Higher Algebraic K-Theory of Schemes and of Derived Categories

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to Alexander Grothendieck on his 60th birthday

In this paper we prove a localization theorem for the K-theory of commutative rings and of schemes, Theorem 7.4, relating the K-groups of a scheme, of an open subscheme, and of the category of those perfect complexes on the scheme which are acyclic on the open subscheme. The localization theorem of Quillen [Q1] for K'- or G-theory is the main support of his many results on the G-theory of noetherian schemes. The previous lack of an adequate localization theorem for K-theory has obstructed development of this theory for the fifteen years since 1973. Hence our theorem unleashes a pack of new basic results hitherto known only under very restrictive hypotheses like regularity. These new results include the "Bass fundamental theorem" 6.6, the Zariski (Nisnevich) cohomological descent spectral sequence that reduces problems to the case of local (hensel local) rings 10.3 and 19.8, the Mayer-Vietoris theorem for open covers 8.1, invariance mod ℓ under polynomial extensions 9.5, Vorst-van der Kallen theory for NK 9.12, Goodwillie and Ogle-Weibel theorems relating K-theory to cyclic cohomology 9.10, mod ℓ Mayer-Vietoris for closed covers 9.8, and mod ℓ comparison between algebraic and topological K-theory 11.5 and 11.9. Indeed most known results in K-theory can be improved by the methods of this paper, by removing now unnecessary regularity, affineness, and other hypotheses.

We also develop the higher K-theory of derived categories, which is an essential tool in the above results. Our techniques here rest on the brilliant work of Waldhausen [W], who has extended and deepened the foundation of K-theory beyond that laid down by Quillen, allowing it to bear a heavier load.

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*partially supported by NSF and the Sloan Foundation.

The key ideas that make all our results possible go back to the theory of K_0 of the derived category, which was conceived by Grothendieck, and was developed by him with Illusie and Berthelot in [SGA 6]. We especially need his concept of a perfect complex, a sheaf of chain complexes that is locally quasi-isomorphic to a bounded complex of algebraic vector bundles. These ideas have remained dormant for some time, especially because Quillen [Q1] discovered the higher K-theory of exact categories in a form which did not immediately extend to define a higher K-theory of derived categories. Thus one worked with the exact category of algebraic vector bundles, and not with the derived category of perfect complexes. Waldhausen's work [W] first made it clear how to define such a K-theory of a derived category, or more precisely, of a category of chain complexes provided with a notion of "weak equivalence" like quasi-isomorphism. Several people, among them Brinkmann [Bri], Gabber, Gillet [Gi2], [Gi4], Hinich and Shekhtman [HS], Landsburg, Waldhausen, and ourselves then became aware of this possibility of returning to the ideas of [SGA 6]. The intrinsic appeal of those ideas did not instantly overcome public inertia, and they did not at once appear strictly necessary to further progress. However, they turn out to be essential to the very statement of our localization theorem, if not to all its consequences. Moreover, the key geometric fact behind the theorem is the fact that the only obstruction to extending up to quasi-isomorphism a perfect complex on the open subscheme to the full scheme is its class in K_0. The naive analogous statement for algebraic vector bundles is false, as shown long ago by Serre ([Se], Section 5, a)). Furthermore, the proof of this extension fact depends essentially on the very Grothendieckian idea that perfect complexes are finitely presented objects in the derived category.

Of course, Grothendieck's ideas completely pervade modern mathematics, and it would be a hopeless task to isolate and acknowledge all intellectual debts to him. But we hope our case illustrates that despite their widespread influence, and nearly two decades after Grothendieck's withdrawal from public mathematical life, many of Grothendieck's ideas are still full of unexhausted potential, and will amply repay further development. Remarkably, his by now classic works can still surprise and instruct the serious reader. We dedicate this paper to him with profound admiration.

The reader may find a brief sketch of the contents of this long paper useful. Section 1 recalls for the convenience of the reader the results of Waldhausen [W]. An expert might skip this section, but should glance at biWaldhausen categories 1.2.4 and 1.2.11 to allow dualization of arguments, the inductive construction of chain complexes 1.9.5, the fact that K-theory is invariant under functors inducing equivalences of derived categories 1.9.8, and the cofinality theorem 1.10.1. Section 2 recalls the theory of perfect complexes on schemes from [SGA 6]. The expert might skip this, but should look at the characterization of perfect complexes as finitely presented objects 2.4, the fact that a complex with quasi-coherent cohomology on a nice scheme is a direct colimit of perfect complexes 2.3, and the basis for the excision theorem laid down in 2.6. Section 3 contains the definitions and basic functorialities of K-theory. Section 4 contains the projective space bundle theorem. Section 5 proves the key extension result for perfect complexes, and contains the first crude form of the localization theorem. Section 6 proves the Bass fundamental theorem, and defines K-groups also in negative degrees. Section 7 extends the previous results into negative degrees, putting them in their final form. In particular, Section 7 contains the good form of the localization theorem. This section gives the best quick summary of the fundamental results. The other basic results occur as consequences in Sections 8 - 11. The appendices contain various results needed in the text, but which are not limited to K-theory. In particular, Appendix B merely summarizes from all points of EGA and SGA the relations between the categories of $\mathcal{O}_X$-modules and of quasi-coherent $\mathcal{O}_X$-modules, as a help to the conscientious reader when he becomes as confused about this as we were.

The paper should be comprehensible to anyone with a good first year graduate knowledge in algebraic geometry, and with a bit of algebraic topology. We must formulate our results in the language of spectra in the sense of topology but this may be picked up easily by skimming through [A] III Sections 1-6 (ignore any pointless examples involving baroque curiosities like "MU," "MSO," "MSpin," or the "Steenrod algebra"), and a glance at [Th1] Section 5 and A. This formulation in terms of spectra is much more powerful than the naive formulation in terms of disembodied abelian groups, and is not subject to certain unstable pathologies like the formulation in terms of spaces as in [Q1] and [W]. Indeed, the spectral formulation works just like Grothendieck's formulation of homological algebra in terms of the derived category, as explained in [Th1] Sections 5. To see the proofs of the results quoted in Section 1, the reader must see [W], although the sufficiently trusting need not. [Q1] is still good reading, although not necessary for this paper, except for the homotopy theory of categories of [Q1] Section 1.

The first author must state that his coauthor and close friend, Tom Trobaugh, quite intelligent, singularly original, and inordinately generous, killed himself consequent to endogenous depression. Ninety-four days later, in my dream, Tom's simulacrum remarked, "The direct limit characterization of perfect complexes shows that they extend, just as one extends a coherent sheaf." Awaking with a start, I knew this idea had to be wrong, since some perfect complexes have a non-vanishing K_0 obstruction to extension. I had worked on this problem for 3 years, and

saw this approach to be hopeless. But Tom's simulacrum had been so insistent, I knew he wouldn't let me sleep undisturbed until I had worked out the argument and could point to the gap. This work quickly led to the key results of this paper. To Tom, I could have explained why he must be listed as a coauthor. During his lifetime, Tom also pointed out the interesting comparison of the careers of Grothendieck and Newton.

For more mundance assistance and useful conversations, I would like to thank Gillet, Grayson, Karoubi, Kassel, Levine, Loday, Nisnevich, Ogle, Soulé, Waldhausen, Weibel, and D. Yao.

1. Waldhausen K-theory and K-theory of derived categories

1.0. In this section we review some definitions and results of Waldhausen's framework for K-theory, [W]. Our only claims to some originality in Section 1 are the general cofinality theorem 1.10.1 which is slightly different from previous results, and the results 1.9.5 and 1.9.8 which make it easier to apply Waldhausen's approximation theorem.

1.1.1. Let $\mathcal{A}$ be an abelian category. Consider chain complexes $C^\cdot$ in $\mathcal{A}$. We use the algebraic geometer's indexing, so differentials increase degree: $\partial : C^n \to C^{n+1}$.

Recall the standard notation $Z^kC^\cdot = \ker \partial : C^k \to C^{k+1}$, and $B^kC^\cdot = \text{im } \partial : C^{k-1} \to C^k$.

A complex $C^\cdot$ is (strictly) bounded above if there is an integer N such that $C^n = 0$ for all $n \geq N$. The category of bounded above complexes is denoted $\mathcal{C}^-(\mathcal{A})$. A complex $C^\cdot$ is cohomologically bounded above if there is an N such that $H^n(C^\cdot) = 0$ for all $n \geq N$. Dually for bounded below, $\mathcal{C}^+(\mathcal{A})$, and cohomologically bounded below. A complex is bounded if it is bounded both above and below. The category of strict bounded complexes is $\mathcal{C}^b(\mathcal{A})$.

A chain map $f : C^\cdot \to D^\cdot$ is a chain homotopy equivalence if there is a chain map $g : D^\cdot \to C^\cdot$ and chain homotopies $fg \simeq 1_D$, $gf \simeq 1_C$. More generally, a chain map $f : C^\cdot \to D^\cdot$ is a quasi-isomorphism if it induces an isomorphism on all cohomology groups $H^*(f) : H^*(C^\cdot) \cong H^*(D^\cdot)$. For any integer m, a chain map f is an m-quasi-isomorphism if $H^k(f)$ is an isomorphism for $k > m$ and an epimorphism for $k = m$.

The derived category $D(\mathcal{A})$ (cf. [H], [V]) is formed from the category of all chain complexes in $\mathcal{A}$ by localizing this category of complexes so that precisely its quasi-isomorphisms become isomorphisms in $D(\mathcal{A})$. The variant subcategories $D^-(\mathcal{A})$, $D^+(\mathcal{A})$, $D^b(\mathcal{A})$ are formed similarly from the categories of cohomologically bounded above, cohomologically bounded below, and cohomologically bounded complexes, respectively. $D(\mathcal{A})$ admits a 2-sided calculus of fractions as a localization of the chain homotopy category that results from the category of complexes by identifying chain homotopic maps, as in 1.9.6 below, or in [V] I Section 2, [H] I Section 3. The additional structure of $D(\mathcal{A})$ as a triangulated category results from the construction of homotopy pushouts and pullbacks, which we review next.

1.1.2. Let $f : A^{\cdot} \to F^{\cdot}$ and $g : A^{\cdot} \to G^{\cdot}$ be chain maps of complexes. The canonical homotopy pushout

$$F^{\cdot} \overset{h}{\underset{A^{\cdot}}{\cup}} G^{\cdot}$$

is the complex given by

$$\left(F^{\cdot} \overset{h}{\underset{A^{\cdot}}{\cup}} G^{\cdot}\right)^n = F^n \oplus A^{n+1} \oplus G^n \tag{1.1.2.1}$$

with differential,

$$\partial(x,a,y) = (\partial_F x + fa,\ -\partial_A a,\ \partial_G y - ga)\,. \tag{1.1.2.2}$$

(We describe ∂ as if objects of $\mathcal{A}$ had "elements," by the standard abuse.)

Chain maps from this canonical homotopy pushout to a complex $C^{\cdot}$ correspond bijectively to data (h,k,H) where $h : F^{\cdot} \to C^{\cdot}$ and $k : G^{\cdot} \to C^{\cdot}$ are chain maps and H is a chain homotopy $hf \simeq kg : A^{\cdot} \to C^{\cdot}$. Thus H consists of maps $A^n \to C^{n-1}$ for all n such that $\partial H + H\partial = hf - kg$. To (h,k,H) corresponds the map from the homotopy pushout to $C^{\cdot}$ sending (x,a,y) to $hx + Ha + ky$.

Given another $f' : A^{\cdot\prime} \to F^{\cdot\prime}$ $g' : A^{\cdot\prime} \to G^{\cdot\prime}$, suppose there are maps $a : A^{\cdot} \to A^{\cdot\prime}$, $b : F^{\cdot} \to F^{\cdot\prime}$, $c : G^{\cdot} \to G^{\cdot\prime}$ and chain homotopies $f'a \simeq bf$, $g'a \simeq cg$. The maps a, b, c, and choice of chain homotopies then determine a map of canonical homotopy pushouts $F \overset{h}{\underset{A^{\cdot}}{\cup}} G \to F' \overset{h}{\underset{A'}{\cup}} G'$, as one sees by the universal mapping property of the preceding paragraph. This map will be a quasi-isomorphism if each of the maps a, b, c is a quasi-isomorphism. This last fact follows from the 5-lemma and the long exact sequence of cohomology groups

(1.1.2.3)
$$\ldots \xrightarrow{\partial} H^n(A) \to H^n(F^{\cdot}) \oplus H^n(G^{\cdot}) \to H^n(F^{\cdot} \overset{h}{\underset{A^{\cdot}}{\cup}} G^{\cdot}) \xrightarrow{\partial} H^{n+1}(A^{\cdot}) \to \ldots$$

which results from the short exact sequence of complexes

$$0 \to F^{\cdot} \oplus G^{\cdot} \to F^{\cdot} \overset{h}{\underset{A^{\cdot}}{\cup}} G^{\cdot} \to A^{\cdot}[1] \to 0. \tag{1.1.2.4}$$

Here $A^\cdot[k]$ is the complex $A^\cdot$ shifted in degree, so $A^\cdot[k]^n = A^{k+n}$, and $H^n(A[k]) = H^{n+k}(A^\cdot)$.

Several special cases of the homotopy pushout construction are particularly important. When $f : A^\cdot = F^\cdot$ is the identity map, the canonical homotopy pushout is the mapping cylinder of $g : A^\cdot \to G^\cdot$, considered in 1.3.4. When $f : A^\cdot \to F^\cdot = 0$ is the map to 0, the canonical homotopy pushout is the mapping cone of $g : A^\cdot \to G^\cdot$.

There is a canonical map from the homotopy pushout to the (strict) pushout, induced by $(x, a, y) \longmapsto (x, y) \bmod A^\cdot$.

$$F^\cdot \overset{h}{\underset{A^\cdot}{\cup}} G^\cdot \to F^\cdot \underset{A^\cdot}{\cup} G^\cdot .$$

This map is a quasi-isomorphism whenever $A^n \to F^n \oplus G^n$ is a monomorphism for all n, as is seen by the 5-lemma applied to the map of the long exact sequence 1.1.2.3 to its analog resulting from the short exact sequence of complexes $0 \to A^\cdot \to F^\cdot \oplus G^\cdot \to F^\cdot \underset{A^\cdot}{\cup} G^\cdot \to 0$.

Dually, given $f : F^\cdot \to A^\cdot$ and $g : G^\cdot \to A^\cdot$ one has a canonical homotopy pullback

$$\text{(1.1.2.5)} \qquad \begin{aligned} \left(F^\cdot \overset{h}{\underset{A}{\times}} G^\cdot\right) &= F^n \oplus A^{n-1} \oplus G^n \\ \partial(x, a, y) &= (\partial_F x,\ -\partial_A a + fx - gy,\ \partial_G y) . \end{aligned}$$

This indeed corresponds to the homotopy pushout in the dual category of complexes in $\mathcal{A}^{\mathrm{op}}$, and so has all the dual properties. As special cases, dual to the mapping cylinder is the mapping path space, and dual to the mapping cone is the homotopy fibre.

1.1.3. Several standard truncation functors are useful. Let $C^\cdot$ be a complex. There is brutal truncation

$$\sigma^k C^\cdot = \sigma^{\geq k} C^\cdot = \cdots \to 0 \to 0 \to C^k \to C^{k+1} \to C^{k+2} \to \dots .$$

This is a subcomplex of $C^\cdot$. The quotient $C^\cdot/\sigma^k C^\cdot$ is another brutal truncation, denoted $\sigma^{\leq k-1} C^\cdot$.

There is also the good truncation

$$C^\cdot\langle -k \rangle = \tau^k C^\cdot = \tau^{\geq k} C^\cdot = \cdots \to 0 \to \ \operatorname{im} \partial C^{k-1} \to C^k \to C^{k+1} \to \dots .$$

There is a quotient map $C^\cdot \twoheadrightarrow \tau^k C^\cdot$ which induces an isomorphism on cohomology H^n for all $n \geq k$. For $n \leq k-1$, $H^n(\tau C^\cdot) = 0$. The kernel of

$C^{\cdot} \to \tau^k C^{\cdot}$ is denoted $\tau^{\leq k-1}C^{\cdot}$. For $n \leq k-1$, $H^n(\tau^{\leq k-1}C^{\cdot}) = H^n(C^{\cdot})$, while $H^n(\tau^{\leq k-1}C^{\cdot}) = 0$ for $n \geq k$.

1.2.1. *Definition* ([W]). A category with cofibrations $\boldsymbol{A}$ is a category with a zero object 0, together with a chosen subcategory co($\boldsymbol{A}$) satisfying the three axioms:

1.2.1.1. Any isomorphism in $\boldsymbol{A}$ is a morphism in co($\boldsymbol{A}$).

1.2.1.2. For every object A in $\boldsymbol{A}$, the unique map $0 \to A$ is in co($\boldsymbol{A}$).

1.2.1.3. If $A \to B$ is a map in co($\boldsymbol{A}$), and $A \to C$ is a map in $\boldsymbol{A}$, then the pushout $B \cup_A C$ exists in $\boldsymbol{A}$, and the canonical map $C \to B \cup_A C$ is in co($\boldsymbol{A}$). In particular, $\boldsymbol{A}$ has finite coproducts.

1.2.2. One calls the morphisms in co($\boldsymbol{A}$) cofibrations. To distinguish them in diagrams, one usually denotes them by a feathered arrow "$\rightarrowtail$." Given $A \rightarrowtail B$, let the quotient B/A be the pushout $B \cup_A 0$ along $A \to 0$. One says that $A \rightarrowtail B \to C$ is a cofibration sequence if $B \to C$ is the canonical map $B \to B/A$ up to an isomorphism $C \cong B/A$. One then says $B \twoheadrightarrow C$ is a quotient map and denotes it by a double- headed arrow. Cofibration sequences are also called exact sequences.

In general, the set of quotient maps need not be closed under composition. Suppose however for a given category with cofibrations $\boldsymbol{A}$ that the set of quotient maps do form a subcategory quot($\boldsymbol{A}$), and moreover that the opposite category $\boldsymbol{A}^{\mathrm{op}}$ is a category with cofibration co($\boldsymbol{A}^{\mathrm{op}}$) = quot$(\boldsymbol{A})^{\mathrm{op}}$. Suppose further that the canonical map $A \cup B \to A \times B$ is always an isomorphism, where $A \times B$ is the product in $\boldsymbol{A}$, and the coproduct in $\boldsymbol{A}^{\mathrm{op}}$. Suppose also that $A \to B \to C$ is a cofibration sequence in $\boldsymbol{A}$ iff the dual sequence $A \leftarrow B \leftarrow C$ is a cofibration sequence in $\boldsymbol{A}^{\mathrm{op}}$. Under these conditions, one says $\boldsymbol{A}$ is a category with bifibrations. This concept is self dual, so $\boldsymbol{A}^{\mathrm{op}}$ is then a category with bifibrations.

Note that in a category with bifibrations, given a cofibration sequence $A \rightarrowtail B \twoheadrightarrow C$, that A is the quotient of B by C in $\boldsymbol{A}^{\mathrm{op}}$, so dually A must be the kernel of $B \twoheadrightarrow C$ in $\boldsymbol{A}$.

1.2.3. *Definition.* A Waldhausen category (in [W], "a category with cofibrations and weak equivalences") is a category with cofibrations $\boldsymbol{A}$, co($\boldsymbol{A}$), together with a subcategory $\boldsymbol{w}(\boldsymbol{A})$ of $\boldsymbol{A}$ satisfying the two axioms:

1.2.3.1. Any isomorphism in $\boldsymbol{A}$ is a morphism in $\boldsymbol{w}(\boldsymbol{A})$.

1.2.3.2. ("gluing lemma") Given a commutative diagram in $\boldsymbol{A}$

$$\begin{array}{ccccc} B & \leftarrowtail & A & \to & C \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\ B' & \leftarrowtail & A' & \to & C' \end{array}$$

with the two maps $A \rightarrowtail B$, $A' \rightarrowtail B'$ being cofibrations, and with the three maps $A \xrightarrow{\sim} A'$, $B \xrightarrow{\sim} B'$, and $C \xrightarrow{\sim} C'$ being in $\boldsymbol{w}(\boldsymbol{A})$, then the induced map $B \underset{A}{\cup} C \to B' \underset{A'}{\cup} C'$ is also in $\boldsymbol{w}(\boldsymbol{A})$.

The Waldhausen category consists of the triple data $\boldsymbol{A}$, co($\boldsymbol{A}$), $\boldsymbol{w}(\boldsymbol{A})$, but one usually abbreviates it as $\boldsymbol{A}$, or as $\boldsymbol{wA}$ when the choice of $\boldsymbol{w}(\boldsymbol{A})$ is particularly important.

One says the maps in $\boldsymbol{w}(\boldsymbol{A})$ are "weak equivalences," and denotes them by arrows with tildes"$\xrightarrow{\sim}$".

1.2.4. *Definition.* A biWaldhausen category is a category with bifibrations $\boldsymbol{A}$, co($\boldsymbol{A}$), quot($\boldsymbol{A}$), together with a subcategory $\boldsymbol{w}(\boldsymbol{A})$ such that both ($\boldsymbol{A}$, co($\boldsymbol{A}$), $\boldsymbol{w}(\boldsymbol{A})$) and the dual ($\boldsymbol{A}^{\mathrm{op}}$ quot$(\boldsymbol{A})^{\mathrm{op}}$, $\boldsymbol{w}(\boldsymbol{A})^{\mathrm{op}}$) are Waldhausen categories. That is, 1.2.3.1, 1.2.3.2, and the dual of 1.2.3.2 concerning pullbacks with $A \twoheadleftarrow B$ and $A' \twoheadleftarrow B'$ being in quot($\boldsymbol{A}$) all hold in the category with bifibrations $\boldsymbol{A}$.

This concept is self-dual, in that if $\boldsymbol{A}$ is a biWaldhausen category, so is $\boldsymbol{A}^{\mathrm{op}}$.

1.2.5. *Definition.* A saturated Waldhausen or biWaldhausen category is one where $\boldsymbol{w}(\boldsymbol{A})$ satisfies the saturation axiom: Given $A \xrightarrow{a} B \xrightarrow{b} C$ composible morphisms in $\boldsymbol{A}$, if any two of a, b, ba are in $\boldsymbol{w}(\boldsymbol{A})$, then so is the third.

1.2.6. *Definition.* An extensional Waldhausen or biWaldhausen category is one that satisfies the extension axiom: Given a commutative diagram whose rows are cofibration sequences

$$\begin{array}{ccccc} A & \rightarrowtail & B & \twoheadrightarrow & C \\ \downarrow a & & \downarrow b & & \downarrow c \\ A' & \rightarrowtail & B' & \twoheadrightarrow & C' \end{array}$$

if both a and c are in $\boldsymbol{w}(\boldsymbol{A})$, then so is b.

1.2.7. *Definition.* A functor $F : \boldsymbol{A} \to \boldsymbol{B}$ between two Waldhausen categories is exact if $F(\mathrm{co}(\boldsymbol{A})) \subseteq \mathrm{co}(\boldsymbol{B})$, if $F(\boldsymbol{w}(\boldsymbol{A})) \subseteq \boldsymbol{w}(\boldsymbol{B})$, and if F preserves pushouts along a cofibration. The last condition means that the canonical map $FC \underset{FA}{\cup} FB \to F(C \underset{A}{\cup} B)$ is an isomorphism whenever $A \rightarrowtail B$ is in co($\boldsymbol{A}$).

A functor $F : \boldsymbol{A} \to \boldsymbol{B}$ between two biWaldhausen categories is exact if both $F : (\boldsymbol{A}\ \mathrm{co}(\boldsymbol{A}),\ \boldsymbol{w}(\boldsymbol{A})) \to (\boldsymbol{B},\ \mathrm{co}(\boldsymbol{B}), \boldsymbol{w}(\boldsymbol{B}))$ and the dual F^{op} : $(\boldsymbol{A}^{\mathrm{op}},\ \mathrm{quot}(\boldsymbol{A})^{\mathrm{op}}, \boldsymbol{w}(\boldsymbol{A})^{\mathrm{op}}) \to (\boldsymbol{B}^{\mathrm{op}},\ \mathrm{quot}(\boldsymbol{B})^{\mathrm{op}}, \boldsymbol{w}(\boldsymbol{B})^{\mathrm{op}})$ are exact functors between Waldhausen categories. This is equivalent to saying that F : $(\boldsymbol{A},\ \mathrm{co}(\boldsymbol{A}), \boldsymbol{w}(\boldsymbol{A})) \to (\boldsymbol{B},\ \mathrm{co}(\boldsymbol{B}), \boldsymbol{w}(\boldsymbol{B}))$ is exact and F preserves pullbacks where one of the maps is a quotient map.

1.2.8. Any category with cofibrations $\boldsymbol{A}$ becomes a Waldhausen category by taking $\boldsymbol{w}(\boldsymbol{A})$ to have as morphisms all the isomorphisms in $\boldsymbol{A}$. Henceforth, we identify all categories with cofibrations to Waldhausen categories in this way.

1.2.9. *Example.* Let $\boldsymbol{A}$ be an abelian category (or more generally an exact category in the sense of Quillen [Q1]). Let $\mathrm{co}(\boldsymbol{A})$ consist of all monomorphisms in $\boldsymbol{A}$ (in the exact category case, let $\mathrm{co}(\boldsymbol{A})$ consist of all admissible monomorphisms [Q1]). Let $\boldsymbol{w}(\boldsymbol{A})$ consist of all isomorphisms in $\boldsymbol{A}$. Then $\boldsymbol{A}$ is a Waldhausen category, and in fact a biWaldhausen category.

1.2.10. *Example* (optional). Let $\boldsymbol{A}$ be the category of simplicial sets. Let $\mathrm{co}(\boldsymbol{A})$ consist of all monomorphisms. Let $\boldsymbol{w}(\boldsymbol{A})$ consist of all "weak equivalences," i.e., all maps that induce homotopy equivalences between geometric realizations. Then $\boldsymbol{A}$ is a Waldhausen category.

1.2.11. *Definition.* A complicial biWaldhausen category is a saturated extensional biWaldhausen category $\boldsymbol{A}$ formed from a category of chain complexes as follows: One takes an abelian category $\mathcal{A}$, whose choice is part of the structure. $\boldsymbol{A}$ is to be a full additive subcategory of the category $\mathcal{C}(\mathcal{A})$ of chain complexes in $\mathcal{A}$. $\mathrm{co}(\boldsymbol{A})$ is to contain at least all maps of complexes in $\boldsymbol{A}$ that are degree-wise split monomorphisms such that the quotient chain complex lies in $\boldsymbol{A}$. That is, $\mathrm{co}(\boldsymbol{A})$ contains all maps $C^{\cdot} \to D^{\cdot}$ in $\boldsymbol{A}$ such that for all integers n the map $C^n \to D^n$ is a split monomorphism in $\mathcal{A}$ and such that moreover the quotient chain complex $D^{\cdot}/C^{\cdot}$ in $\mathcal{C}(\mathcal{A})$ is also isomorphic to a complex in $\boldsymbol{A}$. $\mathrm{co}(\boldsymbol{A})$ may possibly contain other maps. One does require that if $C^{\cdot} \to D^{\cdot}$ is in $\mathrm{co}(\boldsymbol{A})$, then for all n, $C^n \to D^n$ is a monomorphism in $\mathcal{A}$, but not necessarily split.

