British Journal for the History of Mathematics, 2019
Vol. 34, No. 2, 134–142, https://doi.org/10.1080/26375451.2019.1593036

Paul Erdős's mathematics as a social activity

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This essay, which won the 2018 BSHM undergraduate essay prize, investigates the collaborative mathematical practice of Paul Erdős. It draws on new unpublished primary sources and oral history in both English and Hungarian. It raises the question of whether communal mathematics, or mathematics as a social activity, can lead to individual success.

Introduction

Paul Erdős (Erdős Pál) was born in Budapest in 1913. He showed undeniable signs of talent from an early age (Székely 2006) and in 1934 he was awarded a doctorate from the Pázmány Péter University in Budapest. Erdős was an extraordinary man and mathematician. 'Pali bácsi' (Uncle Paul), as he was widely known, collaborated with over 500 mathematicians (O'Connor and Robertson 2000). This is probably the highest number of collaborators any mathematician has ever had. He published over 1500 papers, most of them with collaborators.

Erdős's life and enthusiasm for collaboration have been addressed in detail in the popular biographies by Hoffmann (1998) and Schechter (1998). The current essay adds to these by drawing on hitherto unpublished primary sources and oral history in both English and Hungarian. These provide further evidence for the means by which Erdős created his well-known social net (Figure 1).

Based on Erdős's collaboration graph, discussed in Section 2, Ion and Grossman (2017) have suggested that Erdős inspired his collaborators to become similarly collaborative. This possibly marks a changing trend in mathematics from individual to collaborative work, raising the question of whether communal mathematics, or mathematics as a social activity, can lead to individual success. To answer this question in the case of Erdős himself, I look at his social mathematics from several angles. First I analyse his collaborations and heritage. In that context, I look at his influence on mathematicians and the ways he had for finding the ideal mathematician to work with him on each problem. Finally, I use the Selberg/Erdős controversy over the elementary proof of the Prime Number Theorem to contrast two very different 'social' approaches to mathematics.

Collaborations

The first thing most people in the community of mathematicians would think of upon hearing the name Paul Erdős, would be the 'Erdős number' (Grossman 2015). The Erdős number n is a number associated with a person (essentially anyone in the world), indicating his/her proximity to Erdős in terms of working relationships. Paul Erdős is the only one with Erdős number 0. Everybody else has Erdős number n+1, if they have published a paper with somebody who has Erdős number n. Hence,

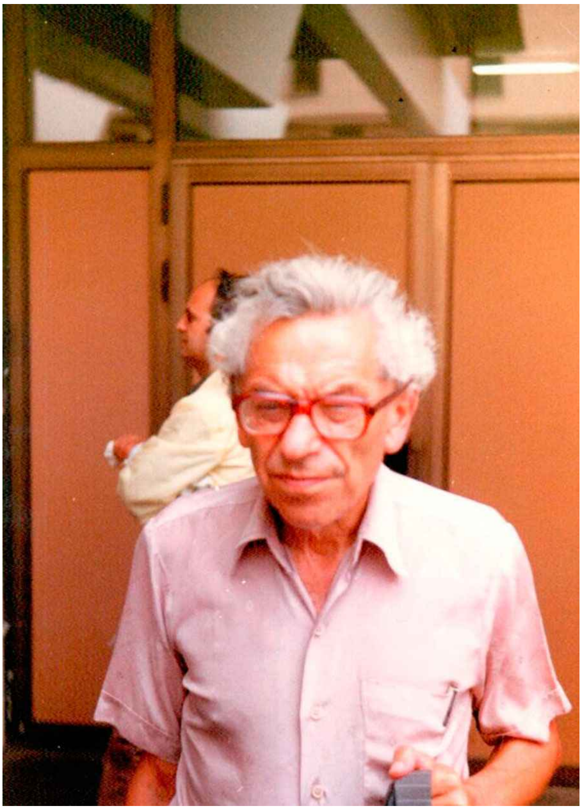


Figure 1. Erdős Pál (Picture by Norbert Hegyvári)

Erdős's direct collaborators have Erdős number 1, and the people they published articles with have Erdős number 2, etc. Having no Erdős number is equivalent to $n = \infty$ (Grossman 2013).

