, thus providing the complete law of motion of the board. However, as mentioned in the introduction, for the scope of the present problem such information turns out to be somewhat redundant. In fact, we recall that the task is to estimate the angular velocity of the board after it left the table edge, as a function of the initial overhang d, as experimentally reported in table I of [6]. Within the present model, the total absence of friction suggests the use the energy conservation law that, with reference to figure 2, should be written between state '1' (corresponding to the board at rest in the initial horizontal position) and state '2', corresponding to a typical position (r, φ). On using König's theorem, we shall first write the total kinetic energy of the board as follows:

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}(a^2 + r^2)\dot{\varphi}^2, \tag{6}$$

so that, on assuming the initial total mechanical energy to be zero, the energy conservation law gives

$$2\,g\,r\,\sin\varphi = \dot{r}^2 + (a^2 + r^2)\,\dot{\varphi}^2. \tag{7}$$

For simplicity, it is worth rewriting all equations that constitutes the final system:

$$\begin{cases} g\,\sin\varphi = \ddot{r} - r\,\dot{\varphi}^2, \\ g\,\cos\varphi - n = r\,\ddot{\varphi} + 2\,\dot{r}\,\dot{\varphi}, \\ nr = a^2\,\ddot{\varphi}, \\ 2\,g\,r\,\sin\varphi = \dot{r}^2 + (a^2 + r^2)\,\dot{\varphi}^2. \end{cases} \tag{8}$$

Since n has to vanish when the board leaves the table, we have from the third equation that the condition $\ddot{\varphi} = 0$ must be fulfilled. To avoid solving the whole system (8) in the time domain, it is possible to use the trick employed by Sommerfeld to determine the shape of Keplerian orbits [9]. Loosely speaking, since the temporal information about the board motion is not required, what we have to do is to eliminate the variable t among the differential equations in (8) *before* solving them. From a mere technical viewpoint, the use of polar coordinates helps such an elimination, which can be easily achieved by using the 'chain rule' for derivative, i.e. by letting[2]

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\varphi} = \dot{\varphi}\,\frac{\mathrm{d}}{\mathrm{d}\varphi}, \tag{9}$$

from which it follows at once that

$$\begin{aligned} \dot{r} &= r'\,\dot{\varphi}, \\ \ddot{r} &= r'\,\ddot{\varphi} + r''\,\dot{\varphi}^2, \end{aligned} \tag{10}$$

and where now r is thought of as a function of φ, so that r' and r'' denotes the first derivative and the second derivative of r with respect to φ, respectively. In other terms, we are going to search for the CM trajectory in space. Substituting equation (10) into the first, second and fourth equations of (8) leads to the following system:

$$\begin{cases} g\,\sin\varphi = r'\ddot{\varphi} + (r'' - r)\dot{\varphi}^2, \\ g\,r\,\cos\varphi = \left(a^2 + r^2\right)\ddot{\varphi} + 2\,r\,r'\,\dot{\varphi}^2, \\ 2\,g\,r\,\sin\varphi = \left(a^2 + r^2 + r'^2\right)\dot{\varphi}^2, \end{cases} \tag{11}$$

where equation (5) was used to eliminate n. Now we have to *formally* eliminate $\dot{\varphi}^2$ and $\ddot{\varphi}$ between the above equations in order to obtain a *single* differential equation for the function $r = r(\varphi)$, which gives the polar equation of the CM trajectory. From the third equation of (11), we have

$$\dot{\varphi}^2 = \frac{2\,g\,r\,\sin\varphi}{a^2 + r^2 + r'^2}, \tag{12}$$

while from the first and second equations, e.g., on using Kramer rule,

$$\dot{\varphi}^2 = g\,\frac{\sin\varphi\,(a^2 + r^2) - rr'\cos\varphi}{(a^2 + r^2)(r'' - r) - 2rr'^2}. \tag{13}$$

[2] Recently, the same trick was used to give a solution of the 'boat time' problem proposed in the *Feynman Lectures of Physics website* (http://feynmanlectures.info/).