Of course $\mathrm{co}(\boldsymbol{A})$ and the corresponding $\mathrm{quot}(\boldsymbol{A})$ must satisfy axiom 1.2.1.3 and its corresponding dual. We also demand that the pushouts and pullbacks required in $\boldsymbol{A}$ by these axioms are also the pushouts and pullbacks in the category of $\mathcal{C}(\mathcal{A})$; i.e., that $\boldsymbol{A}$ is a subcategory closed under the required pushouts and pullbacks.

$\boldsymbol{w}(\boldsymbol{A})$ is to contain all maps in $\boldsymbol{A}$ which are quasi-isomorphisms in the full category of complexes in $\mathcal{A}$. $\boldsymbol{w}(\boldsymbol{A})$ may contain other morphisms. Of course $\boldsymbol{w}(\boldsymbol{A})$ must satisfy the usual axioms 1.2.3.2 and its dual 1.2.4, and also the saturation and extension axioms 1.2.5 and 1.2.6.

In specifying a complicial biWaldhausen category, we make the default convention that unless explicitly specified otherwise, $\boldsymbol{w}(\boldsymbol{A})$ is to consist of exactly the quasi-isomorphisms, and that $\mathrm{co}(\boldsymbol{A})$ is to consist of exactly those degree-wise split monomorphisms whose cokernel is in $\boldsymbol{A}$.

1.2.12. *Example.* Let $\mathcal{E}$ be an exact category [Q1]. Then there is an abelian category $\mathcal{A}$ and a fully-faithful Gabriel-Quillen embedding $\mathcal{E} \to \mathcal{A}$, reflecting exactness and such that $\mathcal{E}$ is closed under extensions in $\mathcal{A}$, (cf Appendix A).

Let $\boldsymbol{A}$ be the full subcategory of complexes $C^{\cdot}$ in $\mathcal{A}$ with each C^k in $\mathcal{E}$, and with $C^k = 0$ unless $k = 0$. Take the cofibrations to be admissible monomorphisms, and the weak equivalences to be the quasi-isomorphisms. As $H^0(C^{\cdot}) = C^0$ for $C^{\cdot}$ in $\boldsymbol{A} \cong \mathcal{E}$, these weak equivalences are just the isomorphisms in $\mathcal{E}$.

This $\boldsymbol{A}$ is a complicial biWaldhausen category, which in fact is the bi-Waldhausen category of 1.2.9.

A more interesting example would be take $\boldsymbol{A}$ the category of complexes in $\mathcal{A}$ which are degree-wise in $\mathcal{E}$. See 1.11.6 below.

1.2.13. *Example.* Let $\mathcal{A}$ be an abelian category, and let $\boldsymbol{A}$ be the category of all chain complexes in $\mathcal{A}$. Then with the default conventions for $\mathrm{co}(\boldsymbol{A})$ and $\boldsymbol{w}(\boldsymbol{A})$, $\boldsymbol{A}$ is a complicial biWaldhausen category.

There are variants where $\boldsymbol{A}$ consists of the bounded complexes, the bounded above complexes, the cohomologically bounded complexes, etc..

1.2.14. *Example.* Let $\mathcal{A}$ be an abelian category, and let $\boldsymbol{A}$ be the category of all bounded below complexes $C^{\cdot}$ in $\mathcal{A}$ such that each C^k is an injective object of $\mathcal{A}$. Then $\boldsymbol{A}$ is complicial biWaldhausen.

1.2.15. *Example.* Let $\mathcal{A}$ be an abelian category with a thick abelian subcategory $\mathcal{B}$. Let $\boldsymbol{A}$ be the category of complexes $C^{\cdot}$ in $\mathcal{A}$ such that $C^{\cdot}$ is cohomologically bounded and with all cohomology groups $H^k(C^{\cdot})$ in $\mathcal{B}$. Then $\boldsymbol{A}$ is complicial biWaldhausen.

1.2.16. *Definition.* Let $\boldsymbol{A}$, $\boldsymbol{B}$ be complicial biWaldhausen categories. A complicial exact functor $F : \boldsymbol{A} \to \boldsymbol{B}$ is an exact functor of biWaldhausen categories in the sense of 1.2.7, with the additional property that the functor F of complexes is induced by degree-wise application of some additive functor $f : \mathcal{A} \to \mathcal{B}$ between the abelian categories chosen as part of the complicial structure.

Hence $F(C^{\cdot}) = \cdots \to f(C^k) \to f(C^{k+1}) \to \ldots.$

1.3.1. *Definition* ([W] 1.6). Let $\boldsymbol{A}$ be a Waldhausen category. A cylinder functor T on $\boldsymbol{A}$ is a functor $T :\ \mathrm{Cat}(\mathbf{1}, \boldsymbol{A}) \to \boldsymbol{A}$ to $\boldsymbol{A}$ from the category of morphisms in $\boldsymbol{A}$, together with three natural transformations p, j_1, j_2, satisfying the conditions below.

Thus to each morphism $f : A \to B$ in $\mathbf{A}$, T assigns an object Tf of $\mathbf{A}$. To each commutative square (1.3.1.1) in $\mathbf{A}$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A' & \xrightarrow{f'} & B' \end{array} \qquad (1.3.1.1)$$

T assigns functorially a morphism $T(a,b) : Tf \to Tf'$.

The natural transformations are maps $j_1 : A \to Tf$, $j_2 : B \to Tf$, and $p : Tf \to B$ such that $pj_1 = f : A \to B$, $pj_2 = 1 : B \to B$, and such that (1.3.1.2) commutes.

$$\begin{array}{ccccc} A \cup B & \xrightarrow{j_1 \cup j_2} & Tf & \xrightarrow{p} & B \\ {\scriptstyle a \cup b}\downarrow & & \downarrow{\scriptstyle T(a,b)} & & \downarrow{\scriptstyle b} \\ A' \cup B' & \xrightarrow[j_1 \cup j_2]{} & Tf' & \xrightarrow[p]{} & B' \end{array} \qquad (1.3.1.2)$$

We also require conditions 1.3.1.3 - 1.3.1.6 to hold.

1.3.1.3. $j_1 \cup j_2 : A \cup B \rightarrowtail Tf$ is in co($\mathbf{A}$).

1.3.1.4. If a and b are in $w(\mathbf{A})$, then $T(a,b)$ is in $w(\mathbf{A})$.

1.3.1.5. If a and b are in co($\mathbf{A}$), then not only is $T(a,b)$ in co($\mathbf{A}$), but also the map $Tf \underset{A \cup B}{\cup} A' \cup B' \to Tf'$ induced by the left square of (1.3.1.2) is in co($\mathbf{A}$).

1.3.1.6. $T(0 \to A) = A$, with p and j_2 the identity map.

To define a "cylinder functor satisfying the cylinder axiom," one imposes the extra cylinder axiom:

1.3.1.7. For all f, $p : Tf \to B$ is in $w(\mathbf{A})$.

(Note our 1.3.1.3 - 1.3.1.5 are equivalent to [W] 1.6 "Cyl 1").

1.3.2. *Definition.* If $\mathbf{A}$ is a biWaldhausen category, a cocylinder functor or mapping path space functor is a functor M : Cat($\mathbf{1}$,$\mathbf{A}$) $\to \mathbf{A}$ and natural transformations

$$\begin{array}{ccccc} & & M(f : A \to B) & & \\ & k_2 \swarrow & & \searrow k_1 & \nwarrow q \\ A & & B & & A \end{array} \qquad (1.3.2.1)$$

such that M is a cylinder functor on the dual $(\boldsymbol{A}^{\mathrm{op}}, \mathrm{quot}(\boldsymbol{A})^{\mathrm{op}}, \boldsymbol{w}(\boldsymbol{A})^{\mathrm{op}})$. M satisfies the cocylinder axiom if the dual of 1.3.1.7 holds, i.e., if $q : A \to M(A \to B)$ is always a weak equivalence.

1.3.3. *Example* (optional). The usual mapping cylinder of algebraic topology is a cylinder functor satisfying the cylinder axiom in the Waldhausen category of simplicial sets 1.2.10.

1.3.4. *Example.* Let $\mathcal{A}$ be an abelian category, and $\boldsymbol{A}$ the biWaldhausen category of all chain complexes in $\mathcal{A}$. Then $\boldsymbol{A}$ has well-known cylinder and cocylinder functors satisfying the cylinder and cocylinder axioms respectively. For given $f : A \to B$, let Tf be the canonical homotopy pushout $A \overset{h}{\underset{A}{\cup}} B$ of $1 : A \to A$ and $f : A \to B$ constructed as in 1.1.2. The maps $j_1 : A \to A \overset{h}{\underset{A}{\cup}} B$ and $j_2 : B \to A \overset{h}{\underset{A}{\cup}} B$ are the canonical inclusions $j_1(a) = (a, 0, 0)$, $j_2(b) = (0, 0, b)$. The map $p : A \overset{h}{\underset{A}{\cup}} B \to B$ is the morphism induced by $f : A \to B$, $1 : B \to B$, and the trivial homotopy $f1 \simeq 1f$ so that $p(a, a', b) = fa + b$. Dually, the canonical homotopy pullback $A \overset{h}{\underset{B}{\times}} B$ provides a cocylinder.

1.3.5. *Example.* Let $\boldsymbol{A}$ be a complicial biWaldhausen category with associated chosen abelian category $\mathcal{A}$. Suppose that those canonical homotopy pullbacks and canonical homotopy pushouts, formed in the category of complexes in $\mathcal{A}$ starting from diagrams in $\boldsymbol{A}$, are in fact objects of $\boldsymbol{A}$. Then we claim that the mapping cylinder and cocylinder functors of 1.3.4 induce mapping cylinder and cocylinder functors on the subcategory $\boldsymbol{A}$, provided only that the cofibration axiom 1.3.1.5 and its dual hold. Moreover, the cylinder axiom 1.3.1.7 and its dual cocylinder axiom hold automatically.

Note 1.3.1.3 holds automatically, as $j_1 \cup j_2$ is a degree-wise split monomorphism whose cokernel is the homotopy pushout of $A \to 0$ along $A \to 0$, and hence in $\boldsymbol{A}$. Thus $j_1 \cup j_2$ is in $\mathrm{co}(\boldsymbol{A})$. As p is a quasi-isomorphism, it is in $\boldsymbol{w}(\boldsymbol{A})$ and so 1.3.1.7 holds. Axiom 1.3.1.4 now follows from saturation 1.2.5, and axiom 1.3.1.6 is trivial. This leaves only 1.3.1.5 in doubt, proving the claim.

To verify 1.3.1.5 it suffices to show $Tf \underset{A \cup B}{\cup} A' \cup B' \to Tf'$ is in $\mathrm{co}(\boldsymbol{A})$, for the canonical map of Tf into $Tf \underset{A \cup B}{\cup} A' \cup B'$ is a cofibration by 1.2.1.3. Hence if the first map is a cofibration so is the composite $Tf \to Tf'$.

The map $Tf \underset{A \cup B}{\cup} A' \cup B' \to Tf'$ is given degree-wise as a sum of $A^{n+1} \to A'^{n+1}$ with identity maps

$$A'^n \oplus A^{n+1} \oplus B'^n \to A'^n \oplus A'^{n+1} \oplus B'^n. \tag{1.3.5.1}$$

If co($\boldsymbol{A}$) consists exactly of the degree-wise split monomorphisms whose quotients lie in $\boldsymbol{A}$, or else consists of all monomorphisms whose quotients lie in $\boldsymbol{A}$ then 1.3.1.5 and its dual also hold automatically. This also works if co($\boldsymbol{A}$) is defined to be all maps that are degree-wise admissible monomorphisms whose quotients lie in $\boldsymbol{A}$ for some exact subcategory $\mathcal{E} \subseteq \mathcal{A}$ that contains A^n for all $A^{\cdot}$ in $\boldsymbol{A}$ and all degrees n. Thus in all the usual examples, and in fact in all cases arising in Sections 2-11, axiom 1.3.1.5 also holds automatically.

1.3.6. *Example.* Let $\boldsymbol{A}$ be a complicial biWaldhausen category with associated abelian category $\mathcal{A}$. Suppose co($\boldsymbol{A}$) is one of the cases listed in 1.3.5 that make 1.3.1.5 automatic, e.g., all degree-wise admissible monomorphisms with quotients lying in $\boldsymbol{A}$. Suppose $\boldsymbol{A}$ is closed under finite degree shifts so that if $A^{\cdot}$ is in $\boldsymbol{A}$, so is $A^{\cdot}[k]$. Suppose $\boldsymbol{A}$ is closed under extensions, i.e., that if $0 \to A^{\cdot} \to B^{\cdot} \to C^{\cdot} \to 0$ is an exact sequence of complexes in $\mathcal{A}$ with $A^{\cdot}$ and $C^{\cdot}$ in $\boldsymbol{A}$ then $B^{\cdot}$ is isomorphic to a complex in $\boldsymbol{A}$. Then $\boldsymbol{A}$ has a mapping cylinder and cocylinder satisfying the cylinder and cocylinder axioms. This will follow from 1.3.5 once we see $\boldsymbol{A}$ is closed under formation of canonical homotopy pushouts and canonical homotopy pullbacks. But this follows from the hypotheses that $\boldsymbol{A}$ is closed under extensions and degree shifts and the exact sequence (1.1.2.4) and is dual.

1.4. We henceforth consider only small Waldhausen categories, those with a set, as opposed to a class of morphisms. Hence, when we speak of a Waldhausen category of all chain complexes of abelian groups or of $\mathcal{O}_X$-modules on a scheme, it is implicit that we are looking at the category of such complexes in a Grothendieck universe so that it is small with respect to a larger universe [SGA 4] I Appendice. (As in [SGA 4], the K-theory spectrum of the biWaldhausen categories of Section 2 and Section 3 will be independent of the choice of universe at least up to homotopy, as these biWaldhausen categories in the various universes will have equivalent derived categories, so 1.9.8 applies. See Appendix F).

1.5.1. *Definition* [W]. For $\boldsymbol{A}$ a Waldhausen category with weak equivalences $\boldsymbol{w} = \boldsymbol{w}(\boldsymbol{A})$ define $\boldsymbol{w}S_{\cdot}\boldsymbol{A}$ to be the following simplicial category.

The objects of the category in degree n, $\boldsymbol{w}S_n\boldsymbol{A}$ are the functors A, meeting the conditions below, to $\boldsymbol{A}$ from the partially ordered set of pairs of integers (i,j), with $0 \le i \le j \le n$. The partial order is defined by $(i,j) \le (i',j')$ iff both $i \le i'$ and $j \le j'$. The functors A must meet the conditions that for all j, $A(j,j) = 0$, and that for all (i,j,k) with $i \le j \le k$, the maps $A(i,j) \rightarrowtail A(i,k) \twoheadrightarrow A(j,k)$ form a cofibration sequence.

The morphisms of $\boldsymbol{w}S_n(\boldsymbol{A})$ are the natural transformations $A \to A'$

such that for all (i,j), $A(i,j) \to A'(i,j)$ is in $\boldsymbol{w}(\boldsymbol{A})$. By the gluing Lemma 1.2.3.2 and the cofibration sequence condition on A for $0 \le i \le j$, it suffices that all $A(0,k) \to A'(0,k)$ are in $\boldsymbol{w}(\boldsymbol{A})$.

Given $\varphi : n \to k$ in Δ^{op} corresponding to a monotone map $\varphi : \{0,1,\ldots,k\} \to \{0,1,\ldots,n\}$, the simplicial operator φ on $\boldsymbol{w}S.\boldsymbol{A}$ is the functor $\varphi : \boldsymbol{w}S_n\boldsymbol{A} \to \boldsymbol{w}S_k\boldsymbol{A}$ that sends the object $(i,j) \mapsto A(i,j)$ to the object $(r,s \mapsto A(\varphi(r),\varphi(s))$.

1.5.1.2. The category $\boldsymbol{w}S_n\boldsymbol{A}$ is equivalent via the forgetful functor to the category of subdiagrams $A_1 \rightarrowtail \cdots \rightarrowtail A_n = A(0,1) \rightarrowtail A(0,2) \rightarrowtail \cdots \rightarrowtail A(0,n)$. Indeed $A(i,j)$ is $A(0,j)/A(0,i) = A_j/A_i$ up to canonical natural isomorphism, so specifying the $A(i,j)$ for $i \neq 0$ just adds choices of objects determined up to isomorphism. However, it is necessary to specify these choices to make the simplicial identities hold strictly in $\boldsymbol{w}S.\boldsymbol{A}$, instead of just up to natural isomorphism.

1.5.2. For $\boldsymbol{A}$ a small Waldhausen category, taking the nerve in each degree of the simplicial category $\boldsymbol{w}S.\boldsymbol{A}$ yields a bisimplicial set $N.\boldsymbol{w}S.\boldsymbol{A}$. Waldhausen defines the K-theory space of $\boldsymbol{A}$ to be the loops on the geometric realization of this bisimplicial set, $K(\boldsymbol{A}) = \Omega|N.\boldsymbol{w}S.\boldsymbol{A}|$. One also denotes this $K(\boldsymbol{w}\boldsymbol{A})$ when it is important to distinguish among several possible choices of weak equivalences. The K-groups of $\boldsymbol{A}$ are the homotopy groups of $K(\boldsymbol{A})$.

This space $K(\boldsymbol{A})$ is in fact an "infinite loop space" by [W] 1.3.3 and 1.5.3, as Waldhausen shows that it is the zero-th space of a spectrum, i.e., of a sequence of spaces each of which is homotopy equivalent by a given map to the loops on the next space in the sequence. It is in fact better to work with this spectrum than with the space. The proofs of [W] (and of [Q1], [Grl], etc.) immediately generalize to give "infinite loop space" versions of its results which are then valid for $K(\boldsymbol{A})$ as a spectrum.

1.5.3. *Definition.* For $\boldsymbol{A}$ a small Waldhausen category with weak equivalences $\boldsymbol{w}$, define $K(\boldsymbol{A}) = K(\boldsymbol{w}\boldsymbol{A})$ to be the spectrum constructed from $\boldsymbol{A}$ by the process of [W] 1.3.3 and remark, 1.5.3, and whose 0th space is $\Omega|N.\boldsymbol{w}S.\boldsymbol{A}|$. Define the Waldhausen K-groups $K_n(\boldsymbol{w}\boldsymbol{A})$ to be the homotopy groups of the spectrum, $\pi_n K(\boldsymbol{w}\boldsymbol{A})$. Note these groups are 0 if $n \le -1$, and are isomorphic to the homotopy groups of the space $\Omega|N.\boldsymbol{w}S.\boldsymbol{A}|$ for $n \ge 0$.

1.5.4. An exact functor $F : \boldsymbol{A} \to \boldsymbol{B}$ induces a simplicial functor $\boldsymbol{w}S.\boldsymbol{A} \to \boldsymbol{w}S.\boldsymbol{B}$, and a map of spectra $KF : K(\boldsymbol{A}) \to K(\boldsymbol{B})$. This makes K a functor.

If $\eta : F \to G$ is a natural transformation of exact functors $\boldsymbol{A} \to \boldsymbol{B}$, and if for all objects A in $\boldsymbol{A}$, $\eta A : FA \xrightarrow{\sim} GA$ is in $\boldsymbol{w}(\boldsymbol{B})$, then η induces a homotopy $\boldsymbol{w}S.F \simeq \boldsymbol{w}S.G$, and in fact a homotopy of maps of spectra

$KF \simeq KG$.

See [W] for details, and [Th3] A for homotopies of maps of spectra.

1.5.5. If $\boldsymbol{A}$ is a biWaldhausen category, we define $K(\boldsymbol{A})$ using the underlying Waldhausen category ($\boldsymbol{A}$, co($\boldsymbol{A}$), $w(\boldsymbol{A})$). But since $A \to B \to C$ is a cofibration sequence in $\boldsymbol{A}$ iff $C \to B \to A$ is a cofibration sequence in $\boldsymbol{A}^{\mathrm{op}}$, there is a canonical isomorphism of simplicial categories $(\boldsymbol{w}S_.\boldsymbol{A})^{\mathrm{op}} \cong \boldsymbol{w}S_.(\boldsymbol{A}^{\mathrm{op}})$. From the canonical isomorphism between the classifying space $|N_.(\ )|$ of a category and of its dual ([Q1] Section 1(3)), we deduce a canonical duality isomorphism of spectra $K(\boldsymbol{A}) \cong K(\boldsymbol{A}^{\mathrm{op}})$. This allows us to dualize all theorems of [W] when applied to biWaldhausen categories.

1.5.6. It is easy to derive the following formula for $K_0(\boldsymbol{wA})$ from the edge-path group presentation of $K_0(\boldsymbol{wA}) = \pi_0\Omega|N_.\boldsymbol{w}S_.\boldsymbol{A}| = \pi_1|N_.\boldsymbol{w}S_.\boldsymbol{A}|$.

$K_0(\boldsymbol{wA})$ is the free group (or the free abelian groups) on generators $[A]$ as A runs over the objects of $\boldsymbol{A}$, modulo the two relations

1.5.6.1. $[A] = [B]$ if there is a map $A \xrightarrow{\sim} B$ in $\boldsymbol{w}(\boldsymbol{A})$.

1.5.6.2. $[B] = [A][B/A]$ for all cofibration sequences $A \rightarrowtail B \twoheadrightarrow B/A$.

Note that relation 1.5.6.2 applied to $A \rightarrowtail A \cup B \twoheadrightarrow B$ and $B \rightarrowtail A \cup B \twoheadrightarrow A$ forces $[A][B] = [A \cup B] = [B][A]$. Thus $K_0(\boldsymbol{wA})$ is abelian and we usually write the relation additively: $[B] = [A] + [B/A]$. (Also 1.5.6.2 forces $[0] = 0$.)

1.5.7. If $\boldsymbol{A}$ has a mapping cylinder satisfying the cylinder axiom, let ΣA be the cone of $A \to 0$. Thus $A \rightarrowtail T(A \to 0) \twoheadrightarrow \Sigma A$ is a cofibration sequence. As $T(A \to 0) \simeq 0$, it follows that $-[A] = [\Sigma A]$ in $K_0(\boldsymbol{A})$. Hence $[B] - [A] = [B \vee \Sigma A]$. Thus every element of $K_0(\boldsymbol{A})$ is the class $[C]$ of some C in $\boldsymbol{A}$.

1.6. We now turn to the basic theorems of Waldhausen K- theory: the additivity, localization, approximation, and cofinality theorems. We will not expose Waldhausen's cell filtration theorem [W] 1.7, but it will be cited later in an optional exercise 5.7.

1.7.1. Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ be small Waldhausen categories, with exact functors $\boldsymbol{A} \to \boldsymbol{C}$ and $\boldsymbol{B} \to \boldsymbol{C}$ which are inclusions of the underlying categories.

Let the category of "exact sequences" $E(\boldsymbol{A},\boldsymbol{C},\boldsymbol{B})$ be the category whose objects are those cofibration in $\boldsymbol{C}$, $A \rightarrowtail C \twoheadrightarrow B$, which have A in $\boldsymbol{A}$ and B in $\boldsymbol{B}$. The morphism of $E(\boldsymbol{A},\boldsymbol{C},\boldsymbol{B})$ are commutative diagrams in $\boldsymbol{C}$

$$\begin{array}{ccccc} A & \rightarrowtail & C & \twoheadrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ A' & \rightarrowtail & C' & \twoheadrightarrow & B' \end{array}$$

with $A \to A'$ a morphism in $\boldsymbol{A}$ and $B \to B'$ a morphism in $\boldsymbol{B}$. Such a morphism in $E(\boldsymbol{A},\boldsymbol{C},\boldsymbol{B})$ is a cofibration if $A \to A'$, $B \to B'$, and $A' \cup_A C \to C'$ are cofibrations in $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ respectively. Note then that $C \to C'$ is a cofibration in $\boldsymbol{C}$, as it is the composite $A \cup_A C \rightarrowtail A' \cup_A C \rightarrowtail C'$.

A morphism in $E(\boldsymbol{A},\boldsymbol{C},\boldsymbol{B})$ is a weak equivalence if $A \to A'$, $B \to B'$, and $C \to C'$ are weak equivalences in $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ respectively.

This gives $E(\boldsymbol{A},\boldsymbol{C},\boldsymbol{B})$ the structure of a Waldhausen category. There are exact functors s, t, q from $E(\boldsymbol{A},\boldsymbol{C},\boldsymbol{B})$ to $\boldsymbol{A}$, $\boldsymbol{C}$, and $\boldsymbol{B}$ respectively, sending $A \rightarrowtail C \twoheadrightarrow B$ to A, C, and B respectively. There is also an exact functor $\cup : \boldsymbol{A} \times \boldsymbol{B} \to E(\boldsymbol{A},\boldsymbol{C},\boldsymbol{B})$ sending (A,B) to $A \rightarrowtail A \cup B \twoheadrightarrow B$. This functor splits $(s,q) : E(\boldsymbol{A},\boldsymbol{C},\boldsymbol{B}) \to \boldsymbol{A} \times \boldsymbol{B}$.

1.7.2. **Additivity Theorem**([W] 1.3.2, 1.4.2). *Take the notation and make the hypotheses of* 1.7.1. *Then the exact functors* (s,q) *induce a homotopy equivalence of* K*- theory spectra*

$$K(s,q) : K(E(\boldsymbol{A},\boldsymbol{C},\boldsymbol{B})) \xrightarrow{\sim} K(\boldsymbol{A}) \times K(\boldsymbol{B}).$$

A natural homotopy inverse to this map is $K(\cup)$ *induced by* $\cup : \boldsymbol{A}\times\boldsymbol{B} \to E(\boldsymbol{A},\boldsymbol{C},\boldsymbol{B})$.

Proof. See [W] 1.3.2, Section 1.4.

1.7.3. **Corollary** ([W] 1.3.2(4)). *Let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be small Waldhausen categories, and let* F, F', F'' : $\boldsymbol{A} \to \boldsymbol{B}$ *be three exact functors. Suppose there are natural transformations* $F' \to F$ *and* $F \to F''$ *such that the following two conditions hold:*

1.7.3.1. *For all* A *in* $\boldsymbol{A}$, $F'A \rightarrowtail FA \twoheadrightarrow F''A$ *is a cofibration sequence.*

1.7.3.2. *For any cofibration* $A' \rightarrowtail A$ *in* $\boldsymbol{A}$, *the induced map* $F'A \cup_{F'A'} FA' \to FA$ *is a cofibration.*

Then there is a homotopy of maps of spectra $KF \simeq KF' + KF''$: $K(\boldsymbol{A}) \to K(\boldsymbol{B})$.