Let C be the collaboration graph, where two vertices (each of them representing an author) are joined by an edge if the two have published a paper together (Ion and Grossman 2017). This graph largely resembles social media platforms, like Facebook, which can be translated to a graph F, where two people are connected by an edge if they are friends. This resemblance suggests that in times before the Internet or Facebook, Erdős provided a network between mathematicians, acting as a human computer (Székely 2006).

If we take a look at the subgraph E_1, where each vertex symbolizes an author with Erdős number 1, we can see even more of his influence on research collaboration. The vertex that symbolizes Erdős himself would be adjacent to all of the vertices in the subgraph E_1. In 1995, this graph had 458 vertices and 1218 edges. This means Erdős had at least 458 collaborators: an enormous number. In E_1 there are only 40 isolated vertices, which suggests that Erdős's style of collaboration inspired his collaborators to do the same (Grossman 2013).

The high number of Erdős's collaborators is due to his love for mathematics, but more importantly his love for mathematicians. He knew them, remembered their work and achievements, and constantly shared problems with them to work on (Székely 2006). One of Erdős's many talents was guessing what type of problem would spark somebody's curiosity. It was very common for him to approach a mathematician and suggest one (or several) of his huge stack of problems, in case they would be interested (Interview with Peter Cameron). His heritage lives on in a unique way, giving clear evidence for his mathematical success.

Influence

Further analysis is required to find the most accurate answer to the question raised in the introduction. Did Erdős seek individual or collective success in mathematics? Did his search for young and talented mathematicians aim to raise potential co-authors, or was it done for more selfless, purely educational reasons? Maybe even a combination?