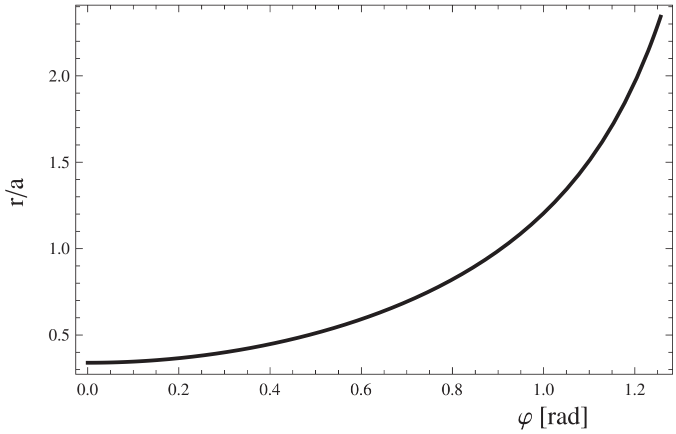

Figure 3. Behaviour of the normalized quantity r/a as a function of φ for $L = 102$ mm and for an overhang $d = 100$ mm. Friction is totally absent.

Finally, on comparing equations (12) and (13), after rearranging, the following nonlinear differential equation for the function $r = r(\varphi)$ is obtained:

$$r'' = \frac{3r}{2} + \frac{a^2}{2r} - \frac{r'}{2\tan\varphi}\left(1 + \frac{r'^2}{a^2 + r^2}\right) + \frac{r'^2}{2r}\,\frac{a^2 + 5r^2}{a^2 + r^2}, \tag{14}$$

which has to be solved together with the initial conditions

$$r(0) = d, \qquad r'(0) = 0. \tag{15}$$

It must be appreciated that, although we have no hope to analytically solve the above Cauchy problem, from a conceptual point of view the use of the energy conservation law and the elimination of the time information have considerably reduced the mathematical complexity of the problem. Once the CM trajectory has been obtained, the angle φ_d corresponding to the board departure can be obtained by the second equation of (11), written for $\ddot{\varphi} = 0$, together with equation (12), which leads to the equation

$$\tan\varphi_d = \frac{a^2 + r_d^2 + r_d'^2}{4 r_d r_d'}, \tag{16}$$

where $r_d = r(\varphi_d)$ and $r_d' = r'(\varphi_d)$. To give an example of practical implementation of the above algorithm, consider the case of $L = 102$ mm and $d = 100$ mm. The size of the board coincides with that used in the experiment of [6]. The differential equation (14) has been solved with the numerical integrator implemented in the symbolic language `Mathematica` through the standard command `NDSOlve`. Figure 3 shows the behaviour of the normalized quantity r/a as a function of the angle φ. To determine the angle φ_d, figure 4 shows the rhs (solid curve) and the lhs (dotted curve) of equation (16) as a function of the angle φ. The abscissa of the intersection point gives φ_d. It could also be worth giving a pictorial description of the CM trajectory in the real space. This is shown in figure 5 together with the initial and final positions of the board (thick lines). Once the detach angle φ_d has been determined, the corresponding angular velocity $\dot{\varphi}_d$ can be directly obtained by using, for instance, equation (12). It should be noted that the shape of the CM trajectory does not depend on the gravity acceleration g and that a universal, dimensionless, version of it can be obtained on dividing both members of equation (14) by a and using the normalized version d/a of the overhang. Figure 6 shows a first comparison between the experimentally found values of the final angular

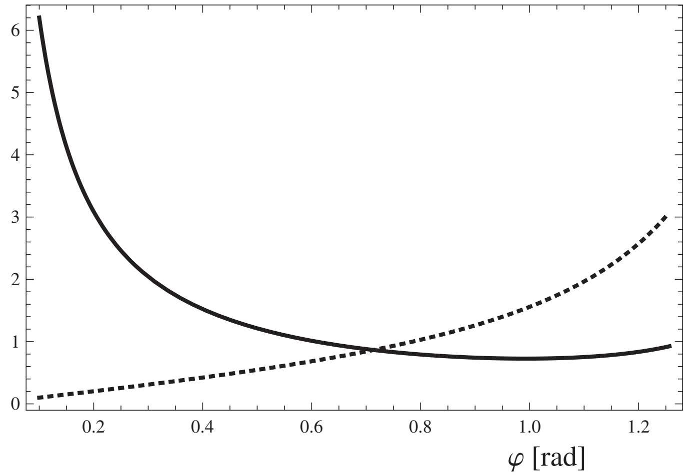

Figure 4. Graphical solution of equation (16) for the case of figure 3. $\varphi_d \simeq 0.713$ rad.