Proof. The natural cofibration sequence $F' \rightarrowtail F \twoheadrightarrow F''$ induces an exact functor $\boldsymbol{A} \to E(\boldsymbol{B},\boldsymbol{B},\boldsymbol{B})$. The additivity theorem 1.7.2 implies a homotopy $Kt \simeq Ks + Kq$ of maps $K(E(\boldsymbol{B},\boldsymbol{B},\boldsymbol{B})) \to K(\boldsymbol{B})$, since these maps become equal after composing with the homotopy equivalence $K(\cup)$. Composing the homotopy $Kt \simeq Ks + Kq$ with the map $K(\boldsymbol{A}) \to K(E(\boldsymbol{B},\boldsymbol{B},\boldsymbol{B}))$ yields a homotopy $KF \simeq KF' + KF''$.

1.7.4. When $\boldsymbol{B}$ is a complicial biWaldhausen category, hypothesis 1.7.3.2 is superfluous as it follows automatically from 1.7.3.1 and exactness of F', F, and F''. For one notes that

$$F'A \cup_{F'A'} FA'/F'A \cong 0 \cup_{F'A'} FA' \cong FA'/F'A' \cong F''A'.$$

Thus the diagram below has cofibration sequences as rows

$$\begin{array}{ccccc} F'A & \rightarrowtail & F'A \underset{F'A'}{\cup} FA' & \twoheadrightarrow & F''A' \\ \| & & \downarrow & & \downarrow \\ F'A & \rightarrowtail & FA & \twoheadrightarrow & F''A \\ & & & & \twoheaddownarrow \\ & & & & F''(A/A') \end{array}$$

As $\boldsymbol{B}$ is biWaldhausen, the composite map $FA \twoheadrightarrow F''(A/A')$ is in quot($\boldsymbol{B}$), and its kernel is a cofibration into FA. But as $\boldsymbol{B}$ is complicial, this kernel is the same as the kernel taken in the category of chain complexes in the associated abelian category $\mathcal{B}$. Applying the snake lemma to the above diagram in the category of chain complexes, we see that the kernel is $F'A \underset{F'A'}{\cup} FA' \to FA$, and so this map is a cofibration as required by 1.7.3.2.

1.8.1. Let $\boldsymbol{A}$ be a small category with cofibrations. Suppose $\boldsymbol{A}$ has two subcategories $\boldsymbol{v}(\boldsymbol{A})$ and $\boldsymbol{w}(\boldsymbol{A})$, each of which is the category of weak equivalences for a Waldhausen category structure on $\boldsymbol{A}$, $\boldsymbol{vA}$ and $\boldsymbol{wA}$. Suppose $\boldsymbol{v}(\boldsymbol{A}) \subseteq \boldsymbol{w}(\boldsymbol{A})$, and that $\boldsymbol{wA}$ satisfies the extension and saturation axioms.

Let $\boldsymbol{A}^w$ be the full subcategory of $\boldsymbol{A}$ whose objects are the A such that $0 \to A$ is in $\boldsymbol{w}(\boldsymbol{A})$, i.e., which are $\boldsymbol{w}$-acyclic. This $\boldsymbol{A}^w$ becomes a Waldhausen category $\boldsymbol{vA}^w$ with $\mathrm{co}(\boldsymbol{A}^w) = \mathrm{co}(\boldsymbol{A}) \cap \boldsymbol{A}^w$ and $\boldsymbol{v}(\boldsymbol{A}^w) = \boldsymbol{v}(\boldsymbol{A}) \cap \boldsymbol{A}^w$. If $\boldsymbol{vA}$ and $\boldsymbol{wA}$ are biWaldhausen, so is $\boldsymbol{vA}^w$. If $\boldsymbol{A}$ has a functor T which is a cylinder functor both for $\boldsymbol{vA}$ and for $\boldsymbol{wA}$, T induces a cylinder functor on $\boldsymbol{A}^w$.

1.8.2. **Localization Theorem** ([W] 1.6.4 Fibration Theorem). *With the notation and hypotheses of* 1.8.1, *suppose also that* $\boldsymbol{A}$ *has a functor* T *which is a cylinder functor both for* $\boldsymbol{vA}$ *and for* $\boldsymbol{wA}$, *and that* T *satisfies the cylinder axiom* 1.3.1.7 *for* $\boldsymbol{wA}$.

Then the exact inclusion functors $\boldsymbol{vA}^w \to \boldsymbol{vA}$, $\boldsymbol{vA} \to \boldsymbol{wA}$, *induce a homotopy fibre sequence of spectra*

$$K(\boldsymbol{vA}^w) \to K(\boldsymbol{vA}) \to K(\boldsymbol{wA}).$$

(The requisite chosen nullhomotopy of $K(\boldsymbol{vA}^w) \to K(\boldsymbol{wA})$ *is induced by the natural weak equivalence* $0 \xrightarrow{\sim} A$ *in* $\boldsymbol{wA}$ *for* A *in* $\boldsymbol{vA}^w$.)

Proof. [W] 1.6.4.

1.9.1. **Approximation Theorem** ([W] 1.6.7). *Let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be small saturated Waldhausen categories. Suppose* $\boldsymbol{A}$ *has a cylinder functor satisfying the cylinder axiom 1.3.1.7. Let* $F : \boldsymbol{A} \to \boldsymbol{B}$ *be an exact functor satisfying the two conditions:*

1.9.1.1. A morphism f *of* $\boldsymbol{A}$ *is in* $\boldsymbol{w}(\boldsymbol{A})$ *if and only if* Ff *is in* $\boldsymbol{w}(\boldsymbol{B})$.

1.9.1.2. Given any A in $\boldsymbol{A}$ and any $x : FA \to B$ in $\boldsymbol{B}$, there is an A' in $\boldsymbol{A}$, a map $a : A \to A'$ in $\boldsymbol{A}$, and a weak equivalence $x' : FA' \xrightarrow{\sim} B$ in $w(\boldsymbol{B})$ such that $x = x' \circ Fa$.

Then under these conditions, F induces a homotopy equivalence $KF : K(\boldsymbol{A}) \xrightarrow{\sim} K(\boldsymbol{B})$.

Proof. This results from Waldhausen's version [W] 1.6.7. Condition 1.9.1.2 appears to be weaker than the corresponding condition "App 2" of [W] in that 1.9.1.2 does not require the map $a : A \to A'$ to be a cofibration. But given $x = x' \circ Fa$ as in 1.9.1.2, one applies the cylinder functor to $a : A \to A'$ to factor $a = a'' \circ a'$, with a' the cofibration $A \rightarrowtail A'' = T(a)$, and a'' the weak equivalence $A'' = T(a) \xrightarrow{\sim} A'$. Then $x'' = x' \circ Fa'' : FA'' \xrightarrow{\sim} B$ is a weak equivalence, $a' : A \rightarrowtail A''$ is a cofibration, and $x = x'' \circ Fa'$. Hence 1.9.1.2 implies "App 2" of [W] in the presence of the other hypotheses.

1.9.2.**Theorem.** *Let $\boldsymbol{A}$ be a small complicial biWaldhausen category. Let $\boldsymbol{A}'$ be the new complicial biWaldhausen structure on $\boldsymbol{A}$ where $w(\boldsymbol{A}') = w(\boldsymbol{A})$, but where co$(\boldsymbol{A}')$ consists exactly of those degree-wise split monomorphisms whose quotient lies in $\boldsymbol{A}$. Suppose that $\boldsymbol{A}'$ has a cylinder functor satisfying the cylinder axiom, as often occurs* (cf. 1.3.5, 1.3.6).

Then the exact inclusion functor $\boldsymbol{A}' \to \boldsymbol{A}$ induces a homotopy equivalence $K(\boldsymbol{A}') \xrightarrow{\sim} K(\boldsymbol{A})$.

Proof. Apply 1.9.1 to the inclusion $\boldsymbol{A}' \to \boldsymbol{A}$. Condition 1.9.1.1 is obvious, and 1.9.1.2 holds trivially with $a = x$, $x' = 1$.

Something much like 1.9.2 was proved in [HS] by Hinich and Shekhtman before the unveiling of Waldhausen's approximation theorem.

1.9.2.1 *Remark.* Theorem 1.9.2 shows it is usually harmless in K-theory to impose the condition that co$(\boldsymbol{A})$ consists of precisely the degree-wise split monomorphisms with quotients lying in $\boldsymbol{A}$, at least in the presence of cylinders. This is convenient, since then any complicial functor $F : \boldsymbol{A} \to \boldsymbol{B}$ induced by an additive $f : \mathcal{A} \to \mathcal{B}$ automatically will preserve cofibrations and pushouts along cofibrations.

1.9.3. In applications of the approximation theorem, most of the effort involved is expended in verifying condition 1.9.1.2. The following lemmas 1.9.4 and 1.9.5 are useful tools for this, and also for some other purposes. Lemma 1.9.5 serves to build bounded above complexes quasi-isomorphic to a given complex, and lemma 1.9.4 serves to truncate them to strict bounded complexes.

1.9.4. **Lemma.** *Let $\mathcal{A}$ be an abelian category, and let $\mathcal{B}$ be a full additive subcategory of $\mathcal{A}$. Suppose that $\mathcal{B}$ is closed under extensions in $\mathcal{A}$, and is closed under kernels of epimorphisms. (More precisely, $\mathcal{B}$ is to be closed under taking kernels of maps in $\mathcal{B}$ that are epimorphisms in $\mathcal{A}$.) Let $C^{\cdot}$ be a strictly bounded above complex in $\mathcal{B} \subseteq \mathcal{A}$. Then:*

1.9.4(a). *If for an integer n, one has $H^k(C^{\cdot}) = 0$ for $k \neq n$ and $C^k = 0$ for $k < n$, then $H^n(C^{\cdot}) = Z^n C^{\cdot}$ is an object of $\mathcal{B}$.*

1.9.4(b). *If for an integer n, one has $H^k(C^{\cdot}) = 0$ ıor $k > n$, then $Z^n C^{\cdot}$ is an object of $\mathcal{B}$, and the complex $C^{\cdot}$ is quasi-isomorphic to the subcomplex $\tau^{\leq n} C^{\cdot}$, which is the complex in $\mathcal{B}$*

$$\cdots \to C^{n-2} \to C^{n-1} \to Z^n C \to 0 \to 0 \to \cdots .$$

1.9.4(c). *If for an integer n, one has $H^k(C^{\cdot}) = 0$ for $k \neq n$, then $H^n(C^{\cdot})$ has a resolution by objects of $\mathcal{B}$*

$$\cdots \to C^{n-2} \to C^{n-1} \to Z^n C \to H^n(C^{\cdot}) \to 0.$$

If $C^{\cdot}$ is also strictly bounded below, this resolution has finite length.

1.9.4(d). *If $C^{\cdot}$ is an acyclic complex, so $H^k(C^{\cdot}) = 0$ for all k, then all $Z^k C^{\cdot} = B^k C^{\cdot}$ are objects of $\mathcal{B}$, and $C^{\cdot}$ has a natural filtration by acyclic complexes in $\mathcal{B}$, $F_n C^{\cdot} = \tau^{\leq n} C^{\cdot}$, so that $F_{n+1} C^{\cdot} / F_n C^{\cdot}$ is isomorphic to the acyclic complex*

$$\cdots \to 0 \to 0 \to B^{n+1} C^{\cdot} \cong B^{n+1} C^{\cdot} \to 0 \to 0 \to \cdots .$$

Proof. As $C^{\cdot}$ is strict bounded above, there is an integer N such that $C^p = 0$ for $p > N$. If $N \leq n$, $Z^n C^{\cdot} = C^n$ and 1.9.4(b) is obvious. One proceeds to prove 1.9.4(b) by induction on $N - n$. If it is already known to be true for smaller values of $N - n$, and $N - n \geq 1$, consider $C^{\cdot} = \cdots \to C^{N-1} \to C^N \to 0 \to \cdots$. As $N > n$, $H^N(C^{\cdot}) = 0$ and $C^{N-1} \to C^N$ is an epimorphism in $\mathcal{A}$ of objects in $\mathcal{B}$. AS $\mathcal{B}$ is closed under taking kernels of such epimorphisms, the kernel $Z^{N-1} C^{\cdot}$ is in $\mathcal{B}$. Then $C^{\cdot}$ is quasi-isomorphic to the shorter subcomplex

$$C^{\cdot\prime} = \tau^{\leq N-1} C^{\cdot} = \cdots \to C^{N-2} \to Z^{N-1} C^{\cdot} \to 0 \to 0 \to \cdots$$

which is also a complex in $\mathcal{B}$. Clearly $Z^k C^{\cdot} = Z^k C^{\cdot\prime}$ for all $k \leq N - 1$, and as $C^{\cdot\prime}$ is shorter we get $Z^k C^{\cdot\prime}$ is in $\mathcal{B}$ for all $k \geq n$ by the induction hypotheses. Thus $Z^k C^{\cdot}$ is in $\mathcal{B}$ for $n \leq k \leq N - 1$. We have that $Z^N C^{\cdot} = C^N$ is in $\mathcal{B}$, and clearly $Z^k C^{\cdot} = 0$ is in $\mathcal{B}$ for $k > N$. Thus $Z^k C^{\cdot}$ is in $\mathcal{B}$ for all $k \geq n$, completing the induction step, and hence the proof of 1.9.4(b). Now 1.9.4(a) and 1.9.4(c) are immediate corollaries and 1.9.4(d) is a porism (i.e., follows from the proof of 1.9.4(b)).

1.9.5. **Lemma** (cf. [SGA 6] I 1.4). *Let $\mathcal{A}$ be an abelian category, let $\mathcal{D}$ be an additive category, and let $F : \mathcal{D} \to \mathcal{A}$ be an additive functor. Let $\mathcal{C}$ be a full subcategory of the category $C(\mathcal{A})$ of chain complexes in $\mathcal{A}$, such that any complex quasi-isomorphic to a complex in $\mathcal{C}$ is also in $\mathcal{C}$. Suppose that every complex in $\mathcal{C}$ is cohomologically bounded above. Suppose that if $D^{\cdot}$ is any strict bounded complex in $\mathcal{D}$, the $F(D^{\cdot})$ is in $\mathcal{C}$, and that $\mathcal{C}$ contains the mapping cone of any map of complexes $F(D^{\cdot}) \to C^{\cdot}$ with $C^{\cdot}$ in $\mathcal{C}$ and $D^{\cdot}$ strict bounded in $\mathcal{D}$.*

Suppose the key condition 1.9.5.1 holds, so "$\mathcal{D}$ has enough objects to resolve":

1.9.5.1. For any integer n, any $C^{\cdot}$ in $\mathcal{C}$ such that $H^i(C^{\cdot}) = 0$ for $i \geq n$, and any epimorphism in $\mathcal{A}$, $A \twoheadrightarrow H^{n-1}(C^{\cdot})$, then there exists a D in $\mathcal{D}$ and a map $FD \to A$ such that the composite $FD \twoheadrightarrow H^{n-1}(C^{\cdot})$ is an epimorphism in $\mathcal{A}$.

Suppose finally that the functor $F : \mathcal{D} \to \mathcal{A}$ satisfies the following two conditions (which trivially hold in the usual case where F is a fully faithful inclusion):

1.9.5.2. Given a morphism $f : FD_2 \to FD_1$ in $\mathcal{A}$, there is a map $d : D_3 \to D_2$ in $\mathcal{D}$ such that $Fd : FD_3 \to FD_2$ is an epimorphism in $\mathcal{A}$ and such that $f \circ Fd = Ff'$ for some map $f' : D_3 \to D_1$ in $\mathcal{D}$.

1.9.5.3. Given $h : D_2 \to D_1$ in $\mathcal{D}$ with $Fh = 0$, there is a map $d : D_3 \to D_2$ in $\mathcal{D}$ with $Fd : FD_3 \to FD_2$ an epimorphism in $\mathcal{A}$ and $hd = 0$ in $\mathcal{D}$.

Then: For any $D^{\cdot}$ in $C^-(\mathcal{D})$ with $F(D^{\cdot})$ in $\mathcal{C}$, any $C^{\cdot}$ in $\mathcal{C}$, and any map $x : FD^{\cdot} \to C^{\cdot}$, there exists a D' in $C^-(\mathcal{D})$ with $F(D'^{\cdot})$ in $\mathcal{C}$, a degree-wise split monomorphism $a : D^{\cdot} \to D'^{\cdot}$, and a quasi- isomorphism $x' : FD'^{\cdot} \xrightarrow{\sim} C^{\cdot}$ such that $x = x' \circ Fa$.

Moreover, if $x : FD^{\cdot} \to C^{\cdot}$ is an n-quasi-isomorphism for some integer n (i.e., $H^i(x)$ is iso for $i > n$ and epi for $i = n$), then one may choose $D'^{\cdot}$ above so that $a : D^k \to D'^k$ is an isomorphism for $k \geq n$.

Proof. We construct $D'^{\cdot}$ by induction, constructing what will become the brutal truncation $\sigma^n D'^{\cdot}$ given $\sigma^{n+1} D'^{\cdot}$ (recall σ from 1.1.3). To start, take n large enough so that $H^i(x)$ is an isomorphism for $i > n$ and an epimorphism for $i = n$. This is possible because $FD^{\cdot}$ and $C^{\cdot}$ are cohomologically bounded above, so $H^i(FD^{\cdot}) = 0 = H^i(C^{\cdot})$ for $i >> 0$. For this large n, we set $\sigma^n D'^{\cdot} = \sigma^n D^{\cdot}$ to begin the induction.

Now assume as induction hypothesis that for some n we have $\sigma^n D'^{\cdot}$ and maps $\sigma^n a$, $\sigma^n x'$ so that

1.9.5.4.1. $\sigma^n D'^{\cdot}$ is bounded complex in $\mathcal{D}$, and $\sigma^n(a) : \sigma^n D^{\cdot} \to \sigma^n D'^{\cdot}$ is a degree-wise split monomorphism. (Moreover, we need to assume that $(\sigma^n D'^{\cdot})^k = 0$ for $k < n$, so that $\sigma^n D'^{\cdot}$ could be the truncation σ^n of some complex.)

1.9.5.4.2. $\sigma^n x = \sigma^n x' \cdot F(\sigma^n a)$.

1.9.5.4.3. $\sigma^n x'$ is an n-quasi-isomorphism.

The induction step consists of defining a complex $\sigma^{n-1}D'^{\cdot}$ in $\mathcal{D}$ whose σ^n is indeed the given $\sigma^n D'^{\cdot}$, and maps $\sigma^{n-1}x'$, $\sigma^{n-1}a$ which restrict to the given $\sigma^n x'$, $\sigma^n a$, and which satisfy 1.9.5.4.1 - 1.9.5.4.3 with n replaced by $n-1$.

Let $M^{\cdot}$ be the mapping cone of $\sigma^n x' : F(\sigma^n D'^{\cdot}) \to C^{\cdot}$

$$(1.9.5.6) \quad M^{\cdot} = \cdots \to C^{n-3} \to C^{n-2} \to FD'^n \oplus C^{n-1} \to FD'^{n+1} \oplus C^n \to \ldots.$$

Then $M^{\cdot}$ is in $\mathcal{C}$. The long exact cohomology sequence for the mapping cone and the fact that $\sigma^n x'$ is an n-quasi-isomorphism yield that $H^i(M^{\cdot}) = 0$ for $i \geq n$. By the key hypothesis 1.9.5.1, there is a $D^{\wedge}$ in $\mathcal{D}$ and a map $FD^{\wedge} \to Z^{n-1}M^{\cdot} \twoheadrightarrow H^{n-1}(M^{\cdot})$ so that $FD^{\wedge} \twoheadrightarrow H^{n-1}(M^{\cdot})$ is an epimorphism. As $M^{n-1} = FD'^n \oplus C^{n-1}$, and (1.1.2.2) shows that the differential is given by $\partial(d,c) = (-\partial d,\ \partial c - (\sigma^n x')(d))$ in $M^n = FD'^{n+1} \oplus C^n$, one easily checks that $Z^{n-1}M^{\cdot} = \ker \partial$ is the fibre product of $\partial : C^{n-1} \to Z^n C$ and $\sigma^n x' : Z^n F\sigma^n D'^{\cdot} \to Z^n C^{\cdot}$

$$(1.9.5.7) \qquad Z^{n-1}M^{\cdot} = Z^n F\sigma^n D'^{\cdot} \underset{Z^n C^{\cdot}}{\times} C^{n-1} \subseteq FD'^n \oplus C^{n-1}.$$

To simplify notation we write $Z^n FD'^{\cdot}$ for $Z^n F\sigma^n D'^{\cdot}$, as if the rest of $D'^{\cdot}$ already existed. Consider now the composite map induced by the canonical projection and inclusion maps

$$FD^{\wedge} \to Z^{n-1}M^{\cdot} \cong Z^n FD'^{\cdot} \underset{Z^n C^{\cdot}}{\times} C^{n-1} \to Z^n FD'^{\cdot} \to FD'^n.$$

By 1.9.5.2, there is a map $D^{\sim} \to D^{\wedge}$ with $FD^{\sim} \twoheadrightarrow FD^{\wedge} \twoheadrightarrow H^{n-1}(M^{\cdot})$ an epimorphism and with the map $FD^{\sim} \to FD^{\wedge} \to FD'^n$ being F of a map $D^{\sim} \to D'^n$. Replacing the old $D^{\wedge}$ by $D^{\sim}$, we may assume that $FD^{\wedge} \to FD'^n$ is F of a map $D^{\wedge} \to D'^n$. As $FD^{\wedge} \to FD'^n$ factors through $Z^n FD'^{\cdot}$, the composite $FD^{\wedge} \to FD'^n \to FD'^{n+1}$ is 0. By 1.9.5.3 there is a $D^{\sim} \to D^{\wedge}$ in $\mathcal{D}$ so that $FD^{\sim} \twoheadrightarrow FD^{\wedge} \twoheadrightarrow H^{n-1}(M^{\cdot})$ is still an epimorphism, and such that the composite $D^{\sim} \to D^{\wedge} \to D'^n \to D'^{n+1}$ is 0. Replacing the old $D^{\wedge}$ with this $D^{\sim}$, we may assume that $D^{\wedge} \to D'^n$ factors through $Z^n D'^{\cdot}$ $(= Z^n \sigma^n D'^{\cdot})$.

Now set D'^{n-1} to be $D^{n-1} \oplus D^{\wedge}$. Define $\partial : D'^{n-1} \to D'^n$ to be $a \circ \partial : D^{n-1} \to D^n \to D'^n$ on the summand D^{n-1}, and to be $D^{\wedge} \to D'^n$ on the summand $D^{\wedge}$. As $D^{\wedge} \to D'^n$ factors through $Z^n D'^{\cdot}$, and as $\partial a \partial = a\partial^2 = 0$ on D^{n-1}, we see that the composite $\partial^2 : D'^{n-1} \to D'^n \to$

D'^{n+1} is 0. Now let $\sigma^{n-1}D'^{\cdot}$ be the chain complex formed from $\sigma^n D'^{\cdot}$ by replacing the old 0 in degree $n-1$ by D'^{n-1}, and with $\partial : D'^{n-1} \to D'^n$ as the new boundary operator here. Let the map $\sigma^{n-1}(a)$ agree with $\sigma^n(a)$ in degrees above $n-1$, and to be the inclusion of the summand $D^{n-1} \to D^{n-1} \oplus D^{\wedge} = D'^{n-1}$ in degree $n-1$. Let the map $\sigma^{n-1}(x')$ agree with $\sigma^n(x')$ in degrees above $n-1$, and to be given in degree $n-1$ on $FD'^{n-1} \cong FD^{n-1} \oplus FD^{\wedge}$ as $x^{n-1} : FD^{n-1} \to C^{n-1}$ on the summand FD^{n-1} and on the summand $FD^{\wedge}$ as the composite of the map $FD^{\wedge} \to Z^{n-1}M^{\cdot}$ and the projection $Z^{n-1}M^{\cdot} \to C^{n-1}$ determined by the isomorphism 1.9.5.7 of $Z^{n-1}M^{\cdot}$ with a fibre product. It is easy to verify that $\sigma^{n-1}(x')$ and $\sigma^{n-1}(a)$ are chain maps and that 1.9.5.4.1 and 1.9.5.4.2 hold with $n-1$ in place of n.

To verify condition 1.9.5.4.3 that $\sigma^{n-1}(x')$ is an $(n-1)$-quasi-isomorphism, consider its mapping cone $\widetilde{M}^{\cdot}$. This is the mapping cone $M^{\cdot}$ of $\sigma^n(x')$ with an additional term $FD^{n-1} \oplus FD^{\wedge} = F(D'^{n-1})$ added:

(1.9.5.8)
$$\widetilde{M}^{\cdot} = \cdots \to C^{n-3} \to FD^{n-1} \oplus FD^{\wedge} \oplus C^{n-2} \to FD'^n \oplus C^{n-1} \to \cdots$$

By construction the boundary map from the summand $FD^{\wedge}$ in $\widetilde{M}^{n-2}$ maps onto $H^{n-1}(M^{\cdot}) = Z^{n-1}(M^{\cdot})/B^{n-1}(M^{\cdot})$. Hence $FD^{\wedge} \oplus B^{n-1}(M^{\cdot})$ and *a fortiori* $FD^{\wedge} \oplus M^{n-2} = FD^{\wedge} \oplus C^{n-2}$ and $FD^{n-1} \oplus FD^{\wedge} \oplus C^{n-2}$ map onto $Z^{n-1}M^{\cdot} = Z^{n-1}\widetilde{M}^{\cdot}$. Then $H^{n-1}(\widetilde{M}^{\cdot}) = 0$, as well as $H^i(\widetilde{M}^{\cdot}) = H^i(M^{\cdot}) = 0$ for $i \geq n$. Now the long exact cohomology sequence for the mapping cone $\widetilde{M}^{\cdot}$ of $\sigma^{n-1}(x')$ shows that $\sigma^{n-1}(x')$ is an $(n-1)$-quasi-isomorphism as required. This completes the induction step.

Now given the inductively constructed $\sigma^n D'^{\cdot}$ for all n, we set $D'^{\cdot} = \varinjlim \sigma^n D'^{\cdot}$ as $n \to -\infty$. As $\sigma^n D'^{\cdot}$ and $\sigma^{n-1}D'^{\cdot}$ agree in degrees above $n-1$, we have $D'^k = (\sigma^n D')^k$ for any $n \leq k$. We define a, x' similarly. It is then clear from 1.9.5.4 that $D'^{\cdot}$, a, and x' meet the requirements, completing the proof of Lemma 1.9.5.

1.9.5.9. *Porism.* If 1.9.5.1 holds only for those $n \geq N+1$ for some fixed N, the proof still constructs a $\sigma^N D'^{\cdot}$ in $\mathcal{C}^b(\mathcal{D})$, and N-quasi-isomorphism $\sigma^N(x') : F(\sigma^N D'^{\cdot}) \to C^{\cdot}$, and a degree-wise split monomorphism $\sigma^N(a) : \sigma^N D^{\cdot} \to \sigma^N(D'^{\cdot})$ such that $\sigma^N(x) = \sigma^N(x') \circ \sigma^N(a)$.