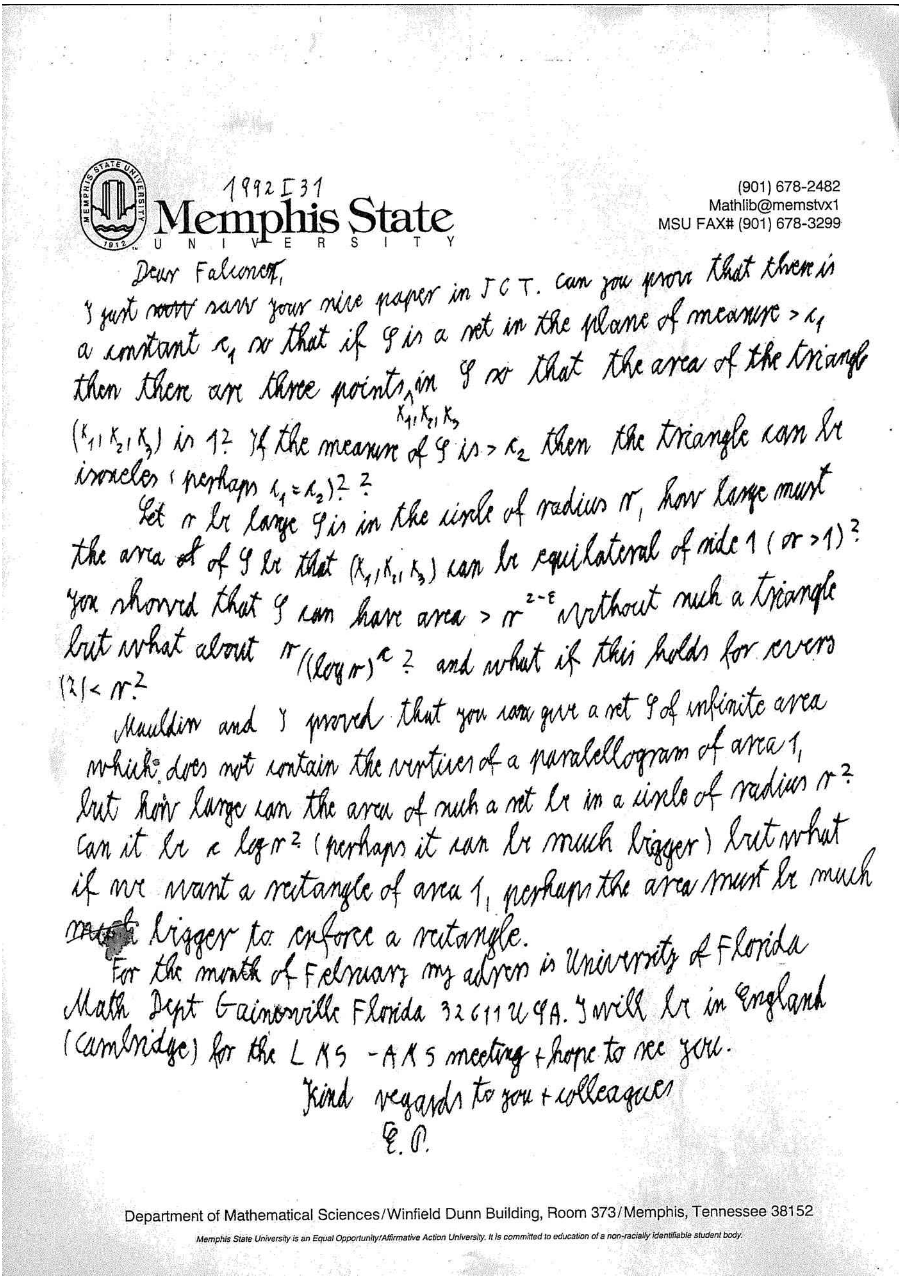
Erdős spent most of his time travelling; hence, he often communicated by mail (Pósa 1997). The extensive travel follows from his main mode of research, that he called 'new roof, new proof' (Székely 2006). He frequently visited institutions and universities, where he often used the phrase 'my brain is open' to initiate the collaboration. If somebody didn't seem interested in his problems, however, he stopped contacting them (Pósa 1997). This raises a further question of whether he was aiming primarily for success, or whether it was his genuine passion that drew him towards those who shared his current interests. Since Paul Erdős liked to discuss mathematics while taking part in several different social activities (such as sailing on Lake Balaton), we might assume the latter (Interview with Norbert Hegyvári).

Between 1981 and 1992 Erdős sent several letters to Kenneth Falconer, who felt very honoured to receive these and added that the first letter to him was 'entirely out of the blue and unsolicited'. These letters illustrate Erdős's style of communication and of collaboration, even though he and Falconer never published together. The letters start with either a thank you for the letter they were replying to (70%) and/or a compliment on mathematical papers authored by Falconer (50%). Thanking Falconer for his papers highlights the fact that, before the Internet, research papers also travelled by mail – the only way of circulating results amongst the mathematical community. The letters then continued with a small paragraph, a sentence or two about Erdős's whereabouts or a brief remark on some news.

Even a sample of these 10 letters illustrates well that Erdős was always on the move (Strauss 1983). In most he mentioned the address where he could receive a reply. In three instances he listed Ronald Graham's address (a mathematician, who also managed Erdős's money), because he would be travelling (Hoffmann 1998). In five letters he expected to receive the reply at the Hungarian Academy of Sciences. This was a base for him, where he often happily returned (Interview with P.C.). Each of the ten letters then continues with a suggestion of a mathematical problem, something Erdős thought Falconer would be interested in. Many of the problems suggested were from the field of measure combinatorial geometry, which fit into Falconer's research interests (Falconer 1992).

Falconer published several papers based on problems posed by Erdős. He now claims that he wishes they had published a paper in collaboration. Even though his proofs weren't done in collaboration, Erdős was rather satisfied with Falconer's results. In 1992, referring to Falconer's paper 'On a Problem of Erdős on Fractal Combinatorial Geometry', published in the *Journal of Combinatorial Theory* earlier that year, Erdős wrote, 'Dear Falconer, I just saw your nice paper in JCT. Can you prove … ', showing he was happy to give him more problems to work on (Figure 2).