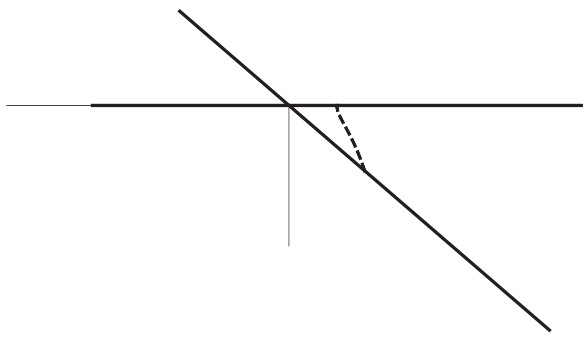

Figure 5. Pictorial representation of the CM trajectory (dashed curve) for the case of figure 3. The thick lines depict the thin board at the initial position and when it leaves the edge.

velocity $\dot{\varphi}_d$ (black dots), extracted from table I of [6], together with the theoretical predictions provided by the above-described simplest model. Although the agreement is far from being perfect, nevertheless the model is able to follow the general behaviour of $\dot{\varphi}_d$ versus d. This, in turn, confirms that the slipping of the toast plays the key role in grasping the whole dynamics, as pointed out in [6].

2.2. A first refinement of the model: dynamics in the presence of static friction

Consider now the first refinement of the model, according to which the toast dynamics is separated into two steps: (i) a pure rotation around the edge table, due to the presence of a static friction between the board and the edge and (ii) a rotation+slipping action analogous to that treated in the above-described simplest model. In other terms, we add the static friction at the beginning and continue to neglect the kinetic friction in the second part. In this way, the whole analysis carried out in the previous section will survive and equation (14) will keep its validity, provided that the initial conditions be replaced by new, more appropriate values obtained by taking into account the presence of the static friction during the first pure rotational

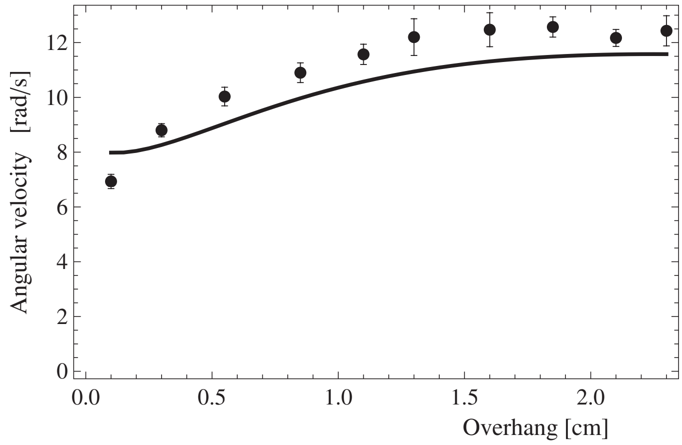

Figure 6. Experimental angular velocity $\dot{\varphi}_d$ (black dots with error bars) versus the initial overhang d for the tumbling board used in [6], together with the theoretical predictions (solid curve) provided by the simplest model of an infinitely thin square board in the absence of any kind of friction.

phase. In particular, since the rotation of the board in the presence of static friction has already been investigated in [1], we limit ourselves to give only the values of the angle φ_0 and of the angular velocity $\dot{\varphi}_0$ corresponding to the start of the slipping phase of the board, which turn out to be [1]