If $\mathcal{A}$ has two additive full subcategories $\mathcal{D}_1 \subseteq \mathcal{D}_2$ so that the inclusion $\mathcal{D}_1 \to \mathcal{A}$ satisfies 1.9.5.1 for $n \geq N+1$ and $\mathcal{D}_2 \to \mathcal{A}$ satisfies 1.9.5.1 for all n, then the proof constructs a quasi-isomorphism $x' : D'^{\cdot} \to C^{\cdot}$ with D'^k in $\mathcal{D}_1$ for $k \geq N$ and D'^k in $\mathcal{D}_2$ for all k. Moreover, if $x : D^{\cdot} \to C^{\cdot}$ is given with $D^{\cdot} \in \mathcal{C}^-(\mathcal{D}_2)$ and with D^k in $\mathcal{D}_1$ for $k \geq N$, then we may construct the quasi-isomorphism $x' : D'^{\cdot} \xrightarrow{\sim} C^{\cdot}$ so that there is a degree-wise split monomorphism $a : D^{\cdot} \to D'^{\cdot}$ with $x' = x \circ a$.

These variants of Lemma 1.9.5 will come to seem less bizarre in Section 2.2 below.

1.9.6. Let $\mathbf{A}$ be a complicial biWaldhausen category with associated abelian category $\mathcal{A}$. So $\mathbf{A}$ is a full subcategory of the category of chain complexes $\mathcal{C}(\mathcal{A})$. Suppose $\mathbf{A}$ is closed under the formation of canonical homotopy pushouts and canonical homotopy pullbacks in $\mathcal{C}(\mathcal{A})$ 1.1.2, as is often the case 1.3.6. Then the "derived" or homotopy category of $\mathbf{A}$ will be a "triangulated category," and there will be a "calculus of fractions." We recall parts of this theory of Grothendieck and Verdier in our context (cf. [V] or [H] I).

Let $\mathbf{A}/\simeq$ be the quotient category where two maps of $\mathbf{A}$ are identified to each other if they are chain homotopic as maps of complexes. The category $\mathbf{A}/\simeq$ is a full subcategory of the chain homotopy category $\mathcal{K}(\mathcal{A}) = \mathcal{C}(\mathcal{A})/\simeq$. If one defines the distinguished triangles in $\mathbf{A}/\simeq$ to be those chain homotopy equivalent to those coming in the usual way from mapping cone sequences in $\mathbf{A}$, then $\mathbf{A}/\simeq$ satisfies the axioms for a triangulated category ([V] I Section 1, nos. 1-2, or [H] I Section 1), and is a subtriangulated category of $\mathcal{K}(\mathcal{A})$ with its usual mapping cone sequence triangulated structure ([V] I Section 1 no. 2, or [H] I Section 2).

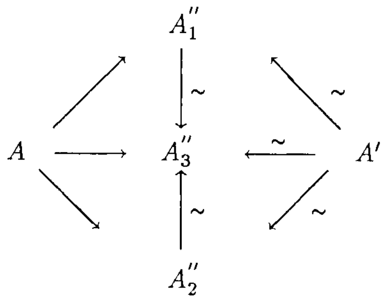
Let $\mathbf{w}$ denote the image of $\mathbf{w}(\mathbf{A})$ in $\mathbf{A}/\simeq$. As the complicial $\mathbf{A}$ is saturated and extensional, it is easy to see that $\mathbf{w}$ is a saturated multiplicative system in $\mathbf{A}/\simeq$ in the sense of ([V] I Section 2, nos. 1-2, or [H] I Section 3). The corresponding thick triangulated subcategory of $\mathbf{A}/\simeq$ is $\mathbf{A}^w/\simeq$, the full subcategory of objects A such that the unique map $0 \to A$ is in $\mathbf{w}$.

Let $\mathrm{Ho}(\mathbf{A}) = \mathbf{w}^{-1}\mathbf{A}$ be the "derived" or homotopy category formed from $\mathbf{A}$ by localizing the category $\mathbf{A}$ to make the maps in $\mathbf{w}(\mathbf{A})$ isomorphisms in $\mathbf{w}^{-1}\mathbf{A}$. As chain homotopy equivalences are quasi-isomorphisms, hence are in $\mathbf{w}(\mathbf{A})$, they become isomorphisms in $\mathbf{w}^{-1}\mathbf{A}$. In particular, in $\mathbf{w}^{-1}\mathbf{A}$ a complex A becomes isomorphic to the complex "$A \times I$" $= A \overset{h}{\underset{A}{\cup}} A$ that parameterizes chain homotopies of maps out of A. Thus $\mathbf{w}^{-1}\mathbf{A}$ is also the localization $\mathbf{w}^{-1}\mathbf{A}/\simeq$ of $\mathbf{A}/\simeq$ at $\mathbf{w}$. The work in ([V] I Section 2 or [H] I Sections 3-4) shows that $\mathbf{w}$ is exactly the set of all morphisms in $\mathbf{A}/\simeq$ (or in $\mathbf{A}$) that become isomorphisms in $\mathbf{w}^{-1}\mathbf{A} = \mathrm{Ho}(\mathbf{A})$. Also $\mathbf{w}^{-1}\mathbf{A}$ has an induced triangulated structure. (Not only mapping cone sequences, but also general cofibration sequences turn out to yield distinguished triangles in $\mathbf{w}^{-1}\mathbf{A}$, as in ([V] II Section 1, nos. 1-5, or [H] I 6.1).)

Furthermore, the passage from $\mathbf{A}/\simeq$ to $\mathbf{w}^{-1}\mathbf{A} = \mathbf{w}^{-1}\mathbf{A}/\simeq$ admits a "calculus of fractions"([H] I Section 3 or [V] I Section 2 no. 3). In particular, the morphisms in $\mathbf{w}^{-1}\mathbf{A}$ from A to A' correspond to the equivalence classes of data (1.9.6.1) in $\mathbf{A}/\simeq$, (1.9.6.1) $$A \to A'' \overset{\sim}{\leftarrow} A'$$

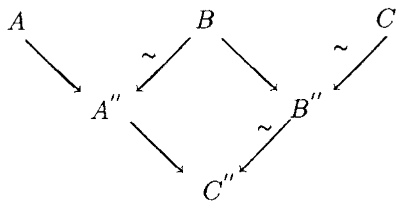
where $A \to A''$ is a morphism in $\boldsymbol{A}/\simeq$, and $A' \overset{\sim}{\to} A''$ is a map in $\boldsymbol{w}$ in $\boldsymbol{A}/\simeq$. Two such data $A \to A_1'' \overset{\sim}{\leftarrow} A'$ and $A \to A_2'' \overset{\sim}{\leftarrow} A'$ are equivalent if there exists a commutative diagram (1.9.6.2) in $\boldsymbol{A}/\simeq$

(1.9.6.2)



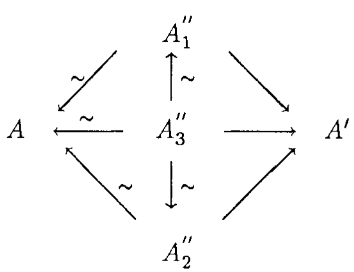
The calculus of fractions insures that this is an equivalence relation, and yields the composition of morphisms represented as data by constructing the C'' and the bottom arrows in (1.9.6.3). Here C'' is the canonical homotopy pushout in $\boldsymbol{A}$ of a choice of maps in $\boldsymbol{A}$ to represent the chain homotopy classes of maps $B \overset{\sim}{\to} A''$ and $B \to B''$ in $\boldsymbol{A}/\simeq$, and the bottom arrows are the canonical maps into the homotopy pushout. (It is easy to check that $B'' \overset{\sim}{\to} C''$ is a weak equivalence, cf. the construction of 1.9.8.4 below.)

(1.9.6.3)



Dually, morphisms in $\boldsymbol{w}^{-1}\boldsymbol{A}$ from A to A' may be represented by equivalence classes of data $A \overset{\sim}{\leftarrow} A'' \to A'$, with equivalence relation

(1.9.6.4)



and with composition coming from homotopy pullbacks.

When there is danger of confusion as to whether a map $A \to A'$ is to be a map in $\boldsymbol{A}$ or in $\mathrm{Ho}(\boldsymbol{A}) = \boldsymbol{w}^{-1}\boldsymbol{A}$, we refer to morphisms in $\boldsymbol{A}$ as strict maps.

1.9.7. Let $F : \boldsymbol{A} \to \boldsymbol{B}$ be a complicial exact functor between biWaldhausen categories (1.2.16). As F is induced by degree-wise application to complexes of an additive functor $f : \mathcal{A} \to \mathcal{B}$ of associated abelian categories, the functor F preserves canonical homotopy pushouts and canonical homotopy pullbacks. In particular, F preserves mapping cones and mapping cylinders. Thus if $\boldsymbol{A}$ and $\boldsymbol{B}$ are closed under the formation of homotopy pushouts and homotopy pullbacks, $F : \boldsymbol{A}/\simeq \to \boldsymbol{B}/\simeq$ and $w^{-1}F : \boldsymbol{w}^{-1}\boldsymbol{A} \to \boldsymbol{w}^{-1}\boldsymbol{B}$ are triangulated functors.

From the calculus of fractions we see that $\boldsymbol{w}^{-1}F : w^{-1}\boldsymbol{A} \to \boldsymbol{w}^{-1}\boldsymbol{B}$ is an equivalence of homotopy categories if $F : \boldsymbol{A} \to \boldsymbol{B}$ satisfies the following four conditions:

1.9.7.0. For any map a in $\boldsymbol{A}$, a is in $\boldsymbol{w}(\boldsymbol{A})$ if and only if Fa is in $\boldsymbol{w}(\boldsymbol{B})$.

1.9.7.1. For any B in $\boldsymbol{B}$, there is an A in $\boldsymbol{A}$ and a map $FA \xrightarrow{\sim} B$ in $\boldsymbol{w}(\boldsymbol{B})$.

1.9.7.2. For any map $b : FA' \to FA''$ in $\boldsymbol{B}$, there are maps $a' : A \xrightarrow{\sim} A'$ in $\boldsymbol{w}(\boldsymbol{A})$ and $a'' : A \to A''$ in $\boldsymbol{A}$, such that there is a chain homotopy $b \circ Fa' \simeq Fa''$ in $\boldsymbol{B}$.

1.9.7.3. For any map $a' : A' \to A''$ in $\boldsymbol{A}$ such that $Fa' \simeq 0$ is chain nullhomotopic in $\boldsymbol{B}$, there is a map $a : A \xrightarrow{\sim} A'$ in $\boldsymbol{w}(\boldsymbol{A})$ such that $a' \cdot a \simeq 0$ is chain nullhomotopic in $\boldsymbol{A}$.

The last two conditions (1.9.7.2 and 1.9.7.3) trivially hold whenever $F : \boldsymbol{A} \to \boldsymbol{B}$ is fully faithful. Note then that F is also full and faithful for chain homotopies between maps, as a chain homotopy between maps $C \to D$ corresponds to a chain map "$C \times I$" $= C \overset{h}{\underset{C}{\cup}} C \to D$.

1.9.8. **Theorem**. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two complicial biWaldhausen categories, each of which is closed under the formation of canonical homotopy pushouts and canonical homotopy pullbacks* (1.9.6, 1.2.11). *Let $F : \boldsymbol{A} \to \boldsymbol{B}$ be a complicial exact functor* (1.2.16). *Suppose that F induces an equivalence of the derived homotopy categories $\boldsymbol{w}^{-1}F : \boldsymbol{w}^{-1}\boldsymbol{A} \to \boldsymbol{w}^{-1}\boldsymbol{B}$. Then F induces a homotopy equivalence of K-theory spectra*

$$K(F) : K(\boldsymbol{A}) \to K(\boldsymbol{B}).$$

Proof. By 1.9.2, we reduce to the case where cofibrations in $\boldsymbol{A}$ and $\boldsymbol{B}$ are precisely the degree-wise split monomorphisms whose quotients lie in $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively. Then $\boldsymbol{A}$ and $\boldsymbol{B}$ have cylinder and cocylinder functors satisfying the cylinder and cocylinder axioms, thanks to 1.3.5.

Let $\boldsymbol{C}$ be the category whose objects are data $(A,\ FA \xrightarrow{\sim} B)$ where A is an object of $\boldsymbol{A}$, B is an object of $\boldsymbol{B}$, and $FA \xrightarrow{\sim} B$ is a map in $\boldsymbol{w}(\boldsymbol{B})$.

A map in $\boldsymbol{C}$ from $(A,\ FA \xrightarrow{\sim} B)$ to $(A',\ FA' \xrightarrow{\sim} B')$ consists of a map $A \to A'$ in $\boldsymbol{A}$ and a map $B \to B'$ in $\boldsymbol{B}$ such that

$$\begin{array}{ccc} FA & \xrightarrow{\sim} & B \\ \downarrow & & \downarrow \\ FA' & \xrightarrow{\sim} & B' \end{array}$$

commutes. Call a map in $\boldsymbol{C}$ a cofibration (or respectively, weak equivalence) if both maps $A \to A'$ in $\boldsymbol{A}$ and $B \to B'$ in $\boldsymbol{B}$ are cofibrations (resp. weak equivalences). This makes $\boldsymbol{C}$ a biWaldhausen category. (Indeed, it is even a complicial biWaldhausen category, with associated abelian category of data $(A,\ fA \to B)$ with A in $\mathcal{A}$, B in $\mathcal{B}$, and $fA \to B$ a map in $\mathcal{B}$.) As in 1.9.7, $F : \boldsymbol{A} \to \boldsymbol{B}$ preserves the mapping cylinders and cocylinders. Thus $\boldsymbol{C}$ has a cylinder functor induced by the cylinder functors of $\boldsymbol{A}$ and $\boldsymbol{B}$. Dually $\boldsymbol{C}$ has a cocylinder functor. These satisfy the cylinder and cocylinder axiom 1.3.1.7.

The functor $F : \boldsymbol{A} \to \boldsymbol{B}$ factors as the composite of exact functors $\boldsymbol{A} \to \boldsymbol{C}$ and $\boldsymbol{C} \to \boldsymbol{B}$. Here $\boldsymbol{A} \to \boldsymbol{C}$ sends A to $(A,\ FA = FA)$, and $\boldsymbol{C} \to \boldsymbol{B}$ sends $(A,\ FA \xrightarrow{\sim} B)$ to B. We will show both these functors induce homotopy equivalences on K-theory spectra.

The functor $\boldsymbol{A} \to \boldsymbol{C}$ is split by an exact functor $\boldsymbol{C} \to \boldsymbol{A}$ sending $(A,\ FA \xrightarrow{\sim} B)$ to A. So $\boldsymbol{A} \to \boldsymbol{C} \to \boldsymbol{A}$ is the identity functor. The composite $\boldsymbol{C} \to \boldsymbol{A} \to \boldsymbol{C}$ is naturally weak equivalent to the identity by the natural transformation $(A,\ FA = FA) \to (A,\ FA \xrightarrow{\sim} B)$ induced by $A = A$ and $FA \xrightarrow{\sim} B$. Thus $K(\boldsymbol{A}) \to K(\boldsymbol{C})$ and $K(\boldsymbol{C}) \to K(\boldsymbol{A})$ are inverse homotopy equivalences by 1.5.4.

Consider now the exact functor $\boldsymbol{C} \to \boldsymbol{B}$ of biWaldhausen categories. We claim it induces a homotopy equivalence $K(\boldsymbol{C}) \xrightarrow{\sim} K(\boldsymbol{B})$ by the dual to the approximation theorem 1.9.1.

First we note that the dual hypothesis to 1.9.1.1 holds. For suppose $(a, b) : (A,\ FA \xrightarrow{\sim} B) \to (A',\ FA' \xrightarrow{\sim} B')$ is a map in $\boldsymbol{C}$, whose image $b : B \xrightarrow{\sim} B'$ in $\boldsymbol{C}$ is a weak equivalence. Then by saturation, $Fa : FA \xrightarrow{\sim} FA'$ is also a weak equivalence. Hence Fa becomes an isomorphism in $\boldsymbol{w}^{-1}\boldsymbol{B}$. As by hypothesis, $\boldsymbol{w}^{-1}F : \boldsymbol{w}^{-1}\boldsymbol{A} \to \boldsymbol{w}^{-1}\boldsymbol{B}$ is an equivalence of categories, the map a becomes an isomorphism in $\boldsymbol{w}^{-1}\boldsymbol{A}$. Hence as in 1.9.6, the map a is in $\boldsymbol{w}(\boldsymbol{A})$. Thus both maps a and b are weak equivalences, so (a, b) is a weak equivalence in $\boldsymbol{C}$, as required by 1.9.1.1.

To verify the dual hypothesis to 1.9.1.2, we must show that given a diagram (1.9.8.1) corresponding to a map $x : B' \to (B \text{ of } (FA \to B))$, this can be completed to a commutative diagram (1.9.8.2) corresponding to the factorization required by 1.9.1.2

(1.9.8.1)
$$\begin{array}{ccc} FA & \xrightarrow{\sim} & B \\ & & \uparrow \\ & & B' \end{array}$$

(1.9.8.2)
$$\begin{array}{ccc} FA & \xrightarrow{\sim} & B \\ {\scriptstyle Fa''}\uparrow & & \uparrow \\ FA'' & \xrightarrow{\sim} & B'' \\ & & \uparrow \\ & & B' \end{array}$$
$$b' : B' \to B$$

As a first approximation, we construct a diagram like (1.9.8.2) in $\boldsymbol{B}/\simeq$, i.e., a version that chain homotopy commutes. We begin by noting that as $\boldsymbol{w}^{-1}F$ is an equivalence of homotopy categories, B' is isomorphic in $\boldsymbol{w}^{-1}\boldsymbol{B}$ to some FA_1. By calculus of fractions, this isomorphism corresponds to a datum in $\boldsymbol{B}$: $FA_1 \xrightarrow{\sim} B_1 \xleftarrow{\sim} B'$. Composing this isomorphism in $\boldsymbol{w}^{-1}\boldsymbol{B}$ with the map $b : B' \to B$ and with the inverse isomorphism to $FA \xrightarrow{\sim} B$ yields a map in $\boldsymbol{w}^{-1}\boldsymbol{B}$ from FA_1 to FA. As $\boldsymbol{w}^{-1}F$ is an equivalence of categories, this map is $\boldsymbol{w}^{-1}F$ of some map from A_1 to A in $\boldsymbol{w}^{-1}\boldsymbol{A}$. This map from A_1 to A is represented by a datum $A_1 \to A_2 \xleftarrow{\sim} A$ in $\boldsymbol{A}$. Applying the formulae of the calculus of fractions for composition and equivalence of data representing maps in $\boldsymbol{w}^{-1}\boldsymbol{B}$, as given in 1.9.6, we deduce that there exists in $\boldsymbol{B}/\simeq$ a commutative diagram, which after removal of intermediate constructions becomes:

(1.9.8.3)
$$\begin{array}{ccc} FA & \xrightarrow{\sim} & B \\ {\scriptstyle F(\ )}\downarrow\wr & & \downarrow\wr \\ FA_2 & \xrightarrow{\sim} & B_2 \\ {\scriptstyle F(\ )}\uparrow & & \uparrow \\ FA_1 & \xrightarrow{\sim} & B_1 \\ & & \nwarrow\wr \\ & & \quad B' \end{array}$$
$$b' : B' \to B$$

We choose representatives of these maps in $\boldsymbol{A}$ and $\boldsymbol{B}$, so that (1.9.8.3) becomes a chain homotopy commutative diagram in $\boldsymbol{B}$, where the indicated maps are F of maps in $\boldsymbol{A}$.

Now let A_3 be the canonical homotopy pullback of $A_1 \to A_2$ and $A \xrightarrow{\sim} A_2$, and let B_3 be the canonical homotopy pullback of $B_1 \to B_2$ and $B \xrightarrow{\sim} B_2$, as in 1.1.2. The FA_3 is the canonical homotopy pullback of $FA_1 \to FA_2$ and $FA \to FA_2$, as in 1.9.7.

By 1.1.2, the chain homotopy commutative diagram (1.9.8.3), after a choice of chain homotopies, induces a map of homotopy pullbacks $FA_3 \to B_3$, which is an edge of a homotopy commutative cube with vertices FA_3, FA_2, FA_1, FA, B_3, B_2, B_1, and B. The map $FA_3 \to B_3$ is a weak equivalence. To see this, we first note that the projection map $B_3 \to B$ from the homotopy pullback is the pullback along $B \to B_2$ of the canonical map

$$k_1 : B_2 \overset{h}{\underset{B_2}{\times}} B_1 = \text{ cocylinder } (B_1 \to B_2) \twoheadrightarrow B_2.$$

As k_1 is a degree-wise split epimorphism, and even a map in quot($\boldsymbol{B}$), it follows that the pullback $B_3 \twoheadrightarrow B$ is in quot($\boldsymbol{B}$). Similarly $FA_3 \twoheadrightarrow FA$ is in quot($\boldsymbol{B}$). Now as $FA_3 \to B_3$ is induced by the horizontal arrows in (1.9.8.3), each of which is a weak equivalence, it follows from the dual of the gluing lemma axiom 1.2.3.2 that $FA_3 \xrightarrow{\sim} B_3$ is a weak equivalence.

Also, as $B \to B_2$ is a weak equivalence, the extension axiom 1.2.6 shows that the induced map

$$B_3 \xrightarrow{\sim} B_2 \overset{h}{\underset{B_2}{\times}} B_1 = \text{ cocylinder } (B_1 \to B_1)$$

is a weak equivalence. As the projection of the cocylinder onto B_1 is even a quasi-isomorphism, it follows that the canonical map $B_3 \xrightarrow{\sim} B_1$ is a weak equivalence.

The homotopy commutative right half of (1.9.8.3), together with a choice of homotopy, determines a map $B' \to B_3$ by the universal mapping property of homotopy pullbacks, dual to the mapping property of homotopy pushouts explained in 1.1.2. As $B_3 \to B_1$ and $b' : B' \to B_1$ are weak equivalences, the saturation axiom implies that $B' \xrightarrow{\sim} B_3$ is a weak equivalence. Thus we have constructed a homotopy commutative diagram (1.9.8.4), the desired first approximation to (1.9.8.2).

$$(1.9.8.4) \qquad \begin{array}{ccccc} FA & \xrightarrow{\sim} & B & & \\ {\scriptstyle F(\ )}\uparrow & & \uparrow & \nwarrow^{b'} & \\ FA_3 & \xrightarrow{\sim} & B_3 & & \\ & & & \hookleftarrow & B' \end{array}$$

To finish, it remains to replace (1.9.8.4) by a strictly commutative diagram in $\boldsymbol{B}$, as opposed to $\boldsymbol{B}/\simeq$. Let B'' be the homotopy pullback of $B \to B$ and $B_3 \to B$. Then the projection $B'' \xrightarrow{\sim} B_3$ is a quasi-isomorphism, hence a weak equivalence. Choices of homotopies in the homotopy commutative diagram (1.9.8.4) determine maps $FA_3 \to B''$ and $B' \to B''$. As $B'' \xrightarrow{\sim} B_3$ and $FA_3 \xrightarrow{\sim} B_3$ are weak equivalences, the saturation axiom shows that $FA_3 \xrightarrow{\sim} B''$ is a weak equivalence. Similarly, $B' \xrightarrow{\sim} B''$ is a weak equivalence. Now consider the map $B'' \to B$ which is the canonical projection of the homotopy fibre product onto B. By construction, the composite $FA_3 \to B'' \to B$ is $FA_3 \to FA \to B$, and $B' \to B'' \to B$ is $b' : B' \to B$. Thus we have a strictly commutative (1.9.8.2) on taking $FA'' = FA_3$ and $B'' = B''$. This completes the verification of the dual of hypothesis 1.9.1.2. Now 1.9.1 applies to complete the proof of the theorem.

1.9.9. The theorem 1.9.8 is very useful in providing K-theoretic equivalences directly from off-the-shelf data, as found in [SGA 6] for example. Morally, it says that $K(\boldsymbol{A})$ essentially depends only on the derived category $\boldsymbol{w}^{-1}\boldsymbol{A}$, and thus that Waldhausen K-theory gives essentially a K-theory of the derived category. However, it is true that to so define a K-theory of a derived category, one must find some underlying model $\boldsymbol{A}$ which is complicial biWaldhausen. Also, we know independence of the choice of model only when the models are related by some additive functor exact in the sense of 1.2.16. These caveats are annoying, but do not seem to cause serious problems in practice.

We also note an equivalence $\boldsymbol{w}^{-1}\boldsymbol{A} \to \boldsymbol{w}^{-1}\boldsymbol{B}$ often induces equivalences of homotopy categories $\boldsymbol{w}^{-1}\boldsymbol{A}' \to \boldsymbol{w}^{-1}\boldsymbol{B}'$ for various naturally defined complicial BiWaldhausen subcategories $\boldsymbol{A}'$, $\boldsymbol{B}'$, of $\boldsymbol{A}$, $\boldsymbol{B}$. Then 1.9.8 shows that $K(\boldsymbol{A}') \to K(\boldsymbol{B}')$ will also be a homotopy equivalence of K-theory spectra. Thus the equivalences of 1.9.8 have a nice tendency towards inheritance by natural subcategories.

1.10.1. **Cofinality Theorem.** *Let $\boldsymbol{vA}$ be a Waldhausen category with a cylinder functor satisfying the cylinder axiom. Let G be an abelian group, and $\pi : K_0(\boldsymbol{vA}) \to G$ an epimorphism. Let $\boldsymbol{A}^w$ be the full subcategory of those A in $\boldsymbol{A}$ for which the class $[A]$ in $K_0(\boldsymbol{vA})$ has $\pi[A] = 0$ in G. Make $\boldsymbol{A}^w$ a Waldhausen category with $\boldsymbol{v}(\boldsymbol{A}^w) = \boldsymbol{A}^w \cap \boldsymbol{v}(\boldsymbol{A})$, $co(\boldsymbol{A}_w) = \boldsymbol{A}^w \cap co(\boldsymbol{A})$. Let "$G$" denote G considered as a Eilenberg-MacLane spectrum whose only non-zero homotopy group is a G in dimension 0.*

Then there is a homotopy fibre sequence

$$K(\boldsymbol{vA}^w) \to K(\boldsymbol{vA}) \to \text{“}G\text{”} \tag{1.10.1.1}$$

In particular,

$$\text{(1.10.1.2)} \qquad \begin{aligned} K_i(\boldsymbol{v}\boldsymbol{A}^w) &= K_i(\boldsymbol{v}\boldsymbol{A}) \qquad \text{for } i > 0 \\ K_0(\boldsymbol{v}\boldsymbol{A}^w) &= \ \textit{Ker } \pi : K_0(\boldsymbol{v}\boldsymbol{A}) \twoheadrightarrow G \end{aligned}$$

Proof. Define $\boldsymbol{w}(\boldsymbol{A})$ to be the set of maps in $\boldsymbol{A}$ whose mapping cones have their K_0 class in the kernel of $\pi : K_0(\boldsymbol{v}\boldsymbol{A}) \to G$. It is easy to check using 1.5.6 that $\boldsymbol{w}\boldsymbol{A}$ is a Waldhausen category and $\boldsymbol{v}(\boldsymbol{A}) \subseteq \boldsymbol{w}(\boldsymbol{A})$. Clearly $\boldsymbol{w}\boldsymbol{A}$ satisfies the extension and saturation axioms. Appealing then to the homotopy fibre sequence given by the localization theorem 1.8.2, it suffices to show that $K(\boldsymbol{w}\boldsymbol{A})$ is homotopy equivalent to "G".