To answer the question posed at the beginning of this section, Erdős's own definition of success can be refined. His reasoning for communal mathematics can be further illustrated by a letter from 1985, in which he presented a problem, also from


1992 I 31

Memphis State University

(901) 678-2482
Mathlib@memstvx1
MSU FAX# (901) 678-3299

Dear Falconer,
I just ~~soon~~ saw your nice paper in JCT. Can you prove that there is
a constant c_1 so that if S is a set in the plane of measure $> c_1$
then there are three points x_1, x_2, x_3 in S so that the area of the triangle
(x_1, x_2, x_3) is 1? If the measure of S is $> c_2$ then the triangle can be
isosceles (perhaps $c_1 = c_2$)? ?
Let r be large S is in the circle of radius r, how large must
the area ~~of~~ of S be that (x_1, x_2, x_3) can be equilateral of side 1 ($r > 1$)?
You showed that S can have area $> r^{2-\varepsilon}$ without such a triangle
but what about $r/(\log r)^c$? and what if this holds for every
$|S| < r^2$
Mauldin and I proved that you can give a set S of infinite area
which does not contain the vertices of a parallelogram of area 1,
but how large can the area of such a set be in a circle of radius r?
Can it be $c \log r^2$ (perhaps it can be much bigger) but what
if we want a rectangle of area 1, perhaps the area must be much
~~much~~ bigger to enforce a rectangle.
For the month of February my adress is University of Florida
Math Dept Gainesville Florida 32611 USA. I will be in England
(Cambridge) for the LMS-AMS meeting + hope to see you.
Kind regards to you + colleagues
E.P.

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Figure 2. Letter from Paul Erdős to Kenneth Falconer from 1992. For a transciption of this letter see the Appendix below.

the field of measure combinatorial geometry. After proposing the question he wrote 'I offered 500 dollars for a proof or a disproof of this'. Erdős was known for his generosity. He shared several of his unsolved problems and offered monetary prizes for their solution. However, most of the mathematicians who solved those problems, instead of collecting the money, kept the more valuable artifact; a cheque from 'Pali bácsi'. These money prizes can still be claimed for successful solutions; they are now administered by Ronald Graham (Interview with P.C.).

The elementary proof of the prime number theorem

In 1948 Paul Erdős sent out some carefully written letters to several people around the world. The message said 'Using a fundamental inequality of Atle Selberg, Selberg and I have succeeded in giving an elementary proof of the Prime Number Theorem' (Spencer and Graham 2009). This message led to a bitter dispute between Erdős and Selberg (Goldfeld 2004). Since the ancient proof that the number of primes is infinite, mathematicians have investigated their distribution (Spencer and Graham 2009). Gauss and Legendre conjectured the Prime Number Theorem: let the number of primes between 0 and a real number $x > 1$ be denoted by $\pi(x)$ (Goldfeld 2004); then

$$\pi(x) \approx \frac{x}{\log(x)}. \tag{1}$$

Chebyshev showed that $\pi(x)$ is of the order $x/\log(x)$ by finding the highest power of a prime dividing $x!$ and hence putting bounds on $\pi(x)/(x/\log(x))$. He applied this to Stirling's asymptotic formula, $\ln x! = x\ln x - x + O(\ln x)$ to deduce (Goldfeld 2004)

$$x\sum_{p\leq x}\frac{\log(p)}{p} = x\log(x) + O(x). \tag{2}$$

In 1896 Hadamard and de la Vallée-Poussin, independently, proved the Prime Number Theorem. However, their proof was not elementary and used complex analysis (Spencer and Graham 2009). The search began for an elementary proof.