$$\varphi_0 = \arctan\left(\frac{\mu_S}{1+3\,\delta^2}\right), \tag{17}$$

and

$$\dot{\varphi}_0^2 = g\,\frac{2\,d}{a^2+d^2}\,\sin\varphi_0, \tag{18}$$

respectively, where μ_S denotes the coefficient of static friction[3]. After this pure rotational phase, the board continues to rotate but at the same time slips on the table edge in the presence of a kinetic friction. However, at least for the moment, we approximate by neglecting the latter friction in order to study the rotational/slipping dynamics of the toast according to the previously developed model. In particular, since during the first phase the total mechanical energy was conserved (static friction does not make any mechanical work on the board), equation (14) is still valid and can be (numerically) solved together with the initial conditions of equation (15), with φ_0 given by equation (17).[4] Figure 7 is the same as figure 6 but with the effects of the static friction included. The value of μ_S used to generate the theoretical curve has been set to 0.32, according to the experimental results obtained in [6]. On comparing figures 6 and 7, the improvement with respect to the no-friction model is evident, although the quantitative agreement between theory and experiment is still not so good. This is due to neglecting the presence of the kinetic friction during the rotational/slipping phase. The inclusion of its effect on the final values of the angular velocity will be the task in the next section.

[3] Equations (17) and (18) can be formally derived directly from equations (7) and (4) of [1], respectively. This is left as a simple exercise for students.
[4] Equation (18) will be used later, when the kinetic friction will be added to the model.

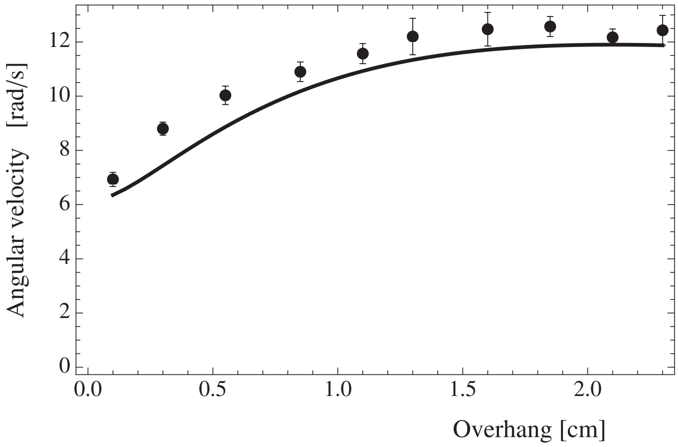

Figure 7. The same as figure 6 but including the effects of the static friction in the first rotational phase of the toast dynamics. Black dots: experimental angular velocity $\dot{\varphi}_d$ from [6]; solid curve: theoretical predictions provided by the model of an infinitely thin square board in the presence of an initial static friction $\mu_S = 0.32$ (see [6]).

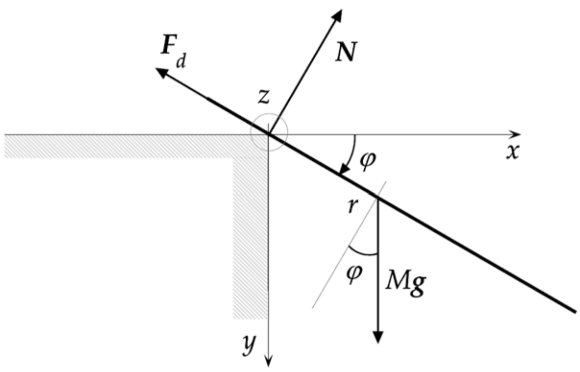

Figure 8. Free-body diagram in the presence of kinetic friction.

2.3. Further refinement: including kinetic friction

The most important consequence of including the kinetic friction in the model is that during such phase the total mechanical energy is no longer conserved. The free-body diagram is shown in figure 8, where the presence of the kinetic friction is represented by the vector $\boldsymbol{F}_d$ whose modulus is given by $F_d = \mu_d N$, with μ_d being the kinetic friction coefficient. The presence of $\boldsymbol{F}_d$ modifies the first equation of Newton's equations given in (2), which will replaced by

$$g \sin\varphi - \mu_d n = \ddot{r} - r\dot{\varphi}^2, \qquad (19)$$

the other being the same, together with the torque–angular momentum equation (5). Accordingly, on taking equation (10) into account, after straightforward algebra, it is possible to eliminate $\ddot{\varphi}$ among the equations to obtain the 2×2 system

$$\begin{cases} g \sin\varphi = \left(\mu_d + \dfrac{r\,r'}{a^2}\right) n + (r'' - r)\,\dot{\varphi}^2, \\ \\ g \cos\varphi = \left(1 + \dfrac{r^2}{a^2}\right) n + 2\,r'\,\dot{\varphi}^2, \end{cases} \tag{20}$$