For each non-negative integer n, consider $\overset{n}{\Pi} G$ as a category whose objects are n-tuples of elements of G, $(g_1, g_2, \ldots, g_n)$. The category $\overset{n}{\Pi} G$ has only identity morphisms. There is a functor $\pi : \boldsymbol{w}S_n\boldsymbol{A} \to \overset{n}{\Pi} G$ induced in terms of the description 1.5.1.2 by sending the object $A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n$ of $\boldsymbol{w}S_n\boldsymbol{A}$ to the n-tuple $(\pi[A_1], \pi[A_2] - \pi[A_1], \pi[A_3] - \pi[A_2], \ldots, \pi[A_n] - \pi[A_{n-1}])$.

(In the more precise description 1.5.1, this functor sends $A(\ ,\ )$ to $(\pi[A(1,0)],\ \pi[A(2,0)] - \pi[A(1,0)],\ \ldots,\ \pi[A(n,0)] - \pi[A(n-1,0)]) = (\pi[A(1,0)],\ \pi[A(2,1)],\ \pi[A(3,2)],\ \ldots,\ \pi[A(n,n-1)])$.)

We claim that for each n, $\pi : \boldsymbol{w}S_n\boldsymbol{A} \to \overset{n}{\Pi} G$ induces a homotopy equivalence of nerves of categories. As $\overset{n}{\Pi} G$ is a discrete category, i.e., has only identity morphisms, it suffices to show for all $(g_1, g_2, \ldots, g_n)$ that the category $\pi^{-1}(g_1, \ldots g_n)$ has contractible nerve. The fibre $\pi^{-1}(0, 0, \ldots, 0)$ has initial object $0 \rightarrowtail 0 \rightarrowtail \cdots \rightarrowtail 0$, and so is contractible. We plead that all fibres $\pi^{-1}(g_1, \ldots, g_n)$ are homotopy equivalent to $\pi^{-1}(0, 0, \ldots, 0)$, and hence are contractible. First note by 1.5.7 and the hypothesis that $\pi : K_0(\boldsymbol{v}\boldsymbol{A}) \to G$ is onto every element g in G is $\pi[C]$ for some C in $\boldsymbol{A}$. Given $(g_1, \ldots, g_n)$ then choose C_i in $\boldsymbol{A}$ so $\pi[C_i] = g_i$. Consider the objects $C_. = C_1 \rightarrowtail C_1 \cup C_2 \rightarrowtail C_1 \cup C_2 \cup C_3 \rightarrowtail \cdots \rightarrowtail C_1 \cup C_2 \cup \cdots \cup C_n$, and $\Sigma C_. = \Sigma C_1 \rightarrowtail \Sigma C_1 \cup \Sigma C_2 \rightarrowtail \cdots \rightarrowtail \Sigma C_1 \cup \Sigma C_2 \cup \cdots \cup \Sigma C_n$ in $\boldsymbol{w}S_n\boldsymbol{A}$. Clearly $\pi(C_.) = (g_1, g_2, \ldots, g_n)$. Also $\pi(\Sigma C_.) = (-g_1, -g_2, \ldots, -g_n)$ as $[\Sigma C] = -[C]$ by 1.5.7. Then the functor $\cup C_. : \boldsymbol{w}S_n\boldsymbol{A} \to \boldsymbol{w}S_n\boldsymbol{A}$ sending $(A_1 \rightarrowtail \cdots \rightarrowtail A_n)$ to $(A_1 \cup C_1 \rightarrowtail A_2 \cup C_1 \cup C_2 \rightarrowtail \cdots \rightarrowtail A_n \cup C_1 \cup \cdots \cup C_n)$ restricts to a functor $\cup C_. : \pi^{-1}(0, 0, \ldots, 0) \to \pi^{-1}(g_1, g_2, \ldots, g_n)$. Similarly $\cup\, \Sigma C_.$ gives a functor $\pi^{-1}(g_1, g_2, \ldots, g_n) \to \pi^{-1}(0, 0, \ldots 0)$. The maps $0 \to C_i \cup \Sigma C_i$, $C_i \cup \Sigma C_i \to 0$ are in $\boldsymbol{w}\boldsymbol{A}$ as $[C \cup \Sigma C] = [C] - [C] = 0$, and they induce natural transformations between the identity functors on $\pi^{-1}(0, 0, \ldots, 0)$ and $\pi^{-1}(g_1, g_2, \ldots, g_n)$ and the composites of the functor $\cup\, C_.$ and $\cup\, \Sigma C_.$. Thus these functors induce homotopy equivalences between $\pi^{-1}(0, 0, \ldots, 0)$ and $\pi^{-1}(g_1, g_2, \ldots, g_n)$, as was to be shown. This completes the proof of the claim that $\pi : \boldsymbol{w}S_n\boldsymbol{A} \to \overset{n}{\Pi}G$ is a homotopy equivalence.

The map $\pi : \boldsymbol{w}S_n\boldsymbol{A} \to \overset{n}{\Pi}G$ for various n are compatible with the simplicial operators, and so induce a functor between simplicial categories which is a homotopy equivalence of classifying spaces in each degree. Here the simplicial operators on $\boldsymbol{w}S_n\boldsymbol{A}$ are as in 1.5.1 - 1.5.2, and on $\overset{\cdot}{\Pi}G$ are defined so that $d_0(g_1, \ldots, g_n) = (g_2, \ldots, g_n)$, $d_i(g_1, \ldots, g_n) = (g_1, g_2, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n)$ for $1 \leq i < n$, $d_n(g_1, \ldots, g_n) = (g_1, \ldots, g_{n-1})$, and $s_i(g_1, \ldots, g_n) = (g_1, \ldots, g_i, 0, g_{i+1}, \ldots, g_n)$. With this structure, the simplicial category $\overset{\cdot}{\Pi}G$ is actually the simplicial set $BG = NG$, the bar construction on G. The nerve of this degree-wise discrete simplicial category thus collapses to NG. The degree-wise homotopy equivalence π induces a homotopy equivalence of spaces

$$|N\boldsymbol{w}S_.(\boldsymbol{A})| \to |NG| \simeq BG.$$

In fact this homotopy equivalence is a map of infinite loop spaces. For one checks easily that the iterated $S_.$ construction on $\boldsymbol{wA}$ that defines the Waldhausen spectrum structure ([W] 1.3.5 Remark) corresponds under π to the iterated bar construction on G that defines the Eilenberg- MacLane spectrum "G". (Or one notes that π is a simplicial symmetric monoidal functor, and feeds it to an infinite loop space machine [Th3] A). Thus π induces a homotopy equivalence of spectra $K(\boldsymbol{wA}) \xrightarrow{\sim}$ "G", as required.

(We found this proof in 1985; it has since become folklore.)

1.10.2. *Exercise* (optional). Theorem 1.10.1 is all the cofinality that we need for Sections 2 - 11. Other well-known cofinality results often have a different flavor, (see [Gr2] 6.1 and [Sta] 2.1 for some latest versions). In particular, the Waldhausen strict cofinality theorem [W] 1.5.9 at first seems quite different in purpose from our 1.10.1. Combining Waldhausen strict cofinality, our 1.10.1 and Grayson's cofinality trick [Gr3] Section 1, prove the following cofinality result:

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Waldhausen categories. Suppose $\boldsymbol{A}$ is a full subcategory of $\boldsymbol{B}$ closed under extensions, that $\boldsymbol{w}(\boldsymbol{A}) = \boldsymbol{A} \cap \boldsymbol{w}(\boldsymbol{B})$, and that a map in $\boldsymbol{A}$ is a cofibration in $\boldsymbol{A}$ iff it is a cofibration in $\boldsymbol{B}$ with quotient isomorphic to an object of $\boldsymbol{A}$. Suppose that $\boldsymbol{B}$ has mapping cylinders satisfying the cylinder axiom, and that $\boldsymbol{A}$ is closed under them. Suppose finally that $\boldsymbol{A}$ is cofinal in $\boldsymbol{B}$ in that for all B in $\boldsymbol{B}$ there is a B' in $\boldsymbol{B}$ such that $B \cup B'$ is isomorphic to an object of $\boldsymbol{A}$.

Then $K(\boldsymbol{A}) \to K(\boldsymbol{B}) \to$ "$K_0(\boldsymbol{B})/K_0(\boldsymbol{A})$" is a homotopy fibre sequence.

1.11.1. To close Section 1 we compare the Quillen K-theory [Q1] of an exact category $\mathcal{E}$ to the Waldhausen K-theory of $\mathcal{E}$, and to the Waldhausen K-theory of a category of bounded complexes in $\mathcal{E}$. The latter category has cylinders and cocylinders and is complicial biWaldhausen. This allows one to apply the results 1.10.1, 1.10.2, 1.9.8, 1.9.1, 1.8.2 to Quillen K-theory of exact categories. See [Gr2] and [Sta] for rederivations of all Quillen's basic results on K-theory of exact categories in the Waldhausen framework. We pause to mention two open problems. First, find a general result for Waldhausen categories that specializes to Quillen's devissage theorem when applied to the category of bounded complexes in an abelian category. Second, show under the conditions 1.2.15, that $K(\boldsymbol{A}) \simeq K(\mathcal{B})$. This would make Quillen's localization theorem for abelian categories an immediate consequence of 1.8.2.

Logically, one should now read Appendix A, and then return to 1.11.2.

1.11.2. **Theorem** (Waldhausen). *Let $\mathcal{E}$ be an exact category in the sense of Quillen* [Q1]. *Consider $\mathcal{E}$ as a biWaldhausen category as in* 1.2.9. *Then the Quillen and the Waldhausen K-theory spectra of $\mathcal{E}$ are naturally homotopy equivalent.*

Proof. [W] 1.9, or [Gi2] 9.3.

1.11.3. Let $\mathcal{A}$ be an abelian category, and let $i : \mathcal{E} \to \mathcal{A}$ be an exact functor which is full and faithful. Assume that $\mathcal{E}$ is closed under extensions in $\mathcal{A}$, and that if a sequence in $\mathcal{E}$ is exact in $\mathcal{A}$, then it must be exact in $\mathcal{E}$. We also make the following stronger assumption (which will be harmless by 1.11.10):

1.11.3.1. If f is a map in $\mathcal{E}$ such that $i(f)$ is an epimorphism in $\mathcal{A}$, then f is an admissible epimorphism in $\mathcal{E}$.

1.11.4. *Example.* Let X be a scheme, and let $\mathcal{E}$ be the exact category of algebraic vector bundles on X. Let $\mathcal{A}$ be either the abelian category of all $\mathcal{O}_X$-modules, or else the abelian subcategory of all quasi-coherent $\mathcal{O}_X$-modules. Let $i : \mathcal{E} \to \mathcal{A}$ be the canonical inclusion. Then this inclusion satisfies all the conditions of 1.11.3.

1.11.5. *Example.* Let $\mathcal{E}$ be an exact category satisfying the condition that all weakly split epimorphisms of $\mathcal{E}$ are admissible epimorphisms. That is, suppose that for any $r : E \to E''$ in $\mathcal{E}$ such that there is an $s : E'' \to E$ with $rs = 1$, then $r : E \twoheadrightarrow E''$ is an admissible epimorphism in $\mathcal{E}$. Let $i : \mathcal{E} \to \mathcal{A}$ be the Gabriel-Quillen embedding (cf. A 7.1), i.e., let $\mathcal{A}$ be the abelian category of left exact additive functors $\mathcal{E}^{\mathrm{op}} \to \mathbf{Z}$-modules, with i the Yoneda embedding $i(E) = \hom_{\mathcal{E}}(\ ,E)$. Then $i : \mathcal{E} \to \mathcal{A}$ satisfies the hypotheses of 1.11.3, including 1.11.3.1. For a proof, see Appendix A.7.1 and A.7.16.

1.11.6. Given $i : \mathcal{E} \to \mathcal{A}$ as in 1.11.3, consider the category $\boldsymbol{E}^{\sim}$ of bounded chain complexes in $\mathcal{E}$ as a full subcategory of the category of chain complexes $\mathcal{C}(\mathcal{A})$. Define $\mathrm{co}(\boldsymbol{E}^{\sim})$ to be the degree-wise admissible monomorphisms. Define $\boldsymbol{w}(\boldsymbol{E}^{\sim})$ to be those maps in $\boldsymbol{E}^{\sim}$ which are quasi-isomorphisms in $\mathcal{A}$. (This appears to depend on the choice of $\mathcal{A}$, but in fact does not given 1.11.3.1, see 1.11.8.)

Then $\boldsymbol{E}^{\sim}$ is a complicial biWaldhausen category. There is a canonical complicial exact functor $\mathcal{E} \to \boldsymbol{E}^{\sim}$ sending E in $\mathcal{E}$ to the complex which is E in degree 0 and 0 in other degrees.

Let $\boldsymbol{E}$ be $\boldsymbol{E}^{\sim}$, but now with co($\boldsymbol{E}$) being the degree-wise split monomorphisms whose quotients lie in $\boldsymbol{E}$. Then $\boldsymbol{E}$ is a complicial biWaldhausen category, and the inclusion functor $\boldsymbol{E} \to \boldsymbol{E}^{\sim}$ is complicial exact.

By 1.3.6, both $\boldsymbol{E}$ and $\boldsymbol{E}^{\sim}$ have cylinder and cocylinder functors, satisfying the cylinder and cocylinder axioms 1.3.1.7.

1.11.7. **Theorem** (Gillet-Waldhausen). *Under the hypotheses of* 1.11.3 *and with the notation of* 1.11.6. *the canonical exact inclusions induce homotopy equivalences of K-theory spectra*

$$K(\mathcal{E}) \xrightarrow{\sim} K(\boldsymbol{E}^{\sim}) \xleftarrow{\sim} K(\boldsymbol{E})$$

Proof. The map $K(\boldsymbol{E}) \to K(\boldsymbol{E}^{\sim})$ is a homotopy equivalence by 1.9.2. The homotopy equivalence $K(\mathcal{E}) \xrightarrow{\sim} K(\boldsymbol{E}^{\sim})$ is due to Gillet, who patterned his argument [Gi2] 6.2 on a proof for special cases due to Waldhausen. Gillet's statement [Gi2] 6.2 does not make the extra hypothesis 1.11.3.1 on the embedding $\mathcal{E} \to \mathcal{A}$, but the proof given there needs 1.11.3.1 in order to work. (Gillet attempts to evade 1.11.3.1 by appealing to Quillen's resolution theorem, but in fact 1.11.3.1 is needed to verify one of the hypotheses of the resolution theorem, [Q1] Section 4 Thm 3i).) We will give the complete proof that $K(\mathcal{E}) \xrightarrow{\sim} K(\boldsymbol{E}^{\sim})$ is a homotopy equivalence.

For integers $a \leq b$, let $\boldsymbol{E}_a^{\sim b}$ be the full subcategory of those complexes $E^{\cdot}$ in $\boldsymbol{E}^{\sim}$ such that $E^i = 0$ for $i \leq a-1$ and for $i \geq b+1$. Hence $\mathcal{E} = \boldsymbol{E}_0^{\sim 0}$, and $\boldsymbol{E}^{\sim}$ is the direct colimit of the $\boldsymbol{E}_a^{\sim b}$ as b goes to $+\infty$ and a goes to $-\infty$. Let $\boldsymbol{w}(\boldsymbol{E}^{\sim})$ be the quasi-isomorphisms of complexes as in 1.11.6, and let $\boldsymbol{i}(\boldsymbol{E}^{\sim})$ be isomorphisms of complexes. Set $\boldsymbol{w}(\boldsymbol{E}_a^{\sim b}) = \boldsymbol{w}(\boldsymbol{E}^{\sim}) \cap \boldsymbol{E}_a^{\sim b}$ $\boldsymbol{i}(\boldsymbol{E}_a^{\sim b}) = \boldsymbol{i}(\boldsymbol{E}^{\sim}) \cap \boldsymbol{E}_a^{\sim b}$, and co$(\boldsymbol{E}_a^{\sim b}) =$ co$(\boldsymbol{E}^{\sim}) \cap \boldsymbol{E}_a^{\sim b}$. Then $\boldsymbol{w}\boldsymbol{E}_a^{\sim b}$ and $\boldsymbol{i}\boldsymbol{E}_a^{\sim b}$ are Waldhausen categories. Let $\boldsymbol{E}_a^{\sim bw}$ be the full subcategory of $\boldsymbol{E}_a^{\sim b}$ of those complexes quasi-isomorphic to 0, with co$(\boldsymbol{E}_a^{\sim bw}) = \boldsymbol{E}_a^{\sim bw} \cap$ co$(\boldsymbol{E}^{\sim})$. Then $\boldsymbol{i}\boldsymbol{E}_a^{\sim bw}$ is a Waldhausen category.

Consider the exact functor

$$\boldsymbol{i}\boldsymbol{E}_a^{\sim b} \to \prod^{b-a+1} \mathcal{E} \tag{1.11.7.1}$$

sending a complex $E^{\cdot}$ to $(E^a, E^{a+1}, \ldots, E^b)$. We claim that this functor induces a homotopy equivalence on Waldhausen K-theory. For $b = a$, this is clear, as the exact functor is then an isomorphism. The proof of the claim now proceeds by induction on $b-a$, and consists of showing that a functor

$$\boldsymbol{iE}_a^{\sim b} \to \boldsymbol{iE}_{a+1}^{\sim b} \times \mathcal{E} \tag{1.11.7.2}$$

induces a homotopy equivalence on K-theory. This functor sends a complex $(E^a \to \cdots \to E^b)$ to the pair consisting of the subcomplex $(0 \to E^{a+1} \to \cdots \to E^b)$ and the quotient $E^a = (E^a \to 0 \to \cdots \to 0)$. This functor does induce a homotopy equivalence on K-theory by the Additivity Theorem 1.7.2 with $\boldsymbol{A} = \boldsymbol{iE}_{a+1}^{\sim b}$, $\boldsymbol{B} = \mathcal{E} = \boldsymbol{iE}_a^{\sim a}$ and $\boldsymbol{C} = \boldsymbol{iE}_a^{\sim b}$, as the canonical filtration defining our functor induces an equivalence of categories $\boldsymbol{iE}_a^{\sim b} \simeq E(\boldsymbol{A},\boldsymbol{C},\boldsymbol{B})$. This proves of claim.

On the other hand, we also claim that similarly, $K(\boldsymbol{iE}_a^{\sim bw})$ is homotopy equivalent to $\overset{b-a}{\Pi} K(\mathcal{E})$. If $b = a$, this holds trivially as $\boldsymbol{E}_a^{\sim aw} = \mathcal{E}^w$ is equivalent to the 0 category. It also holds if $a = b-1$, as the category $\boldsymbol{iE}_{b-1}^{\sim bw}$ is the category of complexes $\partial : E^{b-1} \to E^b$ with ∂ an isomorphism, and this category is equivalent to $\mathcal{E}$. The proof now proceeds by induction on $b-a$ and consists of producing a homotopy equivalence

$$K\left(\boldsymbol{iE}_a^{\sim bw}\right) \xrightarrow{\simeq} K(\boldsymbol{iE}_a^{\sim(b-1)w}) \times K\left(\boldsymbol{iE}_{b-1}^{\sim bw} = \mathcal{E}\right) \tag{1.11.7.4}$$

This homotopy equivalence results by applying the additivity theorem 1.7.2 to an equivalence of categories (cf. 1.9.4(d))

$$\boldsymbol{iE}_a^{\sim bw} \cong E(\boldsymbol{iE}_a^{\sim(b-1)w},\ \boldsymbol{iE}_a^{\sim bw},\ \boldsymbol{iE}_{b-1}^{\sim bw}) \tag{1.11.7.5}$$

To see the equivalence, we must associate an extension to a complex $E^{\cdot}$ in $\boldsymbol{iE}_a^{\sim bw}$. As $E^{\cdot}$ is acyclic and $Z^bE = E^b$ as $E^{b+1} = 0$, the map $E^{b-1} \twoheadrightarrow E^b$ is an epimorphism in $\mathcal{A}$. Hence $E^{b-1} \twoheadrightarrow E^b$ is an admissible epimorphism in $\mathcal{E}$ by 1.11.3.1. Thus its kernel $Z^{b-1}E^{\cdot}$ is in $\mathcal{E}$ and $Z^{b-1}E \rightarrowtail E^{b-1} \twoheadrightarrow E^b$ is an exact sequence in $\mathcal{E}$. The complex $\tau^{\leq b-1}E^{\cdot} = (E^a \to E^{a+1} \to \cdots \to E^{b-2} \to Z^{b-1})$ is thus in $\boldsymbol{iE}_a^{\sim(b-1)w}$. The complex $\tau^bE^{\cdot} = (B^b \xrightarrow{\cong} E^b)$ is in $\boldsymbol{iE}_{b-1}^{\sim(b-1)w}$, and $E^{\cdot}$ fits into a canonical cofibration sequence $\tau^{\leq b-1}E^{\cdot} \rightarrowtail E^{\cdot} \twoheadrightarrow \tau^bE^{\cdot}$ which defines an object of $E(\boldsymbol{iE}_a^{\sim(b-1)w},\ \boldsymbol{iE}_a^{\sim bw},\ \boldsymbol{iE}_{b-1}^{\sim bw})$

(1.11.7.6)

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \\
\downarrow & & \downarrow & & \\
0 & & 0. & & \\
\downarrow & & \downarrow & & \downarrow \\
E^a & = & E^a & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
E^{a+1} & = & E^{a+1} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
E^{b-2} & = & E^{b-2} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
Z^{b-1} & \rightarrowtail & E^{b-1} & \twoheadrightarrow & B^b \\
\downarrow & & \downarrow & & \wr\| \\
0 & \longrightarrow & E^b & = & E^b \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
 & & & & \cdot\,\cdot
\end{array}
$$

The inverse equivalence of the categories takes the total complex $E^{\cdot}$ and forgets the extensions. It is easy to check that both the equivalence of categories and its inverse are exact functors. This completes the proof of our second claim that $K(\boldsymbol{iE}_a^{\sim bw})$ is homotopy equivalent to $\prod^{b-a} K(\mathcal{E})$. In fact our proof shows that for $E^{\cdot}$ in $\boldsymbol{E}_a^{\sim bw}$, the $Z^k E^{\cdot} = B^k E^{\cdot}$ are objects of $\mathcal{E}$, that $E^{\cdot} \mapsto B^k(E^{\cdot})$ is an exact functor, and that our claimed homotopy equivalence is induced by the exact functor $\boldsymbol{iE}_a^{\sim bw} \to \overset{b-a}{\Pi}\,\mathcal{E}$ given by sending $E^\cdot$ to $(B^{a+1}E^\cdot,\ B^{a+2}E^\cdot,\ \dots,\ B^bE)$.

Now consider the exact inclusion $\boldsymbol{iE}_a^{\sim bw} \to \boldsymbol{iE}_a^{\sim b}$. This induces a map on K-theory spectra, which by the two claims above is homotopy equivalent to a map $\overset{b-a}{\Pi} K(\mathcal{E}) \to \overset{b-a+1}{\Pi} K(\mathcal{E})$

$$\text{(1.11.7.7)}\qquad \begin{array}{ccc} K\left(\boldsymbol{iE}_a^{\sim bw}\right) & \longrightarrow & K\left(\boldsymbol{iE}_a^{\sim b}\right) \\ \downarrow\wr & & \downarrow\wr \\ \overset{b-a}{\Pi} K(\mathcal{E}) & \longrightarrow & \overset{b-a+1}{\Pi} K(\mathcal{E}) \end{array}$$

For $E^\cdot$ in $\boldsymbol{iE}_a^{\sim bw}$, the corresponding term in $\overset{b-a+1}{\Pi}\,\mathcal{E}$ is $(E^a,\ E^{a+1},\ \dots,\ E^b)$ while the corresponding term in $\overset{b-a}{\Pi}\,\mathcal{E}$ is $(B^{a+1},\ \dots,\ B^b)$. From the exact sequence $Z^k \rightarrowtail E^k \twoheadrightarrow B^{k+1}$ for $E^\cdot$ and the fact that $Z^k = B^k$ by acyclicity, the Additivity Theorem 1.7.2 shows the map on K-theory spectra induced by sending $E^\cdot$ to E^k is homotopic to the sum of the maps induced on K-theory induced by sending $E^\cdot$ to B^k and to B^{k+1}. Considering also that $B^a = \operatorname{im} E^{a-1} = 0$, we see that our map $\overset{b-a}{\Pi} K(\mathcal{E}) \to \overset{b-a+1}{\Pi} K(\mathcal{E})$ in (1.11.7.7) is that induced by the exact functor:

(1.11.7.8)
$$\left(B^{a+1},\ \dots,\ B^b\right) \longmapsto \left(B^{a+1},\ B^{a+1} \oplus B^{a+2},\ \dots,\ B^{b-1} \oplus B^b\right).$$

The homotopy cofibre of this map is $K(\mathcal{E})$, with $\overset{b-a+1}{\Pi} K(\mathcal{E}) \to K(\mathcal{E})$ induced by $(x_a, \dots, x_b) \mapsto \Sigma(-1)^k x_k$. Taking the direct colimit as a goes to $-\infty$ and b goes to $+\infty$, we get a homotopy cofibre sequence

$$\text{(1.11.7.9)}\qquad K\left(\boldsymbol{iE}^{\sim w}\right) \to K(\boldsymbol{iE}^\sim) \to K(\mathcal{E})$$

where $K(\boldsymbol{iE}^\sim) \to K(\mathcal{E})$ sends $E^\cdot$ to its Euler characteristic $\Sigma(-1)^k E^k$.

But by the localization theorem 1.8.2, the homotopy cofibre spectrum of $K(\boldsymbol{iE}^{\sim w}) \to K(\boldsymbol{iE}^\sim)$ is $K(\boldsymbol{wE}^\sim) = K(\boldsymbol{E}^\sim)$. Thus there is a homotopy equivalence $K(\mathcal{E}) \xrightarrow{\sim} K(\boldsymbol{E}^\sim)$, which in fact is the map induced by the exact functor $\mathcal{E} \to \boldsymbol{E}^\sim$. This proves the theorem.

1.11.8. *Porism.* Under the hypotheses of 1.11.3 and with the notation of 1.11.6, a complex $E^\cdot$ in $\mathcal{E}$ is acyclic in $\mathcal{C}(\mathcal{A})$ if and only if all cycle and boundary objects $Z^kE^\cdot$ and $B^kE^\cdot$ are in $\mathcal{E} \subseteq \mathcal{A}$, $B^kE^\cdot = Z^kE^\cdot$ in $\mathcal{E}$, and the sequences $Z^k \rightarrowtail E^k \twoheadrightarrow B^{k+1}$ are exact in $\mathcal{E}$. In particular, this is independent of the choice of $\mathcal{A}$, provided only it satisfies 1.11.3 and especially 1.11.3.1.

Also, the set $w(E)$ of quasi-isomorphisms of complexes in $\mathcal{E}$ is independent of the choice of $\mathcal{A}$.