In 1948 Atle Selberg proved the asymptotic formula, which he called the 'fundamental formula': let $\vartheta(x) = \sum_{p\leq x}\log(p)$ (Goldfeld 2004); then

$$\vartheta(x)\log(x) + \sum_{p\leq x}\log(p)\vartheta\left(\frac{x}{p}\right) = 2x\log(x) + O(x), \tag{3}$$

which is a refinement of Chebyshev's formula. From Chebyshev's result and the 'fundamental formula' Selberg could prove that

$$\limsup\frac{\vartheta(x)}{x} + \liminf\frac{\vartheta(x)}{x} = 2,$$

and knew that if he could prove that either of the two parts of the sum was 1, the Prime Number Theorem would follow (Goldfeld 2004).

Selberg showed the formula to his fellow mathematician Paul Turán. Turán, with Selberg's permission, mentioned it to a small group of mathematicians, including Erdős. Paul Erdős got interested in the formula and used it to prove that

$$\lim_{n\to\infty} \frac{p_{n+1}}{p_n} = 1. \tag{4}$$

He even succeeded in proving a stronger result. It follows from the Prime Number Theorem that for a number x and a number ϵ there exist prime numbers between x and $x(1+\epsilon)$, if ϵ is fixed and x sufficiently large. Hence (Galvin 2015)

$$\pi(x(1+\epsilon)) - \pi(x) \approx \frac{x(1+\epsilon)}{\log x} - \frac{x}{\log x} \to \infty,$$

which is equivalent to (Goldfeld (2004))

$$\pi(x(1+\epsilon)) - \pi(x) > \delta(\epsilon)\frac{x}{\log x}. \tag{5}$$

However, Erdős discovered that (5) also follows from Selberg's formula, not only from the Prime Number Theorem; in fact, it can be used to find an elementary proof of the latter. With the 'fundamental formula' in hand, Erdős could deduce

$$\pi(x(1+\epsilon)) - \pi(x) > \delta(\epsilon)\frac{x}{\log x}$$

for any ϵ (Goldfeld 2004). Erdős communicated his result to Selberg. Selberg could now prove that $1 = \limsup(\vartheta(x)/x) = \liminf(\vartheta(x)/x)$, a proof he needed to finish his elementary proof of the Prime Number Theorem, which he did a few days later (Goldfeld 2004).

Once the theorem was proved, tension followed over whether the elementary proof of the Prime Number Theorem was Selberg's own achievement, or a collaboration between Selberg and Erdős (Goldfeld 2004; Spencer and Graham 2009). Many sources claim that Selberg's pride suffered when Erdős started telling the world that they jointly established an elementary proof for the theorem (Hoffmann 1998, interview with P.C.). Since Erdős was better known, news travelled in the mathematical world that Paul Erdős and 'some Scandinavian mathematician' (Goldfeld 2004) were behind the proof. According to Selberg, other reports didn't mention any Scandinavian mathematician, and gave the full credit to Erdős (Goldfeld 2004). Hence, Selberg decided to publish the result under only his own name, a decision which played a substantial part in his being awarded the Fields Medal, leaving Erdős out of the spotlight.

However, from a historical perspective, it wasn't such a simple process. Paul Erdős and Atle Selberg not only had very different personalities, but also different publication habits (Interview with P.C.). Erdős was always eager to tell everybody about his mathematics, and worked surrounded by other mathematicians; mathematics was a communal activity for him. But for Selberg, mathematics was ideally done in solitude, and he was known to be secretive about his work (Spencer and Graham 2009).

Thus, the two mathematicians' approach to their general practice of mathematics determined their approach to this specific problem. Selberg, in a letter to Erdős on 20 August 1948 wrote 'I would never have dreamed of forcing you to write a joint paper on this, in spite of the fact that the essential thing in the proof of the result was mine' (Goldfeld 2004). He proposed that each of them publish their own contribution, in two different papers. He was unwilling to offer any further compromise, however, specifying that he would publish his proof regardless of Erdős's wishes (Goldfeld 2004). Erdős, in his reply of 27 September 1948, pointed out that Selberg wouldn't have been able to prove the theorem without his contribution. He claimed that one of the two – equally necessary – parts of the proof was provided by him and shared with Selberg: whereas Selberg kept the other part to himself, not giving Erdős a fair chance to complete the proof at the same time (Goldfeld 2004).