and on expressing n as a function of $\dot{\varphi}^2$,

$$n = g \cos\varphi \, \frac{a^2}{a^2 + r^2} - \frac{2\,a^2\,r'}{a^2 + r^2}\,\dot{\varphi}^2, \tag{21}$$

with some work we arrive to the differential equation

$$g\left[\sin\varphi - \left(\mu_d + \frac{r\,r'}{a^2}\right)\frac{a^2}{a^2 + r^2}\cos\varphi\right] = \dot{\varphi}^2\left[r'' - r - \frac{2a^2\,r'}{a^2 + r^2}\left(\mu_d + \frac{r\,r'}{a^2}\right)\right]. \tag{22}$$

Due to the presence of $\dot{\varphi}^2$, the above equation is not enough to solve our problem and we need another independent relationship. In the absence of kinetic friction such further relation was provided by the energy conservation law that, in the present case, must be suitably modified in order to take into account the mechanical work of the kinetic friction $\boldsymbol{F}_d$ [10]. In particular, since for an infinitesimal displacement, say $\mathrm{d}\boldsymbol{r}$, of the CM such work is given by $-\mu_d\, n\, \mathrm{d}r$ or, on using φ as an independent variable, by $-\mu_d\, n\, r'\, \mathrm{d}\varphi$, the corresponding energy conservation law in equation (7) leads to the differential equation

$$\mathrm{d}[(r'^2 + a^2 + r^2)\,\dot{\varphi}^2 - 2\,g\,r\,\sin\varphi] = -2\mu_d\, n\, r'\, \mathrm{d}\varphi, \tag{23}$$

which, on taking equation (21) into account, after some algebra becomes

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\,[(r'^2 + a^2 + r^2)\,\dot{\varphi}^2 - 2\,g\,r\,\sin\varphi] = 2\mu_d\, r'\,\frac{a^2}{a^2 + r^2}\,(2r'\dot{\varphi}^2 - g\cos\varphi). \tag{24}$$

Equations (22) and (24) constitute the final system of two differential equations with respect to the unknown r and $\dot{\varphi}^2$, which can be solved together with the following initial conditions:

$$\begin{cases} r(\varphi_0) = d, \\ r'(\varphi_0) = 0, \\ \dot{\varphi}^2(\varphi_0) = \dot{\varphi}_0^2, \end{cases} \tag{25}$$

with $\dot{\varphi}_0^2$ being given by equation (18). To find the detach angle φ_d, it is sufficient to put into equation (21) the condition $n = 0$, which gives at once

$$g \cos\varphi_d = 2\,r'_d\,\dot{\varphi}_d^2. \tag{26}$$

On using the same numerical techniques employed in the previous sections, we can easily calculate the values of the final angular velocities $\dot{\varphi}_d$ as functions of the initial overhang d. Before showing the results, it is worth pointing out the fact that solving the system of equations (22) and (24) present at least a pair of advantages with respect to work in the time domain, namely the degree of the system, which is 2+1 instead of 2+2 for the time domain, and that one of the two outcomes is just the (squared) angular velocity required for the comparison with the experimental data. The results are presented in figure 9, which shows the same results as figures 6 and 7 but including the effect of kinetic friction during the slipping phase of the toast dynamics. In particular, the coefficient of kinetic friction has been set to $\mu_d \simeq 0.24$, according to [6]. Now the agreement with the experimental data is acceptable for overhangs greater than 1 cm, whereas for smaller values the theoretical model still does not work properly.