For the first paragraph is a porism of the proof of 1.11.7. To verify the second paragraph, we note that a map in E is a quasi-isomorphism (with respect to $\mathcal{A}$) iff its mapping cone is acyclic. The formation of mapping cones uses only the additive structure of $\mathcal{E}$, and acyclicity is independent of the choice of $\mathcal{A}$ by the first paragraph.

1.11.9. *Remark.* An exact functor $f : \mathcal{E} \to \mathcal{E}'$ induces a compatible additive functor of the associated Gabriel-Quillen abelian categories $f^* : \mathcal{A} \to \mathcal{A}'$, by A.8.2. Although $f^* : \mathcal{A} \to \mathcal{A}'$ need not preserve exact sequences in $\mathcal{A}$, it does preserve exact sequences in $\mathcal{E}$, A.8.5. Hence it preserves quasi-isomorphisms of complexes in $\mathcal{E}$ by 1.11.8, and induces complicial exact functors $f^* : E \to E'$, $E^\sim \to E'^\sim$. Thus 1.11.7 becomes natural in $\mathcal{E}$ with $\mathcal{E} \to \mathcal{A}$ chosen as the Gabriel- Quillen embedding 1.11.5.

1.11.10. *Remark.* If $\mathcal{E}$ is an exact category not satisfying the hypothesis of 1.11.5 that weakly split epimorphisms are admissible epimorphisms, we let $\mathcal{E}'$ be its Karoubianization A.9.1. Then 1.11.7 applies to the Gabriel-Quillen embedding of $\mathcal{E}'$, as $\mathcal{E}'$ satisfies the hypothesis of 1.11.5. As $K(\mathcal{E})$ is a cover of $K(\mathcal{E}')$ so $K_i(\mathcal{E}) = K_i(\mathcal{E}')$ for $i > 0$, by classical cofinality A.9.1, this shows the extra hypothesis of 1.11.3.1 is essentially harmless. Indeed by 1.11.1 we get in general that $K(\mathcal{E})$ is homotopy equivalent to the K-theory of the category of those bounded chain complexes in $\mathcal{E}'$ whose Euler characteristic lies in $K_0(\mathcal{E}) \subseteq K_0(\mathcal{E}')$.

2. Perfect complexes on schemes

2.0. In this section, we review and extend the theory of perfect complexes on a scheme. This theory was discovered and developed by Grothendieck and his school (especially Illusie) in [SGA 6] as a more flexible replacement for the naive theory of algebraic vector bundles on a scheme in K-theory. The somewhat new results of this section are the characterizations of pseudo-coherence and perfection in 2.4, and some of the functoriality statements in 2.6 and 2.7. Aside from a few minor improvements, the rest of this material is already in [SGA 6].

2.1.1. *Definition.* A scheme with an ample family of line bundles is a scheme X, which is quasi-compact and quasi-separated, and which has a family of line bundles $\{\mathcal{L}_\alpha\}$ which satisfy any one of the following equivalent conditions ([SGA 6] II 2.2.3 and proof thereof).

(a) Let n run over all positive integers $n \geq 1$ and let $\mathcal{L}_\alpha$ run over the family of line bundles. Let $f \in \Gamma(X, \mathcal{L}_\alpha^{\otimes n})$ run over all the global sections of all the $\mathcal{L}_\alpha^{\otimes n}$. Then the resulting family of open set $X_f = \{x \in X | f(x) \neq 0\}$ is a basis for the Zariski topology of X.

(b) One may choose a set of positive integers $n \geq 1$, a set of line bundles $\mathcal{L}_\alpha$ in the family, and a set of global sections $f \in \Gamma(X, \mathcal{L}_\alpha^{\otimes n})$ such that the set $\{X_f\}$ is a basis for the Zariski topology of X and all these X_f are affine schemes.

(c) One may choose a set of positive integers $n \geq 1$, line bundles $\mathcal{L}_\alpha$ in the family, and global sections $f \in \Gamma(X, \mathcal{L}_\alpha^{\otimes n})$ such that $\{X_f\}$ is a cover of X by affine schemes.

(d) For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, the evaluation map

$$\bigoplus_{\alpha, n \geq 1} \Gamma\left(X, \mathcal{F} \otimes \mathcal{L}_\alpha^{\otimes n}\right) \otimes \mathcal{L}_\alpha^{-\otimes n} \twoheadrightarrow \mathcal{F}$$

is an epimorphism.

2.1.2. *Examples.* (a) Any scheme with an ample line bundle in the sense of [EGA] II 4.5.3 and IV 1.7 has an ample family (consisting of one line bundle) in the present sense. As special cases we have (b) and (c):

(b) Any affine scheme has an ample family of line bundles.

(c) Any scheme quasi-projective over an affine scheme has an ample family of line bundles. In particular, any quasi-affine scheme has an ample family of line bundles.

(d) Any separated regular noetherian scheme has an ample family of line bundles ([SGA 6] II 2.2.7.1).

(e) If Y has an ample family of line bundles, and $U \subseteq Y$ is a quasi-compact open, then U has an ample family of line bundles, given as the restriction of the family on Y. This is clear by 2.1.1(a).

(f) Let Y be a scheme with an ample family of line bundles $\{\mathcal{L}_\alpha\}$. Let $f : X \to Y$ be a quasi-compact and quasi-separated map of schemes. Suppose X has an f-ample family of line bundles $\{\mathcal{K}_\beta\}$. That is, suppose Y is covered by (affine) opens $U \subseteq Y$ such that $\{\mathcal{K}_\beta | f^{-1}(U)\}$ is an ample family on each $f^{-1}(U)$. Then $\{f^*\mathcal{L}_\alpha^{\otimes k} \otimes \mathcal{K}_\beta^{\otimes n} | k \geq 1,\ n \geq 1\}$ is an ample family on X, as clearly follows from criterion 2.1.1(a) and [EGA] I 6.8.1. As special cases we have (g) and (h):

(g) If Y has an ample family of line bundles and $f : X \to Y$ is an affine map of schemes, then X has an ample family of line bundles.

(h) If Y has an ample family of line bundles and $f : X \to Y$ is a quasi-projective map of schemes, then X has an ample family of line bundles.

2.1.3. **Lemma.** *Let X have an ample family of line bundles. Then*

(a) *For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, there is a locally free $\mathcal{O}_X$-module $\mathcal{E}$ and an epimorphism $\mathcal{E} \twoheadrightarrow \mathcal{F}$.*

(b) *For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ of finite type, there is an algebraic vector bundle (i.e., a locally free $\mathcal{O}_X$-module of finite type) $\mathcal{E}$ on X, and an epimorphism $\mathcal{E} \twoheadrightarrow \mathcal{F}$.*

(c) *For any epimorphism $\mathcal{G} \to \mathcal{F}$ of quasi-coherent $\mathcal{O}_X$-modules with $\mathcal{F}$ of finite type, there is an algebraic vector bundle $\mathcal{E}$ and a map $\mathcal{E} \to \mathcal{G}$ such that the composite map $\mathcal{E} \to \mathcal{G} \to \mathcal{F}$ is an epimorphism onto $\mathcal{F}$.*

In all cases (a), (b), (c), *$\mathcal{E}$ may be taken to be a direct sum of tensor powers of line bundles of the family.*

Proof. First note by quasi-compactness of X and 2.1.1(c) that X has an ample finite subfamily of line bundles, $\mathcal{L}_{\alpha_1}, \ldots, \mathcal{L}_{\alpha_n}$. As each $\mathcal{L}_{\alpha_i}$ is locally free, X is covered by open sets where all the $\mathcal{L}_{\alpha_i}$ are simultaneously free. Thus any direct sum of $\mathcal{L}_{\alpha_i}^{-\otimes n}$ is locally free on X, even if it is an infinite sum.

To prove (a) we appeal to 2.1.1(d), and let $\mathcal{E}$ be the sum with one factor $\mathcal{L}_{\alpha_i}^{-\otimes n}$ for each global section in $\Gamma(X, \ \mathcal{F} \otimes \mathcal{L}_{\alpha_i}^{\otimes n})$.

Case (b) follows from case (c) on setting $\mathcal{G} = \mathcal{F}$.

To prove case (c), we consider the epimorphism $\oplus \mathcal{L}_{\alpha_i}^{-\otimes n} \twoheadrightarrow \mathcal{G}$ constructed in the proof of case (a). The composite of this epimorphism with $\mathcal{G} \twoheadrightarrow \mathcal{F}$ is also an epimorphism. As $\mathcal{F}$ has finite type and X is quasi-compact, some finite subsum of the factors $\mathcal{L}_{\alpha_i}^{-\otimes n}$ maps epimorphically to $\mathcal{F}$. We let $\mathcal{E}$ be this finite subsum, and take $\mathcal{E} \subseteq \oplus \mathcal{L}_{\alpha_i}^{-\otimes n} \twoheadrightarrow \mathcal{G}$ the induced map.

2.2. Logically, the reader should now examine Appendix B before returning to 2.2.1. We note the convention that the word "$\mathcal{O}_X$-module" means "a sheaf on the scheme X which is a sheaf of modules over the sheaf of rings $\mathcal{O}_X$," and does not apply to a general presheaf of modules over the sheaf of rings $\mathcal{O}_X$. That is, an "$\mathcal{O}_X$-module" is a $\mathcal{O}_X$-module in the Zariski topos of X.

2.1.1. *Definition* ([SGA 6] I 2.1). For any integer m, a chain complex $E^{\cdot}$ of $\mathcal{O}_X$-modules on a scheme X is said to be strictly m-pseudo-coherent if E^i is an algebraic vector bundle on X for all $i \geq m$ and $E^i = 0$ for all i sufficiently large. A complex $E^{\cdot}$ is strictly pseudo-coherent if it is strictly m-pseudo-coherent for all m, i.e., if it is a bounded above complex of algebraic vector bundles.

2.2.2. *Definition* ([SGA 6] I 2.1). A complex $E^{\cdot}$ of $\mathcal{O}_X$-modules is strictly perfect if it is strictly pseudo-coherent and strictly bounded below. That is, a strict perfect complex is a strict bounded complex of algebraic vector bundles.

2.2.3. **Lemma** ([SGA 6] I 2.10b). *Let $A^\cdot$ be a complex of $\mathcal{O}_X$-modules with $H^i(A^\cdot) = 0$ for $i \geq m+1$. Then $H^m(A^\cdot)$ is an $\mathcal{O}_X$-module of finite type iff for each $x \in X$ there is an open neighborhood U of x and an isomorphism in the derived category $D(\mathcal{O}_U\text{-Mod})$ between the restriction $A^\cdot|U$ and a strictly m-pseudo-coherent complex on U.*

Proof. Suppose $H^m(A^\cdot)$ is of finite type, and let $x \in X$. Then for some positive integer k there is an epimorphism of stalks $\oplus^k \mathcal{O}_{X,x} \twoheadrightarrow H^m(A)_x$. As $\oplus^k \mathcal{O}_{X,x}$ is a free module over $\mathcal{O}_{X,x}$, this map lifts to the group of cycles $Z^m(A^\cdot)_x \twoheadrightarrow H^m(A^\cdot)_x$. The lifted map extends over some open *nbd* U of x to a map $\oplus^k \mathcal{O}_U \to Z^m(A^\cdot)|U \subseteq A^m|U$. Shrinking U, we may assume that $\oplus^k \mathcal{O}_U \to H^m(A^\cdot)|U$ is epimorphic. Then the complex consisting only of $\oplus^k \mathcal{O}_U$ in degree m maps to the complex $A^\cdot|U$ by a map inducing an isomorphism on the cohomology $H^p = 0$ for $p > m$, and an epimorphism $\oplus^k \mathcal{O}_U \twoheadrightarrow H^m(A^\cdot)|U$ on H^m. We now apply Lemma 1.9.5 with $\mathcal{A} = \mathcal{D} =$ category of $\mathcal{O}_U$-modules and $\mathcal{C} = \mathcal{C}^-(\mathcal{A})$. This inductive construction lemma produces a complex $D^\cdot$ on U and a quasi-isomorphism $D^\cdot \xrightarrow{\sim} A^\cdot|U$. Moreover, $D^\cdot$ satisfies $D^m = \oplus^k \mathcal{O}_U$, and $D^p = 0$ for $p > m$. Thus $D^\cdot$ is strict m-pseudo-coherent on U, as required.

Conversely, suppose $A^\cdot$ is locally quasi-isomorphic to a strict m-pseudo-coherent complex. As $H^m(A^\cdot)$ is a quasi-isomorphism invariant, and as being of finite type is a local question, passing to $U \subseteq X$ we may assume that $A^\cdot$ is strict m-pseudo-coherent. Recall that $H^i(A^\cdot) = 0$ for $i \geq m+1$. Applying 1.9.4(a), with $\mathcal{A} = \mathcal{O}_U$-Mod and $\mathcal{B} =$ the category of algebraic vector bundles on U, to the truncated complex of vector bundles $\sigma^m A^\cdot$, we see that $Z^m(A^\cdot) = Z^m(\sigma^m A^\cdot)$ is an algebraic vector bundle on U, and hence is of finite type. As $H^m(A^\cdot)$ is a quotient of $Z^m A^\cdot$, it is also of finite type, as required.

2.2.4. **Lemma** ([SGA 6]). *Let U be a scheme, and x a point of U. Consider the solid arrow diagram of complexes of $\mathcal{O}_U$-modules, $F^\cdot$, $G^\cdot$, $E^\cdot$:*

(2.2.4.1)
$$
\begin{array}{ccc}
 & & E'^\cdot \\
 & \overset{d}{\nearrow} & \downarrow c \\
F^\cdot & \xrightarrow{a} & G^\cdot \\
 & & \downarrow b \\
 & & E^\cdot
\end{array}
$$

Then under any of the following sets of conditions, there exists a smaller nbd V of x, a complex $E'^\cdot$ of $\mathcal{O}_V$-modules on V, and maps $c : E'^\cdot \to$

$G^\cdot|V$, $d : F^\cdot|V \to E'^\cdot$ such that $cd = a|V$, and which satisfy the extra conclusions attached to the corresponding set of conditions:

(a) If $E^\cdot$ and $F^\cdot$ are strict m-pseudo-coherent, and the truncation $\tau^m(b) : \tau^m G^\cdot \xrightarrow{\sim} \tau^m E^\cdot$ is a quasi-isomorphism, then we may take $E'^\cdot$ to be strict m-pseudo-coherent and c to be a quasi-isomorphism.

(b) If $E^\cdot$ is strict m-pseudo-coherent, and $F^\cdot$ is strict perfect, with $\tau^m(b) : \tau^m G^\cdot \xrightarrow{\sim} \tau^m E^\cdot$ a quasi-isomorphism, then we may take $E'^\cdot$ to be strict perfect and c to be an m-quasi-isomorphism.

(c) If $E^\cdot$ and $F^\cdot$ are strict perfect and $b : G^\cdot \xrightarrow{\sim} E^\cdot$ is a quasi-isomorphism, then we may take $E'^\cdot$ to be strict perfect and c to be a quasi- isomorphism.

Proof. In all cases $E^\cdot$ and $F^\cdot$ are strictly bounded above. Hence $G^\cdot$ is cohomologically bounded above, and replacing $G^\cdot$ by the subcomplex $\tau^{\leq k}G^\cdot$ for some sufficiently large k, we may assume that $G^\cdot$ is strictly bounded above.

Consider the germs of complexes $E^\cdot_x$, $F^\cdot_x$, $G^\cdot_x$ at the point $x \in U$. We apply to $F^\cdot_x \to G^\cdot_x$ the inductive construction Lemma 1.9.5 with $\mathcal{A}$ the category of modules over $\mathcal{O}_{X,x}$, $\mathcal{D}$ the category of free modules of finite type over $\mathcal{O}_{X,x}$, and $\mathcal{C}$ the category of complexes in $\mathcal{A}$ with a map b to a strict m-pseudo-coherent complex such that $\tau^m(b)$ is a quasi- isomorphism. The quasi-isomorphism $\tau^m(b)$ and Lemma 2.2.3 show that hypothesis 1.9.5.1 is met for $n-1 \geq m$. For if $C^\cdot$ is in $\mathcal{C}$ with $H^i(C^\cdot) = 0$ for $i \geq n \geq m+1$, $H^{n-1}(C^\cdot)$ is isomorphic to an $\mathcal{O}_{X,x}$ module of finite type via $\tau^m(b)$ and 2.2.3. So there is an epimorphism $\oplus^k \mathcal{O}_{X,x} \to H^{n-1}(C^\cdot)$. As $\oplus^k \mathcal{O}_{X,x}$ is a free module over the local ring $\mathcal{O}_{X,x}$, this epimorphism lifts along any epimorphism of modules over $\mathcal{O}_{X,x}$, $A_x \twoheadrightarrow H^{n-1}(C^\cdot)$. This verifies 1.9.5.1 as long as $n-1 \geq m$.

Now the variant 1.9.5.9 of 1.9.5 provides a strict perfect complex $\sigma^m E'^\cdot_x$, a map $\sigma^m d : \sigma^m F^\cdot_x \to \sigma^m E'^\cdot_x$, and an m-quasi-isomorphism $\sigma^m E'^\cdot_x \to \sigma^m G^\cdot_x \to G^\cdot_x$, which is the germ at x of a truncated version of (2.2.4.1).

As $\sigma^m E'^\cdot_x$, $\sigma^m F^\cdot$, and $\sigma^m E^\cdot$ are all bounded complexes of free modules of finite type, there is a small open *nbd* V of x in U over which $\sigma^m E'^\cdot_x$ extends to a strict perfect complex $\sigma^m E'^\cdot$, over which the maps d and c extend, and over which the map $bc : E'^\cdot \to E^\cdot$ is an m-quasi-isomorphism (i.e., its mapping cone is acyclic in degrees $\geq m$). Then as $\tau^m(b)$ is a quasi-isomorphism, it follows that $\sigma^m(c) : \sigma^m E'^\cdot \to G^\cdot$ is an m-quasi-isomorphism on V.

Now we apply 1.9.5 again, now with $\mathcal{A} = \mathcal{O}_V$-Mod, $\mathcal{D} = \mathcal{O}_V$-Mod to construct the rest of $E'^\cdot$ and d, c, leaving $\sigma^m E'^\cdot$ unchanged. Then E' is strict m-pseudo-coherent, as $\sigma^m E'^\cdot$ is strict perfect. This proves 2.2.4(a).

To prove (b), let $E^{\cdot\prime}$ be the pushout of $\sigma^m F^\cdot \to F^\cdot$ and the map $\sigma^m F^\cdot \to \sigma^m E^{\cdot\prime}$ constructed above. Let $c : E^{\cdot\prime} \to G^\cdot$ be the map induced by $a : F^\cdot \to G^\cdot$ and $\sigma^m(c) : \sigma^m E^{\cdot\prime} \to G^\cdot$. Then $E^{\cdot\prime} \to G^\cdot$ is an m-quasi-isomorphism as both $\sigma^m(c) : \sigma^m E'^{\cdot} \to G^{\cdot}$ and $\sigma^m E'^{\cdot} \to E'^{\cdot}$ are. As the pushout of strict perfect complexes along a degree-wise split monomorphism, $E'^{\cdot}$ is strict perfect.

To prove (c), we take $m \ll 0$ so that $E^i = 0 = F^i$ for $i \leq m+1$. We apply the proof of (b), which gives a strict perfect $E'^{\cdot}$ such that $\sigma^m E'^{\cdot} = E'^{\cdot}$ (as $\sigma^m F^{\cdot} = F^{\cdot}$), and a map $bc : E'^{\cdot} \to E^{\cdot}$ which is an m-quasi-isomorphism on V. The mapping cone $M^{\cdot}$ of bc is strict perfect, is 0 below degree $m-1$ and is acyclic except for $H^{m-1}(M^{\cdot}) = \ker(H^m(E'^{\cdot}) \twoheadrightarrow H^m(E^{\cdot}))$. By 1.9.4(a) $H^{m-1}(M^{\cdot}) = Z^{m-1}M^{\cdot}$ is an algebraic vector bundle. As $E'^{m-1} = 0 = E^m$, we note that $Z^{m-1}M^{\cdot} = \ker(Z^m E'^{\cdot} \to Z^m E^{\cdot}) = Z^m E'^{\cdot}$. As the map $b : G^{\cdot} \to E^{\cdot}$ is a quasi-isomorphism, $H^{m-1}(M^{\cdot})$ goes to 0 in $H^m(G^{\cdot}) \cong H^m(E^{\cdot})$. Then $Z^m E'^{\cdot}_x \to Z^m G^{\cdot}_x$ lifts along $\partial G^{m-1}_x \to Z^m G_x$ as a map of germs of $\mathcal{O}_X$-modules over the local ring $\mathcal{O}_{X,x}$. As $Z^m E'^{\cdot} = Z^{m-1}M^{\cdot}$ is locally free of finite type, this lift extends to a map of $\mathcal{O}_V$-modules $Z^m E'^{\cdot} \to G^{m-1}$ lifting $Z^m E'^{\cdot} \to Z^m G^{\cdot}$ on some smaller open *nbd* V of x. We now extend $\sigma^m E'^{\cdot}$ to a new $E'^{\cdot}$ by $E'^{m-1} = Z^m E'^{\cdot}$ with boundary $\partial : E'^{m-1} \to E'^m$ given by the inclusion $Z^m E'^{\cdot} \subseteq E'^m$. As $Z^m E'^{\cdot} = Z^{m-1}M^{\cdot}$ is a vector bundle, the new $E'^{\cdot}$ is still strict perfect. The map $\sigma^m c : \sigma^m E'^{\cdot} \to G^{\cdot}$ is extended to the new $E'^{\cdot}$ by using $Z^m E'^{\cdot} \to G^{m-1}$ in degree $m-1$. Now clearly $c : E'^{\cdot} \to G^{\cdot}$ is a quasi- isomorphism, and the other conditions of (c) are met.

2.2.5. **Lemma** ([SGA 6] I 2.2). *On a scheme X, the following conditions are equivalent for any complex $E^{\cdot}$ of $\mathcal{O}_X$-modules:*

2.2.5.1. For every point $x \in X$, there is a nbd U of x, a strict n-pseudo-coherent complex $F^{\cdot}$, and a quasi-isomorphism $F^{\cdot} \xrightarrow{\sim} E^{\cdot}|U$.

2.2.5.2. For every point $x \in X$, there is a nbd U of x, a strict perfect complex $F^{\cdot}$, and an n-quasi-isomorphism $F^{\cdot} \to E^{\cdot}|U$.

2.2.5.3. For every point $x \in X$, there is a nbd U of x, a strict n-pseudo-coherent complex $F^{\cdot}$, and an isomorphism between $F^{\cdot}$ and $E^{\cdot}|U$ in the derived category $D(\mathcal{O}_U$-Mod).

2.2.5.4. For every point $x \in X$, there is a nbd U of x, a strict perfect complex $F^{\cdot}$, and an n-quasi-isomorphism $F^{\cdot} \to E^{\cdot}|U$ in the derived category $D(\mathcal{O}_U$-Mod) (that is, there is a map in the derived category inducing an epimorphism on H^n and an isomorphism on H^k for $k \geq n+1$).

Proof. We see that (1) $\Rightarrow$ (2) by replacing $F^{\cdot}$ in (1) by the strict perfect $\sigma^n F^{\cdot}$. Clearly (2) $\Rightarrow$ (4) and (1) $\Rightarrow$ (3). It suffices then to show that (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1).

To see that (3) $\Rightarrow$ (1), we consider an isomorphism $F^{\cdot} \to E^{\cdot}|U$ in $D(\mathcal{O}_U$-Mod). By the calculus of fractions, this is represented by a datum of strict maps which are quasi-isomorphisms $F^{\cdot} \xleftarrow{\sim} G^{\cdot} \xrightarrow{\sim} E^{\cdot}|U$. By 2.2.4(a), after shrinking the *nbd* U, there is a strict n-pseudo-coherent

$F'^{\cdot}$ and a quasi-isomorphism $F'^{\cdot} \xrightarrow{\sim} G^{\cdot}$. Then the composite $F'^{\cdot} \xrightarrow{\sim} G^{\cdot} \xrightarrow{\sim} E^{\cdot}|U$ is the strict quasi-isomorphism required by (1).

To see that (4) $\Rightarrow$ (1), we represent the n-quasi-isomorphism in $D(\mathcal{O}_U\text{-Mod})$ by a datum of strict maps $F^{\cdot} \xleftarrow{\sim} G^{\cdot} \to E^{\cdot}|U$ where $G^{\cdot} \to E^{\cdot}|U$ is an n-quasi- isomorphism. After shrinking U, by 2.2.4(c) there is a strict perfect $F'^{\cdot}$ and a quasi-isomorphism $F'^{\cdot} \to G^{\cdot}$. Then $F'^{\cdot} \to E^{\cdot}|U$ is an n-quasi-isomorphism. Applying the Inductive Construction Lemma 1.9.5 with $\mathcal{A} = \mathcal{D} = \mathcal{O}_U$-Mod and $\mathcal{C} = \mathcal{C}^-(\mathcal{A})$, we obtain a complex F'' and a quasi-isomorphism $F'' \xrightarrow{\sim} E^{\cdot}|U$, such that $\sigma^n F'' = \sigma^n F'$. Thus $\sigma^n F''$ is strict perfect, so F'' is strict n-pseudo-coherent as required.

2.2.6. *Definition* ([SGA 6] I 2.3). A complex $E^{\cdot}$ of $\mathcal{O}_X$-modules on a scheme X is said to be n-pseudo-coherent if any of the equivalent conditions 2.2.5.1 - 2.2.5.4 hold. The complex $E^{\cdot}$ is said to be pseudo-coherent if it is n- pseudo-coherent for all integers n.

2.2.7. Clearly pseudo-coherence of $E^{\cdot}$ depends only on the quasi-isomorphism class of $E^{\cdot}$, and is a local property on X. The cohomology sheaves $H^*(E^{\cdot})$ of a pseudo-coherent complex are all quasi-coherent $\mathcal{O}_X$-modules. If X is quasi-compact, and $E^{\cdot}$ is pseudo-coherent, there is an N such that $H^k(E^{\cdot}) = 0$ for all $k > N$, as this is true locally on X and since any open cover of X has a finite subcover.

A strict pseudo-coherent complex is pseudo-coherent. It will follow from 2.3.1(b) below that any pseudo-coherent complex of quasi-coherent $\mathcal{O}_X$-modules is locally quasi-isomorphic to a strict pseudo-coherent complex. In fact, it will be quasi-isomorphic to a strict pseudo-coherent complex on any affine open subscheme. For a pseudo-coherent complex of general $\mathcal{O}_X$-modules, there will locally be n-quasi-isomorphisms with a strict pseudo-coherent complex, but the local *nbds* where the n-quasi-isomorphisms are defined may shrink as n goes to $-\infty$, and so may fail to exist in the limit. So there may not be a local quasi-isomorphism with a strict pseudo-coherent complex. This phenomenon renders the auxiliary concept of n-pseudo-coherent necessary to our work.