This led to a bitter dispute between Erdős and Selberg, which only eased slightly in 1993, when Selberg went to a number theory conference informally dedicated to Erdős for his eightieth birthday, in Lillafüred, Hungary. This was a gesture from Selberg that showed his respect towards Erdős, softening their relationship; but there was never a real reconciliation (Interview with N.H.) (Figure 3).

Since Erdős was very talented in matching mathematical problems with the optimal mathematician, one might ask whether he was a good judge of character. Did he suspect, knowing Selberg's personality, that he might not give him the credit

Figure 3. Selberg in Lillafüred (photograph by Norbert Hegyvári)

he expected? Could that have been the reason he suggested the joint paper? Or did the idea of collaboration follow from his communal mathematical habits? The evidence makes it hard to judge, but we can be sure that this event had a negative impact on him (Spencer and Graham 2009). Receiving the Cole Prize in 1952 for his contributions, therefore, was a small consolation (Goldfeld 2004).

Conclusion

Paul Erdős left an indelible mark on mathematical research, as well as giving his name to the Erdős number, and his methods greatly influenced many to adopt similarly collaborative methods. The style and scope of Erdős's own collaborations was imposing. The problems that he generously shared with others, whether they led to a joint publication or not when solved, were the source of a great deal of pleasure for Erdős.

As I have suggested in the previous two sections, he believed success lay in solving beautiful mathematical problems: his focus was on the solution instead of on earning fame and credit. The case of the elementary proof of the Prime Number Theorem, however – and perhaps a different definition of success, more concerned with individual credit for important discoveries – leaves a bitter taste. Why did Erdős insist on a joint paper in this case? Why didn't he congratulate Selberg instead, and celebrate the proof itself? Was there a real pride in him, aiming to add his name to an important new proof? These questions are hard to answer, and they do seem to show a different side of Erdős.

Is communal mathematics, or mathematics as a social activity, compatible with individual success? This question continues to interest mathematicians. I would like to finish my essay with a quote from Paul Erdős, translated from Hungarian.

> I probably have more papers than anyone else, and I have collaborated with the highest number of people, but in the old Hungarian Parliament [...] they used to say the votes shouldn't be counted, but evaluated. This wasn't correct and democratic in politics, but it surely is in science (Erdős 1997).

Acknowledgments

I would like to thank Peter Cameron, Kenneth Falconer, Norbert Hegyvári, Lajos Pósa and Ödön Vancsó for sharing with me their memories of Paul Erdős, and further personal information that I have used in my essay.

Disclosure statement

No potential conflict of interest was reported by the authors.

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Appendix. Erdős's letter to Falconer, 31 January 1992

Dear Falconer,

I just saw your nice paper in JCT. Can you prove that there is a constant λ_1 so that if S is a set in the plane of measure $> \lambda_1$ then there are three points x_1,x_2,x_3 in S so that the area of the triangle (x_1,x_2,x_3) is 1? If the measure of S is $> \lambda_1$ then the triangle can be perhaps isosceles, perhaps $\lambda_1 = \lambda_2$??

Let r be large S is in the circle of radius r, how large must the area of S be that (x_1,x_2,x_3) can be equilateral of side 1 or > 1? You showed that S can have area $> r^{2-\epsilon}$ without such a triangle but what about $r/(\log r)^{\lambda}$? And what if this holds for every $|\lambda| < r$?

Muuldin and I proved that you can give a set S of infinite area which does not contain the vertices of a parallelogram of area 1, but how large can the area of such a set be in a circle of radius r? Can it be $\lambda \log r^2$? (perhaps it can be much bigger) but what if we want a rectangle of area 1, perhaps the area must be much bigger to enforce a rectangle.

For the month of February my adress is University of Florida Math Dept Gainsville Florida 32611UYA. I will be in England (Cambridge) for the LMS – AMS meeting + hope to see you.

Kind regards to you + colleagues

E.P.