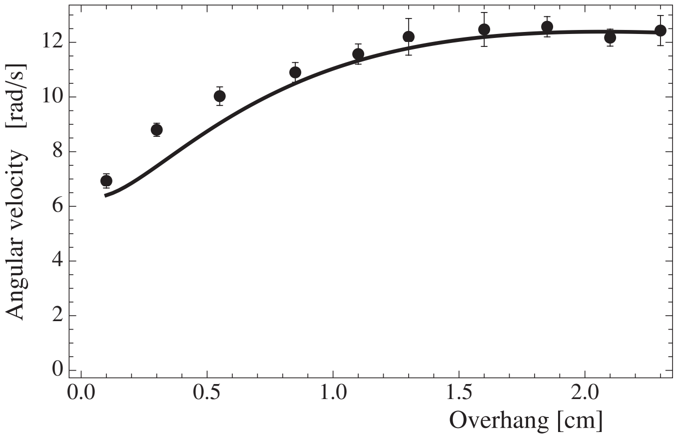

Figure 9. The same as figure 7 but also including the effect of kinetic friction during the slipping phase of the toast dynamics. Black dots: experimental angular velocity $\dot{\varphi}_d$ from [6]; solid curve: theoretical predictions provided by the model of an infinitely thin square board in the presence of an initial static friction $\mu_S = 0.32$ and a kinetic friction $\mu_d = 0.24$ during the slipping phase (see [6] for the corresponding numerical values).

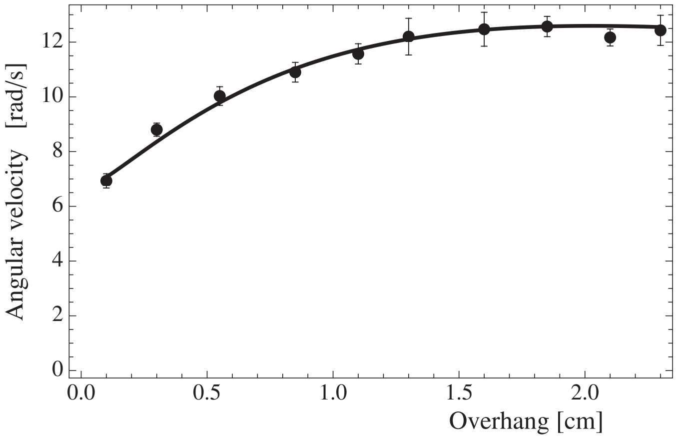

Figure 10. The same as figure 9 but also including the effect of the nonzero thickness of the toast. Black dots: experimental angular velocity $\dot{\varphi}_d$ from [6]; solid curve: theoretical predictions provided by the model of an infinitely thin square board in the presence of an initial static friction $\mu_S = 0.32$ and a kinetic friction $\mu_d = 0.24$ during the slipping phase, with the thickness of the board set being $b = 13$ mm (see [6] for the corresponding numerical values).

As suggested in [6], this is because the thickness of the board has been neglected. To complete our analysis, but at the same time to avoid an excessive growth of the mathematical complexity of the paper, the inclusion of a nonzero thickness (13 mm from [6]) has been confined to the appendix, while the corresponding theoretical predictions are reported in figure 10, which now show a nearly perfect agreement with the experimental measurements within the whole range of variability of the overhang.

3. Conclusion

The problem of tumbling toast has been reconsidered here as a good example of what could be called a 'laboratory exercise' for undergraduate courses in mechanics. The most attractive feature, in our opinion, of such problem is that the overall dynamics of the toast (from the start to the instant at which it leaves the edge table) is ruled by several physical mechanisms (static friction and kinetic friction, board thickness, etc) whose effects on the global behaviour of the toast can be sequentially incorporated into theoretical models of growing complexity, starting from the simplest one, i.e. a frictionless slipping of an infinitely thin homogeneous square on an ideal edge. To show the students how to reduce the mathematical complexity of the problem, Newton's equations have been numerically solved (via computational platforms now commercially available) in the 'space domain' by first eliminating the time variable through the use of the energy conservation law and of the derivative 'chain rule' in a suitably chosen polar reference frame. Finally, for each model considered, the agreement between the theoretical predictions obtained for the values of the angular velocity of the board when it leaves the table edge and the experimental values obtained in [6] has been shown, and in particular, its continuous improvement with respect to the subsequent refinements of the theoretical model has been evidenced.

Acknowledgment

I wish to thank Turi M Spinozzi for his help during preparation of the manuscript.

Appendix. A thick board

Figure A1 shows the geometry for a thick board.