2.2.8. *Example* ([SGA 6] I Section 3). A complex $E^{\cdot}$ of $\mathcal{O}_X$-modules on a noetherian scheme X is pseudo-coherent iff $E^{\cdot}$ is cohomologically bounded above and all the $H^k(E^{\cdot})$ are coherent $\mathcal{O}_X$-modules.

Proof. If E is pseudo-coherent, then $E^{\cdot}$ is locally $(k-1)$-quasi-isomorphic to a complex of coherent locally free sheaves. Computing $H^k(E^{\cdot})$ as H^k of the latter complex, we see that $H^k(E^{\cdot})$ is coherent.

Conversely, suppose $E^{\cdot}$ is cohomologically bounded above with coherent cohomology. Let m be an integer and x a point of X. We apply the Inductive Construction Lemma 1.9.5 with $\mathcal{A} = \mathcal{O}_{X,x}$-modules, $\mathcal{D} =$ free $\mathcal{O}_{X,x}$-

modules of finite type, and $\mathcal{C}$ = cohomologically bounded above complexes with coherent cohomology to produce a strict pseudo-coherent complex of $\mathcal{O}_{X,x}$-modules $F_x^{\cdot}$ and a quasi-isomorphism $F_x^{\cdot} \xrightarrow{\sim} E_x^{\cdot}$. Then $\sigma^m F_x^{\cdot}$ is strict perfect, and $\sigma^m F_x^{\cdot} \to E_x^{\cdot}$ is an m-quasi-isomorphic. The strict perfect complex $\sigma^m F_x^{\cdot}$ and the map $\sigma^m F_x^{\cdot} \to E_x^{\cdot}$ extend to a strict perfect complex $\sigma^m F^{\cdot}$ and map $\sigma^m F^{\cdot} \to E^{\cdot}|V$ on some small *nbd* V of x, as this requires only finitely many extensions of germs of sections of various sheaves at x. Choosing V smaller and using coherence of cohomology, we can arrange that $\sigma^m F^{\cdot} \to E^{\cdot}$ is an m-quasi-isomorphism on V. This shows criterion 2.2.5.2 holds for $E^{\cdot}$.

2.2.9. **Lemma** ([SGA] I Section 4). *The following conditions on a complex $E^{\cdot}$ of $\mathcal{O}_X$-modules on a scheme are equivalent:*

2.2.9.1. For each point $x \in X$, there is an nbd U of x, a strict perfect complex $F^{\cdot}$ on U, and a quasi-isomorphism $F^{\cdot} \xrightarrow{\sim} E^{\cdot}|U$.

2.2.9.2. For each point $x \in X$, there is an nbd U of x, a strict perfect complex $F^{\cdot}$ on U, and an isomorphism in $D(\mathcal{O}_X\text{-}Mod)$ between $E^{\cdot}|U$ and $F^{\cdot}$.

Proof. Clearly (1) $\Rightarrow$ (2). To show (2) $\Rightarrow$ (1), we represent the isomorphism in the derived category via calculus of fractions as a datum of strict quasi-isomorphisms $F^{\cdot} \xleftarrow{\sim} G^{\cdot} \xrightarrow{\sim} E^{\cdot}|U$. Then we apply 2.2.4 (c) to $G^{\cdot} \xrightarrow{\sim} F^{\cdot}$ to produce after shrinking U a strict perfect $F'^{\cdot}$, and a strict quasi-isomorphism $F'^{\cdot} \xrightarrow{\sim} G^{\cdot} \xrightarrow{\sim} E^{\cdot}|U$.

2.2.10. *Definition* ([SGA 6] I 4.2). A complex $E^{\cdot}$ of $\mathcal{O}_X$-modules on a scheme is perfect if it is locally quasi-isomorphic to a strict perfect complex, i.e., if 2.2.9.1 or 2.2.9.2 hold for $E^{\cdot}$.

2.2.11. *Definition* ([SGA 6] I 5.1). A complex $E^{\cdot}$ of $\mathcal{O}_X$-modules has Tor-amplitude contained in $[a,b]$ for integers $a \leq b$ if for all $\mathcal{O}_X$-modules $\mathcal{F}$, $H^k(E^{\cdot} \otimes^L_{\mathcal{O}_X} \mathcal{F}) = 0$ unless $a \leq k \leq b$. If such an a and b exist, one says $E^{\cdot}$ has (globally) finite Tor-amplitude. If X is covered by opens U such that a and b exist on each U for $E^{\cdot}|U$, one says that $E^{\cdot}$ has locally finite Tor-amplitude.

2.2.12. **Proposition** ([SGA 6] I 5.8.1). *A complex $E^{\cdot}$ of $\mathcal{O}_X$-modules is perfect iff $E^{\cdot}$ is pseudo-coherent and has locally finite Tor-amplitude.*

Proof. A strict perfect complex E' is flat and strictly bounded, so that $E'^{\cdot} \otimes^L \mathcal{F} = E'^{\cdot} \otimes \mathcal{F}$ is cohomologically bounded. So $E'^{\cdot}$ is pseudo-coherent and of finite Tor-amplitude. As a perfect complex $E^{\cdot}$ is locally quasi-isomorphic to a strict perfect $E'^{\cdot}$, it is pseudo-coherent and of locally finite Tor-amplitude.

Conversely, suppose $E^\cdot$ is pseudo-coherent of locally finite Tor-amplitude. Take any point $x \in X$, and pick a *nbd* U of x on which $E^\cdot$ has Tor-amplitude in $[a,b]$. By 2.2.5.3, after shrinking U, we may replace $E^\cdot$ up to quasi-isomorphism and assume that $E^\cdot$ is strict $(a-2)$-pseudo-coherent. Then as $\sigma^a E^\cdot$ is a strict perfect complex strictly bounded below by a, it has Tor-amplitude bounded below by a. Applying 2.2.11 with $\mathcal{F} = \mathcal{O}_X$, we see that $E^\cdot$ is cohomologically bounded between a and b. Thus $\sigma^a E^\cdot \to E^\cdot$ induces an isomorphism on cohomology H^k for $k \neq a$, and on H^a induces an epimorphism $H^a(\sigma^a E^\cdot) = Z^a E^\cdot \twoheadrightarrow H^a(E^\cdot)$. Let B be the kernel of this epimorphism, and $B[a]$ the complex consisting of B in degree a. Then $B[a] \to \sigma^a E^\cdot \to E^\cdot$ is a homotopy fibre sequence, i.e., 2 sides of a distinguished triangle in the derived category. Considering the induced long exact sequence for $H^*(\mathcal{F} \otimes^L (\ ))$, we get that $\mathrm{Tor}^k(\mathcal{F}, B) = H^{a-k}(\mathcal{F} \otimes^L B[a])$ is 0 for $k \geq 1$. Thus B is a flat $\mathcal{O}_U$-module. B is also the only non-vanishing cohomology group H^{a-1} of the mapping cone $M^\cdot$ of $\sigma^a E^\cdot \to E^\cdot$. As $\sigma^a E^\cdot$ is strictly perfect and $E^\cdot$ is strict $(a-2)$-pseudo-coherent, the mapping cone $M^\cdot$ is strict $(a-2)$-pseudo-coherent. By 1.9.4(b), $Z^{a-1}(M^\cdot)$ is a vector bundle, as is M^{a-2}. The exact sequence $M^{a-2} \to Z^{a-1}(M^\cdot) \to H^{a-1}(M^\cdot) \to 0$ shows that $B = H^{a-1}(M^\cdot)$ is a finitely presented $\mathcal{O}_U$-module. But as B is also flat, it is then a locally free $\mathcal{O}_U$-module of finite type, i.e., a vector bundle (e.g., [SGA 6] I 5.8.3, or Bourbaki). Now consider the truncated complex $\tau^a E^\cdot$. This differs from the strict perfect $\sigma^a E$ only in degree $a-1$, where $\tau^a E^\cdot$ has $B^a(E^\cdot) = \ker(Z^a E^\cdot \to H^a(E^\cdot)) = B$. As B is a vector bundle, $\tau^a E^\cdot$ is strict perfect. But as $E^\cdot$ is cohomologically bounded below by a, $E^\cdot \xrightarrow{\sim} \tau^a E^\cdot$ is a quasi-isomorphism on U. Thus $E^\cdot$ is perfect locally, and hence perfect, as required.

2.2.13. **Proposition** ([SGA 6] I). *Let x be a scheme.*

(a) *Suppose $A^\cdot \to B^\cdot \to C^\cdot$ is a homotopy fibre sequence in $D(\mathcal{O}_X\text{-}Mod)$, i.e., forms two sides of a distinguished triangle. Then:*

If $A^\cdot$ is $(n+1)$-pseudo-coherent and $B^\cdot$ is n-pseudo-coherent, then $C^\cdot$ is n-pseudo-coherent.

If $A^\cdot$ and $C^\cdot$ are n-pseudo-coherent, then $B^\cdot$ is n-pseudo-coherent.

If $B^\cdot$ is n-pseudo-coherent, and $C^\cdot$ is $(n-1)$-pseudo-coherent, then $A^\cdot$ is n-pseudo-coherent.

(b) *If $A^\cdot$, $B^\cdot$, $C^\cdot$ are the three vertices of a distinguished triangle in $D(\mathcal{O}_X\text{-}Mod)$, and 2 of these 3 vertices are pseudo-coherent (resp. perfect), then the third vertex is also pseudo-coherent (resp. perfect).*

(c) *The complex $F^\cdot \oplus G^\cdot$ is n-pseudo-coherent (resp. pseudo-coherent, resp. perfect) iff both summands $F^\cdot$ and $G^\cdot$ are n-pseudo-coherent (resp. pseudo-coherent, resp. perfect).*

Proof. To prove (a), we first note by "rotating the triangle" ([V]

TR2 I Section 1 no. 1-1, or [H] I Section 1, TR2) that $A^\cdot \to B^\cdot \to C^\cdot$ is a homotopy fibre sequence iff $C^\cdot[-1] \xrightarrow{\partial} A^\cdot \to B^\cdot$ and $B^\cdot[-1] \to C^\cdot[-1] \to A^\cdot$ are. Also, it is clear that a shifted complex $F^\cdot[k]$ is $(n+k)$-pseudo-coherent iff $F^\cdot$ is n-pseudo-coherent. So it suffices to prove the first statement of (a). So assume that $A^\cdot$ is $(n+1)$-pseudo-coherent and that $B^\cdot$ is n-pseudo-coherent. We need to show that $C^\cdot$ is n-pseudo-coherent. This question is local, so it suffices to show it in an arbitrarily small open *nbd* U of each point $x \in X$. By definition, for U small, we may choose representatives of the quasi-isomorphism classes of $A^\cdot$ and $B^\cdot$ so that $A^\cdot$ is strict $(n+1)-$pseudo-coherent and $B^\cdot$ is strict n-pseudo-coherent. The map $A^\cdot \to B^\cdot$ in the derived category $D(\mathcal{O}_U\text{-Mod})$ is represented by a datum of strict maps $A^\cdot \xleftarrow{\sim} G^\cdot \to B^\cdot$. Applying 2.2.4(a) and shrinking U, there is a strict $(n+1)$-pseudo-coherent $A'^\cdot$, and a quasi- isomorphism $A'^\cdot \xrightarrow{\sim} G^\cdot \xrightarrow{\sim} A^\cdot$. Replacing $A^\cdot$ by $A'^\cdot$, we may assume we have a strict map $A^\cdot \to B^\cdot$. Now $C^\cdot$ is quasi-isomorphic to the mapping cone of $A^\cdot \to B^\cdot$. In degree k this cone is $A^{k+1} \oplus B^k$, and so is a vector bundle for $k \geq n$. So the cone is strict n-pseudo-coherent. So $C^\cdot$ is n-pseudo-coherent as required, proving (a).

Clearly (a) implies (b) for pseudo-coherence. To prove (b) for perfection, we reduce by rotating the triangle to show that if $A^\cdot$ and $B^\cdot$ are perfect, then the mapping cone $C^\cdot$ is perfect. We now argue as in the proof of (a), locally taking strict perfect representatives for $A^\cdot$ and $B^\cdot$, using 2.2.4(c) to reduce to the case where there is a strict map of complexes $A^\cdot \to B^\cdot$. Then the mapping cone is strict perfect, and is quasi-isomorphic to $C^\cdot$, which is hence perfect. This proves (b).

The non-trivial part of (c) is to show that if $F^\cdot \oplus G^\cdot$ is n-pseudo-coherent (resp. pseudo-coherent, resp. perfect) then both factors $F^\cdot$ and $G^\cdot$ have the same property.

Suppose $F^\cdot \oplus G^\cdot$ is n-pseudo-coherent. We must show that for a small *nbd* U of $x \in X$ that $F^\cdot|U$ is n-pseudo-coherent. As the question is local, we may assume that $F^\cdot \oplus G^\cdot$ is quasi-isomorphic to a strict n-pseudo-coherent complex. Then there is an integer $N \gg 0$ such that $H^k(F^\cdot) \oplus H^k(G) = H^k(F \oplus G) = 0$ for $k \geq N$. Then $F^\cdot$ and $G^\cdot$ are trivially N-pseudo-coherent, as $0 \to F$ is an N-quasi-isomorphism. By descending induction on p for $n \leq p \leq N$ we show that $F^\cdot$ and $G^\cdot$ are p-pseudo-coherent. To do the induction step, suppose we already know that $F^\cdot$ and $G^\cdot$ are $(p+1)$-pseudo-coherent. Shrinking U, we may assume that $F^\cdot$ and $G^\cdot$ are strict $(p+1)$-pseudo-coherent, on replacing their representatives up to quasi-isomorphism. Consider the canonical homotopy fibre sequence, noting that $\sigma^{\leq p-1}A = A/\sigma^p A$

$$\sigma^{p+1}F^\cdot \oplus \sigma^{p+1}G^\cdot \to F^\cdot \oplus G^\cdot \to \sigma^{\leq p}F^\cdot \oplus \sigma^{\leq p}G^\cdot. \tag{2.2.13.1}$$

As $F^\cdot$ and $G^\cdot$ are strict $(p+1)$-pseudo-coherent, $\sigma^{p+1}F^\cdot \oplus \sigma^{p+1}G^\cdot$ is strict perfect. As $p \geq n$, $F^\cdot \oplus G^\cdot$ is p-pseudo-coherent, by 2.2.13(a) proved above, $\sigma^{\leq p}F^\cdot \oplus \sigma^{\leq p}G^\cdot$ is p-pseudo-coherent. By 2.2.3, $H^p(\sigma^{\leq p}F^\cdot) \oplus H^p(\sigma^{\leq p}G^\cdot) = H^p(\sigma^{\leq p}F^\cdot \oplus \sigma^{\leq p}G^\cdot)$ is an $\mathcal{O}_U$-module of locally finite type. Thus the summand $H^p(\sigma^{\leq p}F^\cdot)$ is locally of finite type. By 2.2.3 again, $\sigma^{\leq p}F^\cdot$ is p-pseudo-coherent. As $\sigma^{p+1}F^\cdot$ was even strict perfect, 2.2.13(a) now shows that $F^\cdot$ is p-pseudo-coherent. Similarly, $G^\cdot$ is p-pseudo-coherent. This completes the induction step. When the induction stops at $n = p$, it has proved (c) for the n-pseudo-coherent case. The pseudo-coherent case follows immediately.

It remains to do the perfect case. But if $F^\cdot \oplus G^\cdot$ is perfect, it is pseudo-coherent and of locally finite Tor-amplitude by 2.2.12. It is clear from definition 2.2.11 that the summands $F^\cdot$ and $G^\cdot$ have locally finite Tor-amplitude. Both are pseudo-coherent by the above. Thus by 2.2.12, $F^\cdot$ and $G^\cdot$ are perfect, as required.

2.3. In the presence of an ample family of line bundles, we have "global resolution" results:

2.3.1. **Proposition** (cf. [SGA 6] II). . *Let X be a quasi-compact and quasi-separated scheme with an ample family of line bundles. Then*

(a) *If $E^\cdot$ is a strict perfect complex, $F^\cdot$ any perfect strictly bounded below complex of quasi-coherent $\mathcal{O}_X$-modules, and $x : E^\cdot \to F^\cdot$ is any strict map of complexes, then there exists a strict perfect complex $F'^\cdot$ on X, a map $a : E^\cdot \to F'^\cdot$, and a quasi-isomorphism $x' : F'^\cdot \xrightarrow{\sim} F^\cdot$ such that $x = x' \cdot a$.*

(b) *If $E^\cdot$ is any strict pseudo-coherent complex, $F^\cdot$ any pseudo-coherent complex of quasi-coherent $\mathcal{O}_X$-modules, and $x : E^\cdot \to F^\cdot$ any strict map, then there exists a strict pseudo-coherent complex $F'^\cdot$ on X, a map $a : E^\cdot \to F'^\cdot$, and a quasi-isomorphism $x' : F'^\cdot \xrightarrow{\sim} F^\cdot$ such that $x' \cdot a = x$.*

(c) *If $E^\cdot$ is any strict n-pseudo-coherent complex, $F^\cdot$ any n-pseudo-coherent complex of quasi-coherent $\mathcal{O}_X$-modules, and $x : E^\cdot \to F^\cdot$ is any map, then there is a strict n-pseudo-coherent complex $F'^\cdot$ on X, a map $a : E^\cdot \to F'^\cdot$, and a quasi-isomorphism $x' : F'^\cdot \xrightarrow{\sim} F^\cdot$ such that $x' \cdot a = x$.*

(d) *If $F^\cdot$ is any perfect complex of $\mathcal{O}_X$-modules (perhaps not quasi-coherent), then there is a strict perfect complex $E^\cdot$ and an isomorphism in the derived category $D(\mathcal{O}_X\text{-Mod})$ between $E^\cdot$ and $F^\cdot$.*

(e) *Let $F^\cdot$ be any pseudo-coherent complex of $\mathcal{O}_X$-modules (perhaps not quasi-coherent). Suppose either that $F^\cdot \in D^b(\mathcal{O}_X\text{-Mod})$ is cohomologically bounded, or else that X is noetherian of finite Krull dimension. Then there is a strict pseudo-coherent complex $E^\cdot$ on X and an isomorphism in $D(\mathcal{O}_X\text{-Mod})$ between $E^\cdot$ and $F^\cdot$.*

Proof. To prove (b), we apply the Inductive Construction Lemma 1.9.5 with $\mathcal{A} =$ Qcoh(X), $\mathcal{D} =$ the category of algebraic vector bundles, and $\mathcal{C} =$ the category of pseudo-coherent complexes in Qcoh(X). The hypothesis 1.9.5.1 holds by 2.2.3 and the ample family 2.1.3(c).

The proof of (c) is similar, using 1.9.5.9.

To prove (a), we note that (b) yields $E \to F''^{\cdot}$ and $F''^{\cdot} \xrightarrow{\sim} F^{\cdot}$ with F'' strict pseudo-coherent. $F''^{\cdot}$ is perfect as $F^{\cdot}$ is. We take an integer $n \ll 0$ so that $E^k = F^k = 0 = H^k(F''^{\cdot})$ for all $k \leq n$. As in the proof of 2.2.12, $B^n = \ker Z^n(F'') \to H^n(F'')$ is perfect, and hence a module of locally finite Tor-dimension, as $B^n[n]$ is quasi-isomorphic to the homotopy fibre of the map of perfect complexes $\sigma^n F''^{\cdot} \to F''^{\cdot}$. As X is quasi-compact, $B^n = B^n(F'')$ has globally finite Tor-dimension, say N. As $H^k(F'') = 0$ for $k \leq n$, $Z^k F'' = B^k F''$ for $k \leq n$, and $0 \to B^k F'' \to F''^k \to B^{k+1} F'' \to 0$ is exact for $k \leq n$. Considering the induced long exact sequence for $\mathrm{Tor}^*_{\mathcal{O}_x}(\ , \mathcal{F})$ and the fact that F''^k is a vector bundle and hence is flat, we see that $B^{n-1}(F''^{\cdot})$ has Tor-dimension $N-1$, etc. Thus $B^{n-N}(F''^{\cdot})$ is flat. The exact sequence $F''^{n-N-2} \to F''^{n-N-1} \to B^{n-N}(F''^{\cdot}) \to 0$ shows that $B^{n-N}(F''^{\cdot})$ is also finitely presented. Thus $B^{n-N}(F''^{\cdot})$ is a vector bundle ([SGA 6] I 5.6.3, or Bourbaki). Thus the good truncation $\tau^{n-N}(F''^{\cdot})$ is strict perfect. By choice of n, $F''^{\cdot} \to F^{\cdot}$ factors into quasi-isomorphisms $F''^{\cdot} \xrightarrow{\sim} \tau^{n-N}(F''^{\cdot}) \xrightarrow{\sim} F^{\cdot}$. Setting $F'^{\cdot} = \tau^{n-N}(F''^{\cdot})$ then proves (a).

To prove (d), we note that the perfect $F^{\cdot}$ is cohomologically bounded. The coherator B.16 yields an isomorphism in $D(\mathcal{O}_X$-Mod) between $F^{\cdot}$ and a perfect complex of quasi-coherent modules $RQ(F^{\cdot})$. (Note X satisfies the semiseparation hypothesis of B.16 because of the ample family of line bundles, cf. B.7.) For $n << 0, RQ(F^{\cdot})$ is quasi-isomorphic to $\tau^n RQ(F^{\cdot})$, which is still quasi-coherent. Now we apply (a) to $0 \to \tau^n RQ(F^{\cdot})$ to produce a strict perfect complex $E^{\cdot}$ quasi-isomorphic to $RQ(F^{\cdot})$ and $F^{\cdot}$.

The proof of (e) is similar to that of (d), using (b) instead of (a) at the last step.

2.3.2. **Proposition.** *Let X be a scheme with an ample family of line bundles. Let $E^{\cdot}$ be a complex of quasi-coherent $\mathcal{O}_X$-modules. Then there is a direct system of strict perfect complexes $\{F^{\cdot}_\alpha\}$, and a quasi-isomorphism*

(2.3.2.1)
$$\varinjlim F^{\cdot}_\alpha \xrightarrow{\sim} E^{\cdot}.$$

Proof. Let $E^{\cdot}(n)$ be the subcomplex $\tau^{\leq n}(E^{\cdot})$ of $E^{\cdot}$:

(2.3.2.2)
$$E^{\cdot}(n) = \ldots \to E^{n-2} \to E^{n-1} \to Z^n E^{\cdot} \to 0 \to 0 \to \ldots .$$

Then $E^\cdot$ is the direct colimit $E^\cdot = \varinjlim E^\cdot(n)$ as n goes to ∞.

By increasing induction on n, starting with $n = 0$, we construct a complex $E'^\cdot\{n\}$, which in each degree is an infinite direct sum of line bundles, and which is 0 in degrees above n. We construct this so that there is a quasi-isomorphism $E'^\cdot\{n\} \xrightarrow{\sim} E^\cdot(n)$, and a degree-wise split monomorphism $E'^\cdot\{n-1\} \rightarrowtail E'^\cdot\{n\}$ such that (2.3.2.3) commutes.

$$
\text{(2.3.2.3)} \qquad \begin{array}{ccc}
E'^\cdot\{n-1\} & \xrightarrow{\sim} & E^\cdot(n-1) \\
\downarrow & & \downarrow \\
E'^\cdot\{n\} & \xrightarrow{\sim} & E^\cdot(n)
\end{array}
$$

This construction is possible by the Inductive Construction Lemma 1.9.5 with $\mathcal{A} = \mathrm{Qcoh}(X)$ and $\mathcal{D}$ = the category of sums of line bundles. Hypothesis 1.9.5.1 holds because of the ample family 2.1.3(a).

Now we consider the directed system whose objects consist of an integer $n \geq 0$ and a strict bounded subcomplex $F_\alpha^\cdot$ of $E'^\cdot\{n\}$ such that in each degree $F_\alpha^\cdot$ is a finite subsum of the given direct sum of line bundles which is $E'^\cdot\{n\}$ in that degree. The morphisms in the direct system are the obvious increases in n with inclusions of subcomplexes of $\varinjlim E'^\cdot\{n\}$. Given any finite subsum in $E'^\cdot\{n\}$, ∂ of it is of finite type, so is contained in a finite subsum with all degrees shifted one higher. Continuing in this way until we hit the bounding degree n, we see that any finite subsum in $E'^\cdot\{n\}$ is contained in a complex $F_\alpha^\cdot$ in the directed system. Thus for the subsystem with n fixed, $\varinjlim F_\alpha^\cdot = E'^\cdot\{n\}$. Thus over the full directed system $\varinjlim F_\alpha^\cdot = \varinjlim E'^\cdot\{n\}$, which is quasi-isomorphic to $\varinjlim E^\cdot(n) = E^\cdot$, as required.

2.3.2.4. *Porism.* If $E^\cdot$ in 2.3.2 also has $H^k(E^\cdot) = 0$ for $k > N$, we can choose the $F_\alpha^\cdot$ to be strictly 0 in degrees $k > N$. Indeed, then $E'^\cdot\{N\} \simeq E^\cdot(n) \simeq E^\cdot$ are quasi-isomorphisms, and we take the subsystem of $F_\alpha^\cdot$ in $E'^\cdot\{N\}$.

2.3.2.5. *Remark.* A general scheme is locally affine, and hence locally has an ample family of line bundles. Thus 2.3.2 holds locally on a general scheme. In Deligne's terms ([SGA 4] V 8.2) a quasi-coherent complex is a local inductive limit of strict perfect complexes.

2.3.3. **Corollary**. *Let X have an ample family of line bundles. Let $E^\cdot$ be a complex of $\mathcal{O}_X$-modules with quasi-coherent cohomology. Suppose either $E^\cdot \in D^+(\mathcal{O}_X\text{-Mod})$ is cohomologically bounded below, or else that X is noetherian of finite Krull dimension.*

Then there is a direct system of strict perfect complexes $F_\alpha^\cdot$ in $\mathcal{C}(\mathrm{Qcoh}(X))$, and an isomorphism in $D(\mathcal{O}_X\text{-Mod})$ between $\varinjlim F_\alpha^\cdot$ and

$E^\cdot$. If $E^\cdot$ is also cohomologically bounded above (by N), then all the $F^\cdot_\alpha$ may be chosen to be strictly bounded above (by N).

Proof. Recall that X is semi-separated because of the ample family, B.7. Then by B.16, B.17, the coherator gives a quasi-isomorphism of $E^\cdot$ to a complex of quasi-coherent modules $RQ(E^\cdot)$. We conclude by applying 2.3.2 to this $RQ(E^\cdot)$.