In this case, the new parameter of the model is the angle β that is given by

$$\sin\beta = \frac{b}{2r}, \tag{A.1}$$

where b denotes the board thickness, which of course depends on r.[5] From a physical viewpoint, the most important difference with respect to the case of an infinitely thin board is that the velocity of rotation of the CM *does not coincide* with the velocity of rotation of the board around its CM, which is that measured in the experiments of [6]. This can be appreciated from figure A1. The angular momentum $\mathcal{L}_{\mathrm{CM}}$ thus becomes

$$\mathcal{L}_{\mathrm{CM}} = M a^2 \dot{\alpha}, \tag{A.2}$$

where a is now defined through

$$a = \frac{L}{2\sqrt{3}}\sqrt{1+\frac{b^2}{L^2}} \tag{A.3}$$

and the angle α is given by

$$\alpha = \varphi + \beta. \tag{A.4}$$

Accordingly, on taking equation (A.1) into account, after some algebra it is easy to show that [6]

$$\dot{\alpha} = A\dot{\varphi} \tag{A.5}$$

and

$$\ddot{\alpha} = A\ddot{\varphi} + B\dot{\varphi}^2, \tag{A.6}$$

[5] To facilitate the comparison with [6], we use the same notation.

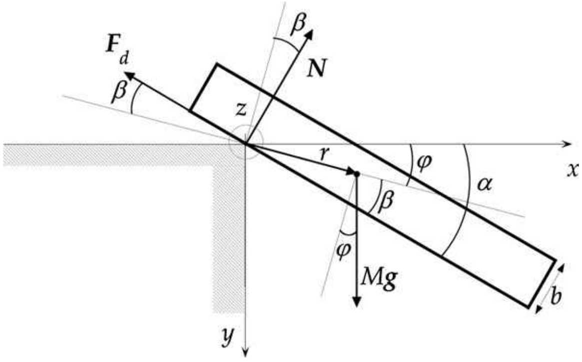

Figure A1. Geometry for a thick board.

where

$$A = 1 - \frac{r'}{r} \tan\beta,$$
$$B = \tan\beta \left(\frac{r'^2}{r^2} \frac{1+\cos^2\beta}{\cos^2\beta} - \frac{r''}{r} \right). \tag{A.7}$$

Newton's second law, applied to the CM, gives

$$\begin{aligned} g \sin\varphi + n(\sin\beta - \mu_d \cos\beta) &= r'\ddot{\varphi} + (r'' - r)\dot{\varphi}^2, \\ g\cos\varphi - n(\cos\beta + \mu_d \sin\beta) &= r\ddot{\varphi} + 2r'\dot{\varphi}^2, \end{aligned} \tag{A.8}$$

while the torque–angular momentum equation becomes

$$n\, r(\cos\beta + \mu_d \sin\beta) = \ddot{\alpha}, \tag{A.9}$$

which, thanks to equation (A.6), leads to

$$n = \frac{a^2}{r} \frac{A\ddot{\varphi} + B\dot{\varphi}^2}{\cos\beta + \mu_d \sin\beta}. \tag{A.10}$$

Now, we proceed to the formal elimination of $\ddot{\varphi}$ from our equations. To this end, substitution of equation (A.10) into the second equation of (A.8) gives

$$\ddot{\varphi} = \frac{gr\cos\varphi - (a^2 B + 2rr')\dot{\varphi}^2}{r^2 + a^2 A}, \tag{A.11}$$

which, once inserted into equation (A.10), after some algebra leads to

$$n = \frac{a^2}{r^2 + a^2 A} \frac{A g \cos\varphi + (Br - 2Ar')\dot{\varphi}^2}{\cos\beta + \mu_d \sin\beta}. \tag{A.12}$$

Then, on substituting equations (A.11) and A.12 into equation (A.8), after long but straightforward algebra, we obtain the differential equation

$$\begin{aligned} & g\left[\sin\varphi - \left(\eta_d A + \frac{r r'}{a^2} \right) \frac{a^2}{a^2 A + r^2} \cos\varphi \right] \\ & \quad = \dot{\varphi}^2 \left[r'' - r - \frac{2a^2 r'}{a^2 + r^2} \left(\eta_d A + \frac{r r'}{a^2} \right) + \frac{a^2 B}{a^2 A + r^2} (\eta_d r - r') \right], \end{aligned} \tag{A.13}$$

where[6]

$$\eta_d = \frac{\mu_d \cos\beta - \sin\beta}{\mu_d \sin\beta + \cos\beta}. \tag{A.14}$$

[6] It is a trivial, but didactically useful, algebraic exercise to verify that equation (A.13), written in the limit of $\beta \to 0$, coincides with equation (22).

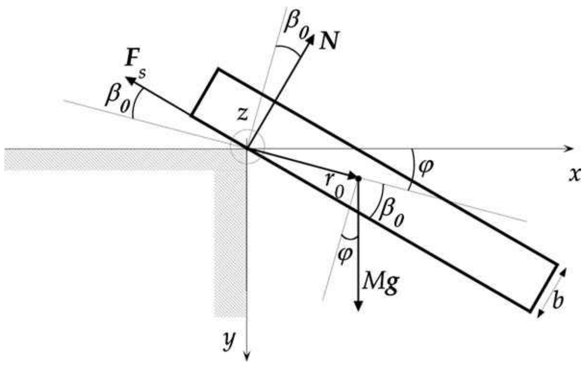

Figure A2. Geometry for a thick board in the rotational phase.

As far as the energy conservation law is concerned, we start from the geometry of figure A1 and note that equation (23) keeps its validity provided that it is modified as follows:

$$\mathrm{d}[(r'^2 + a^2A^2 + r^2)\dot{\varphi}^2 - 2gr\sin\varphi] = -2\mu_d\, n\, \mathrm{d}(r\cos\beta), \tag{A.15}$$

or, on dividing both sides by $\mathrm{d}\varphi$,

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}[(r'^2 + a^2A^2 + r^2)\dot{\varphi}^2 - 2gr\sin\varphi] = -2\mu_d\, n\frac{\mathrm{d}}{\mathrm{d}\varphi}(r\cos\beta). \tag{A.16}$$

Equations (A.13) and (A.16) generalize the system of ordinary differential equations which has to be solved to find the final angular velocity. To establish the initial conditions, i.e. the values of the angle φ_0 and the angular velocity $\dot{\varphi}_0$ at which the board begins to slip, the initial pure rotational phase must be studied. This is characterized by a constant value, say β_0, of the angle β and by the replacement of the kinetic friction force by the static friction force, say $\boldsymbol{F}_s$, as depicted in figure A2. Newton's equations and the torque–momentum equation (written now with respect to the z-axis) become

$$\begin{cases} g\sin\varphi + n\sin\beta_0 - f_s\cos\beta_0 = -r_0\dot{\varphi}^2 \\ g\cos\varphi - n\cos\beta_0 - f_s\sin\beta_0 = r_0\ddot{\varphi}, \\ g r_0\cos\varphi = \left(a^2 + r_0^2\right)\ddot{\varphi}, \end{cases} \tag{A.17}$$

where $r_0 = \sqrt{d^2 + b^2/4}$. The energy-conservation law instead reads

$$\left(a^2 + r_0^2\right)\dot{\varphi}^2 = 2gr_0(\sin\beta_0 + \sin\varphi), \tag{A.18}$$

and on imposing that for $\varphi = \varphi_0$, $f_s = \mu_S n$, after some algebra, the following transcendental equation for the angle φ_0 is obtained:

$$\eta\cos\varphi_0 = \left(1 + \frac{3r_0^2}{a^2}\right)\sin\varphi_0 + \frac{2r_0^2}{a^2}\sin\beta_0, \tag{A.19}$$

where

$$\eta = \frac{\mu_S\cos\beta_0 - \sin\beta_0}{\mu_S\sin\beta_0 + \cos\beta_0}. \tag{A.20}$$

Equation (A.19) can be analytically solved with respect to the initial angle φ_0, so that the angular velocity $\dot{\varphi}_0$ can be derived directly by substitution into equation (A.18), which gives

$$\dot{\varphi}_0^2 = \frac{2gr_0}{r_0^2 + a^2}(\sin\beta_0 + \sin\varphi_0). \tag{A.21}$$

Finally, it must be stressed that, in the case of a thick board, the final angular velocity to be compared with the experimental data of [6] is no longer $\dot{\varphi}$, which is the angular velocity of the CM, but rather $\dot{\alpha} = A\dot{\varphi}$, which is the angular velocity of the board itself.

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