2.4.1. **Theorem.** *Let X be a scheme, and $E^\cdot$ a perfect complex of $\mathcal{O}_X$-modules. Then:*

(a) *The derived functor $RHom(E^\cdot, \ ) : D(\mathcal{O}_X\text{-}Mod) \to D(\mathcal{O}_X\text{-}Mod)$ is defined on all $D(\mathcal{O}_X\text{-}Mod)$, not just on $D^+(\mathcal{O}_X\text{-}Mod)$.*

(b) *$RHom(E^\cdot, \ )$ is locally of finite cohomological dimension. That is, for each point $x \in X$, there is an open nbd U of x and integers $a \leq b$ such that if $F^\cdot$ is any complex of $\mathcal{O}_X$-modules on U with $H^k(F^\cdot) = 0$ unless $c \leq k \leq d$ (resp. unless $c \leq k$; resp. unless $k \leq d$), then $H^k(R\mathrm{Hom}(E^\cdot, F^\cdot)) = 0$ unless $c - b \leq k \leq d - a$ (resp. unless $c - b \leq k$, resp. unless $k \leq d - a$).*

(c) *If $F^\cdot$ is a complex with quasi-coherent cohomology, the $RHom(E^\cdot, F^\cdot)$ has quasi-coherent cohomology.*

(d) *For any U open in X, for any direct system $F^\cdot_\alpha$ of complexes of $\mathcal{O}_U$-modules, and for any integer k, the canonical map (2.4.1.1) is an isomorphism of sheaves of $\mathcal{O}_U$-modules:*

$$(2.4.1.1) \quad \varinjlim_\alpha H^k\left(R\mathrm{Hom}\left(E^\cdot|U, F^\cdot_\alpha\right)\right) \xrightarrow{\cong} H^k\left(R\mathrm{Hom}\left(E^\cdot|U, \varinjlim F^\cdot_\alpha\right)\right).$$

(e) *If X has noetherian underlying space of finite Krull dimension, and $F^\cdot_\alpha$ is any direct system of complexes of $\mathcal{O}_X$-modules, then the canonical map (2.4.1.2) is an isomorphism*

(2.4.1.2)

$$\varinjlim_\alpha \mathrm{Mor}_{D(\mathcal{O}_X-\mathrm{Mod})}\left(E^\cdot, F^\cdot_\alpha\right) \xrightarrow{\cong} \mathrm{Mor}_{D(\mathcal{O}_X-\mathrm{Mod})}\left(E^\cdot, \varinjlim_\alpha F^\cdot_\alpha\right).$$

This says roughly that $\mathrm{Mor}_D(E^\cdot, \ )$ preserves direct colimits, with the qualification that the direct system and its colimit are taken in the strict category of complexes $C(\mathcal{O}_X-\mathrm{Mod})$, not in $D(\mathcal{O}_X-\mathrm{Mod})$.

(f) *If X is quasi-compact and quasi-separated, and $F^\cdot_\alpha$ is any direct system of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology, then (2.4.1.2) is an isomorphism.*

(g) *If X is quasi-compact and quasi-separated, and $F^\cdot_\alpha$ is any direct system of complexes of $\mathcal{O}_X$-modules which is uniformly cohomologically* *bounded below (i.e.,* $\exists n \forall \alpha \forall k < n \; H^k(F^{\cdot}_\alpha) = 0$*), then* (2.4.1.2) *is an isomorphism.*

Proof. Recall the construction of the mapping complex $\mathrm{Hom}(E^{\cdot}, F^{\cdot}) = \prod \mathrm{Hom}\,(E^p, F^q)$. (See [H] II Section 3 for the usual details). The statement (a) on extension from D^+ to D follows from the finite cohomological dimension given by (b) in the usual way, (cf. [H] I 5.3 γ).

To prove (b), (c), and (d), we note these are local questions. Given a point $x \in X$, we take V to be a small *nbd* so that $E^{\cdot}|V$ is quasi-isomorphic to a strict perfect complex, and so reduce to the case where $E^{\cdot}$ is a bounded complex of vector bundles

(2.4.1.3) $$E^{\cdot} = \ldots \to 0 \to E^a \to E^{a+1} \to \ldots \to E^b \to 0 \to \ldots.$$

Shrinking V further, we may assume all the E^i are free of finite ranks k_i. Then $\mathrm{Hom}(E^i, F^{\cdot})$ is $\oplus^{k_i} F^{\cdot}$, and $\mathrm{Hom}(E^{\cdot}, F^{\cdot})$ is the total complex of the bicomplex (2.4.1.4) consisting of finitely many finite sums of shifted copies of $F^{\cdot}$:

(2.4.1.4)
$$\mathrm{Hom}(E^{\cdot}, F^{\cdot})|V = \oplus^{k_a} F^{\cdot}[-a] \leftarrow \oplus^{k_{a+1}} F^{\cdot}[-a-1] \leftarrow \ldots \leftarrow \oplus^{k_b} F^{\cdot}[-b].$$

Clearly this $\mathrm{Hom}(E^{\cdot}, F^{\cdot})|V$ is exact in $F^{\cdot}$, and so represents $R\mathrm{Hom}(E^{\cdot}, F^{\cdot})|V$. Now it is clear that (b) holds with a and b the given strict bounds on $E^{\cdot}$. Also (c) is clear, and (d) is clear as (2.4.1.4) commutes with direct colimits.

To prove (g) we recall ([H] I 4.7, [V] II Section 1 no. 2-3 4, or our 1.9.5 dualized) that any $F^{\cdot}$ in $D^+(\mathcal{O}_X\text{-Mod})$ is quasi-isomorphic to a complex of injective $\mathcal{O}_X$-modules $I^{\cdot}$. Then $\mathrm{Hom}(E^{\cdot}, I^{\cdot})$ is a complex of flasque sheaves ([SGA 4] V 4.10), and thus is deployed for computing $R\Gamma(X, \;)$. Hence $R\Gamma(X, \; R\mathrm{Hom}(E^{\cdot}, F^{\cdot}))$ is represented by $\Gamma(X, \; \mathrm{Hom}(E^{\cdot}, I^{\cdot})) = \mathrm{Hom}_X(E^{\cdot}, I^{\cdot})$, which also represents $R\mathrm{Hom}_X(E^{\cdot}, F^{\cdot})$. The cohomology in degree 0, H^0, of the complex $\mathrm{Hom}_X(E^{\cdot}, I^{\cdot})$ is the group of chain homotopy classes of maps $E^{\cdot} \to I^{\cdot}$, and as $I^{\cdot}$ is injective this is exactly $\mathrm{Mor}_{D(\mathcal{O}_X\text{-Mod})}(E^{\cdot}, F^{\cdot})$.

Consider the Grothendieck spectral sequence

(2.4.1.5) $$E_2^{p,q} = H^p\left(X; H^q R\mathrm{Hom}\left(E^{\cdot}, F^{\cdot}_\alpha\right)\right) \Rightarrow H^{p+q} R\mathrm{Hom}_X\left(E^{\cdot}, F^{\cdot}_\alpha\right).$$

By (d), $H^q R\mathrm{Hom}(E^{\cdot}, \;)$ preserves direct colimits, and by B.6, $H^p(X; \;)$ preserves direct colimits, so the $E_2^{p,q}$ term of the spectral sequence for $\varinjlim F^{\cdot}_\alpha$ is the direct colimit of the $E_2^{p,q}$ terms for the F_α. By the hypothesis of (g) that $\exists n \forall \alpha \forall k < n \; H^k(F^{\cdot}_\alpha) = 0$, the convergence of the spectral sequences is uniform in α, and it follows that $H^* R\mathrm{Hom}_X(E^{\cdot}, \;)$

preserves direct colimits of such systems $F^{\cdot}_{\alpha}$ (cf. [Th1] 5.50, 1.40). For $H^0(R\mathrm{Hom}_X(E^{\cdot},\ ))$, this yields the desired conclusion of (g).

It remains to prove (e) and (f). First we do this under the extra hypothesis that $E^{\cdot}$ is globally quasi-isomorphic to a strict perfect complex. Then on replacing $E^{\cdot}$ by a strict perfect complex, we get that $R\mathrm{Hom}(E^{\cdot},\ )$ is represented by $\mathrm{Hom}(E^{\cdot},\ )$, which is a well-defined functor into the category of complexes, not just into $D(\mathcal{O}_X\text{-Mod})$. We consider the Grothendieck spectral sequence. As above, the $E_2^{p,q}$ term, $H^p(X; H^q R\mathrm{Hom}(E^{\cdot},\ )$ preserves direct colimits. Also, there is an integer N such that $H^p(X; H^q R\mathrm{Hom}(E^{\cdot}, F^{\cdot}_{\alpha}) = 0$ for $p > N$. In case (e), this is because X of finite Krull dimension has finite Zariski cohomological dimension by [Gro] 3.6.5. In case (f), this is because all $H^q R\mathrm{Hom}(E^{\cdot}, F^{\cdot}_{\alpha})$ are quasi-coherent $\mathcal{O}_X$-modules by (c), and because X has finite cohomological dimension for quasi-coherent modules by B.11 applied to $X \to \mathrm{Spec}(\mathbb{Z})$. This N gives uniform convergence of the Grothendieck spectral sequence to $H^{p+q}(R\Gamma(X; R\mathrm{Hom}(E^{\cdot}, F^{\cdot}_{\alpha}))$. From this uniform convergence and the fact that the $E_2^{p,q}$ terms preserve direct colimits, it follows that the $H^*(R\Gamma(X; R\mathrm{Hom}(E^{\cdot},\ ))$ preserve the appropriate direct colimits (cf. [Th1] 1.40).

It remains in cases (e) and (f) to identify $R\Gamma(X, R\mathrm{Hom}(E^{\cdot}, F^{\cdot}))$ to $R\mathrm{Hom}_X(E^{\cdot}, F^{\cdot})$, or more precisely to show that the canonical augmentation map (2.4.1.6) is an isomorphism

$$\text{(2.4.1.6)} \qquad \mathrm{Mor}_{D(\mathcal{O}_X\text{-Mod})}(E^{\cdot}, F^{\cdot}) \to H^0(X, R\mathrm{Hom}(E^{\cdot}, F^{\cdot})).$$

As we are considering the case where $E^{\cdot}$ is strict perfect, the obvious devissage shows it suffices to prove (2.4.1.6) is an isomorphism when $E^{\cdot}$ is a single vector bundle E^i. But as $H^0(X;\ )$ is clearly isomorphic to $\mathrm{Mor}_{D(\mathcal{O}_X\text{-Mod})}(\mathcal{O}_X\ \ )$, this reduces to the obvious adjointness of the functors $\otimes E^i$ and $\mathrm{Hom}(E^i,\ )$. This proves (e) and (f) when $E^{\cdot}$ satisfies the extra hypothesis that it is globally quasi-isomorphic to a strict perfect complex.

This extra hypothesis is always met if X has an ample family of line bundles 2.3.1(d). We prove (e) and (f) that the map (2.4.1.2) is an isomorphism without the extra hypothesis by induction on the number of open quasi-affine schemes needed to cover X. The induction starts because a quasi-affine scheme has an ample line bundle $\mathcal{O}_X$. The induction step follows immediately by comparing the exact sequence of 2.4.1.7 below for $\varinjlim F^{\cdot}_{\alpha}$ and the direct colimit of these exact sequences for the $F^{\cdot}_{\alpha}$. Thus the proof of Lemma 2.4.1.7 below will complete the proof of Theorem 2.4.1.

2.4.1.7. **Lemma.** *Let $U \cup V$ be a scheme, covered by open subschemes U and V. Denote the open immersions by $j : U \to U \cup V$, k :*

$V \to U \cup V$, and $\ell : U \cap V \to U \cup V$. Then for any complexes $E^{\cdot}$ and $F^{\cdot}$ of $\mathcal{O}_{U \cup V}$-modules, there is a long exact sequence Mayer-Vietoris of groups of morphisms in the derived categories $D(X) = D(\mathcal{O}_X\text{-modules})$ for $X = U \cup V$, U, V, and $U \cap V$;

(2.4.1.8)

$$\begin{array}{c}
\cdots \\
\downarrow \\
\mathrm{Mor}_{D(U)}(j^* E^{\cdot}[1], j^* F^{\cdot}) \quad \oplus \quad \mathrm{Mor}_{D(V)}(k^* E^{\cdot}[1], k^* F^{\cdot}) \\
\downarrow \\
\mathrm{Mor}_{D(U \cap V)}(\ell^* E^{\cdot}[1], \ell^* F^{\cdot}) \\
\downarrow \partial \\
\mathrm{Mor}_{D(U \cup V)}(E^{\cdot}, F^{\cdot}) \\
\downarrow \\
\mathrm{Mor}_{D(U)}(j^* E^{\cdot}, j^* F^{\cdot}) \quad \oplus \quad \mathrm{Mor}_{D(V)}(k^* E^{\cdot}, k^* F^{\cdot}) \\
\downarrow \\
\mathrm{Mor}_{D(U \cap V)}(\ell^* E^{\cdot}, \ell^* F^{\cdot}) \\
\downarrow \partial \\
\mathrm{Mor}_{D(U \cup V)}(E^{\cdot}[-1], F^{\cdot}) \\
\downarrow \\
\cdots
\end{array}$$

Proof. Recall that j is flat, so $j^* = Lj^*$ is exact. Let $j! : \mathcal{O}_U\text{-Mod} \to \mathcal{O}_{U \cup V}\text{-Mod}$ extension by 0, the functor left adjoint to j^*, ([SGA 4] IV 11.3.1). Then $j!$ is also exact. As both functors are exact and adjoint, they induce adjoint functors on the derived category. So there is a canonical isomorphism for all complexes $G^{\cdot}$ in $D(U)$ and $H^{\cdot}$ in $D(U \cup V)$

$$\mathrm{Mor}_{D(U \cup V)}\left(j!G^{\cdot}, H\right) \cong \mathrm{Mor}_{D(U)}\left(G^{\cdot}, j^* H\right) \tag{2.4.1.9}$$

There is a canonical exact sequence of complexes on $U \cup V$

$$0 \to \ell!\ell^* E^{\cdot} \to j!j^* E^{\cdot} \oplus k!k^* E \to E^{\cdot} \to 0 \tag{2.4.1.10}$$

The maps in (2.4.1.10) are induced by the adjunction maps. Locally, and in fact after restriction to U and to V, this sequence is split exact, hence exact globally (cf. 3.20.4 below). This exact sequence induces a long exact sequence of $\mathrm{Ext}^*_{D(U \cup V)}(\ , F^{\cdot}) = \mathrm{Mor}_{D(U \cup V)}((\ )[-*], F^{\cdot})$ in the usual way ([V] I Section 1 1-2, or [H] I 6.1). Interpreting this long exact sequence by means of the isomorphisms (2.4.1.9) for j, k, and ℓ yields the long exact sequence (2.4.1.8).

2.4.2. **Theorem.** *Let X have an ample family of line bundles. Let $E^\cdot$ be a complex of $\mathcal{O}_X$-modules, with quasi-coherent cohomology. Then the following are equivalent:*

(a) *$E^\cdot$ is pseudo-coherent.*

(b) *For all integers n and k, and all direct systems $\{F^\cdot_\alpha\}$ of complexes of $\mathcal{O}_X$-modules, the canonical map (2.4.2.1) is an isomorphism of $\mathcal{O}_X$-modules*

(2.4.2.1)

$$\varinjlim_\alpha H^k\left(R\mathrm{Hom}\left(E^\cdot, \tau^n F^\cdot_\alpha\right)\right) \xrightarrow{\cong} H^k\left(R\mathrm{Hom}\left(E^\cdot, \varinjlim_\alpha \tau^n F^\cdot_\alpha\right)\right).$$

(c) *For all n, k, and all direct systems of strict perfect complexes $F^\cdot_\alpha$, the map (2.4.2.1) is an isomorphism.*

(d) *$E^\cdot$ is in $D^-(\mathcal{O}_X\text{-}Mod)$, and for all n, k and all direct systems of strict perfect complexes $F^\cdot_\alpha$ which are uniformly bounded above ($\exists\, m\, \forall\, \alpha$ $\forall\, i > m$ $F^i_\alpha = 0$), the map (2.4.2.1) is an isomorphism.*

(e) *For all integers n, and all direct systems $\{F^\cdot_\alpha\}$ of complexes of $\mathcal{O}_X$-modules, the canonical map (2.4.2.2) is an isomorphism*

(2.4.2.2)

$$\varinjlim_\alpha \mathrm{Mor}_{D(\mathcal{O}_X-\mathrm{Mod})}\left(E^\cdot, \tau^n F^\cdot_\alpha\right) \xrightarrow{\cong} \mathrm{Mor}_{D(\mathcal{O}_X-\mathrm{Mod})}\left(E^\cdot, \varinjlim_\alpha \tau^n F^\cdot_\alpha\right)$$

That is, roughly speaking, $Mor_D(E^\cdot,\ )$ preserves direct colimits of uniformly cohomologically bounded below systems.

(f) *For all integers n, and all direct systems $\{F^\cdot_\alpha\}$ of strict perfect complexes, the map (2.4.2.2) is an isomorphism.*

(g) *$E^\cdot$ in $D^-(\mathcal{O}_X\text{-}Mod)$, and for all n and all direct systems of strict perfect complexes $F^\cdot_\alpha$ which are uniformly bounded above, the map (2.4.2.2) is an isomorphism.*

Proof. We fix an integer n, and note that the good truncation τ^n preserves direct colimits, so $\tau^n(\varinjlim F^\cdot_\alpha) = \varinjlim \tau^n F^\cdot_\alpha$. Also, any map $f: E^\cdot \to \tau^n F^\cdot$ factors uniquely through $E^\cdot \to \tau^n E^\cdot$ as the composite with $\tau^n f: \tau^n E^\cdot \to \tau^n \tau^n F^\cdot = \tau^n F^\cdot$. Similarly, $R\mathrm{Hom}(E^\cdot, \tau^n F^\cdot)$ is quasi-isomorphic to $R\mathrm{Hom}(\tau^{n-k-1}E^\cdot, \tau^n F^\cdot)$ in degrees less than or equal to k. So all the conclusions of statements (b), (c), ..., (g) depend only on $\tau^m E^\cdot$ for some m, where m possibly depends on n and k.

Now as $E^\cdot$ has quasi-coherent cohomology, $\tau^{m-1}E^\cdot$ is quasi-isomorphic to a complex of quasi-coherent modules by the coherator B.16.

We show (a) $\Rightarrow$ (b). As $E^\cdot$ is pseudo-coherent, $\tau^{m-1}E^\cdot$ is $(m-1)$-pseudo-coherent. By 2.3.1(c) applied to a complex of quasi-coherent modules quasi-isomorphic to $\tau^{m-1}E^\cdot$, $\tau^{m-1}E^\cdot$ is quasi-isomorphic to some strict $(m-1)$-pseudo-coherent complex $E'^{\cdot}$. Then $\sigma^{m-1}E'^{\cdot}$ is strict perfect, and $\tau^m\sigma^{m-1}E'^{\cdot}$ is quasi-isomorphic to $\tau^m E^{\cdot}$. So to prove (b) for given n and k, we take an appropriate m, and replace $E^{\cdot}$ with a strict perfect complex which has truncation τ^m quasi-isomorphic to $\tau^m E^{\cdot}$. Then it suffices to prove (b) for $E^{\cdot}$ perfect, which follows from 2.4.1(d).

Clearly (b) $\Rightarrow$ (c).

To show (c) $\Rightarrow$ (d), the problem is to show that $E^{\cdot}$ is cohomologically bounded above. It suffices to show $\tau^0 E^{\cdot}$ is so. As $\tau^0 E^{\cdot}$ is in $D^+(\mathcal{O}_X\text{-Mod})$ with quasi-coherent cohomology, it is quasi-isomorphic to the direct colimit of a direct system of strict perfect complexes $F^{\cdot}_\alpha$ by 2.3.3. Consider the induced quasi- isomorphism in $D^+(\mathcal{O}_X\text{-Mod})$, $\tau^0 E \to \tau^0(\varinjlim F^{\cdot}_\alpha) = \varinjlim \tau^0 F^{\cdot}_\alpha$. This gives a section of the sheaf $H^0(R\mathrm{Hom}(\tau^0 E^{\cdot}, \tau^0(\varinjlim F^{\cdot}_\alpha)))$, and hence via the map induced by $E^{\cdot} \to \tau^0 E^{\cdot}$, it gives a section of $H^0(R\mathrm{Hom}(E^{\cdot}, \tau^0(\varinjlim F^{\cdot}_\alpha)))$. By hypothesis (c), at every point $x \in X$, the germ of this section comes from some section of a $H^0(R\mathrm{Hom}(E^{\cdot}, \tau^0(F^{\cdot}_\alpha)))_x$. Replacing $\tau^0 F^{\cdot}_\alpha$ by a quasi-isomorphic complex of injectives $I^{\cdot}$ to compute $R\mathrm{Hom}$, we get a germ in $H^0(\mathrm{Hom}(E^{\cdot}, I^{\cdot}))_x$. This means there is a *nbd* U of x, and a chain map $E^{\cdot} \to I^{\cdot}$ on U, representing the class of a map $E \to \tau^0 F^{\cdot}_\alpha$ in $D^+(\mathcal{O}_U\text{-Mod})$, and such that $E^{\cdot} \to \tau^0 F^{\cdot}_\alpha \to \tau^0(\varinjlim F^{\cdot}_\alpha) \simeq \tau^0(E^{\cdot})$ is the canonical map $E^{\cdot} \to \tau^0 E^{\cdot}$ in $D^+(\mathcal{O}_U\text{-Mod})$. Thus $\tau^0 E^{\cdot}$ splits off $\tau^0 F^{\cdot}_\alpha$ in $D^+(\mathcal{O}_U\text{- Mod})$. As $F^{\cdot}_\alpha$ is strict perfect, it is bounded above. Hence on U, $\tau^0 F^{\cdot}_\alpha$ and so $\tau^0 E^{\cdot}$ are cohomologically bounded above. Thus $E^{\cdot}$ is locally cohomologically bounded above. As X is quasi-compact (2.1.1), it follows that $E^{\cdot}$ is globally cohomologically bounded above, as required.

To see that (b) $\Rightarrow$ (e), (c) $\Rightarrow$ (f) and (d) $\Rightarrow$ (g), we note that the hypotheses imply that $E^{\cdot}$ is in $D^-(\mathcal{O}_X\text{-Mod})$ as above. Also, the $\tau^n F^{\cdot}_\alpha$ are uniformly cohomologically bounded below. Combining these facts, we see that $R\mathrm{Hom}(E, \tau^n F^{\cdot}_\alpha)$ is uniformly cohomologically bounded below. This shows that the Grothendieck spectral sequence (2.4.1.5) computing $H^0(R\mathrm{Hom}_X(E^{\cdot}, \tau^n F^{\cdot})) = \mathrm{Mor}_{D(X)}(E^{\cdot}, \tau^n F^{\cdot})$ converges uniformly in α. Now (b) $\Rightarrow$ (e), (c) $\Rightarrow$ (f), and (d) $\Rightarrow$ (g) follow as in the proof of 2.4.1(g).

Clearly, (e) $\Rightarrow$ (f) and (f) $\Rightarrow$ (g) as (f) implies that $E^{\cdot}$ is cohomologically bounded above similarly to the proof of (d) above.

Finally (g) $\Rightarrow$ (a). For it suffices to show for all n that $\tau^n E^{\cdot}$ is n-pseudo-coherent, since then $E^{\cdot}$ is $(n-1)$-quasi-isomorphic to the n-pseudo-coherent $\tau^n E^{\cdot}$, and thus locally n-quasi-isomorphic to a strict perfect complex, 2.2.5.4. As $\tau^n E^{\cdot}$ is cohomologically bounded, 2.3.3 shows that $\tau^n E^{\cdot}$ is quasi-isomorphic with a direct colimit of a direct system of strict perfect complexes $F^{\cdot}_\alpha$ which are uniformly bounded above. Then there are quasi-isomorphisms $\tau^n E^{\cdot} \simeq \varinjlim F^{\cdot}_\alpha \simeq \varinjlim \tau^n F^{\cdot}_\alpha$. By hypothesis (g), the map $E^{\cdot} \to \tau^n E^{\cdot}_\alpha \simeq \varinjlim \tau^n F^{\cdot}_\alpha$ factors $E^{\cdot} \to \tau^n F^{\cdot}_\alpha \to \varinjlim \tau^n F^{\cdot}_\alpha$ for some α.

Then $\tau^n E^{\cdot}$ is a retract of $\tau^n F^{\cdot}_\alpha$. As $F^{\cdot}_\alpha$ is strict pseudo-coherent, $\tau^n F^{\cdot}_\alpha$ is n-pseudo-coherent, and hence $\tau^n E^{\cdot}$ is n-pseudo-coherent by 2.2.13(c). This completes the proof.

2.4.2.3. As pseudo-coherence is a local property, and any scheme is locally affine, hence locally has an ample family of line bundles, 2.4.2 is a useful criterion to apply locally on a general scheme.

2.4.3. **Theorem**. *Let X be a scheme with an ample family of line bundles. Let $E^{\cdot}$ be a complex of $\mathcal{O}_X$-modules. Suppose that $E^{\cdot}$ has quasi-coherent cohomology. Then the following are equivalent:*

(a) *$E^{\cdot}$ is perfect.*

(b) *$E^{\cdot}$ is quasi-isomorphic with a strict perfect complex.*

(c) *$E^{\cdot}$ is cohomologically bounded below, and for any direct system of complexes $F^{\cdot}_\alpha$ with quasi-coherent cohomology, $Mor_{D(X)}(E^{\cdot},\ )$ preserves the direct colimit in that the map (2.4.1.2) is an isomorphism.*

(d) *$E^{\cdot}$ is cohomologically bounded, and the map (2.4.1.2) is an isomorphism for any direct system $F^{\cdot}_\alpha$ of strict perfect complexes which are uniformly cohomologically bounded above.*

Proof. (a) $\Rightarrow$ (b) by 2.3.1(d). We have (b) $\Rightarrow$ (c) by 2.4.1(f). To show (c) $\Rightarrow$ (d), the main point is to show that $E^{\cdot}$ is also cohomologically bounded above. But this follows from the proof of 2.4.2(c) $\Rightarrow$ (d).

Now to show (d) $\Rightarrow$ (a), we note by 2.3.3 that $E^{\cdot}$ is quasi-isomorphic to the direct colimit of a direct system of strict perfect complexes which are uniformly bounded above. By (d), the quasi-isomorphism in $D(\mathcal{O}_X\text{-Mod})$, $E^{\cdot} \to \varinjlim F^{\cdot}_\alpha$, factors through some $F^{\cdot}_\alpha$. Thus in $D(\mathcal{O}_X\text{-Mod})$, $E^{\cdot}$ is a summand of the perfect $F^{\cdot}_\alpha$. Hence by 2.2.13(c), $E^{\cdot}$ is perfect as required.

2.4.3.1. As perfection is a local property, the theorem may be applied to the open affines on a general scheme to test perfection there.

2.4.4. To summarize, 2.4.3 roughly characterizes perfect complexes on schemes with ample families of line bundles as the finitely presented objects (in the sense of Grothendieck [EGA] IV 8.14 that Mor out of them preserves direct colimits) in the derived category $D(\mathcal{O}_X\text{-Mod})_{qc}$ of complexes with quasi-coherent cohomology. On a general scheme, the prefect complexes are the locally finitely presented objects in the "homotopy-stack" of derived categories. (We must say "roughly characterizes" as we always take our direct systems in the category $\mathcal{C}(\mathcal{O}_X\text{-Mod})$ of chain complexes, and have not examined the question of lifting a direct system if $D(\mathcal{O}_X\text{-Mod})$ to $\mathcal{C}(\mathcal{O}_X\text{-Mod})$ up to cofinality.)

This characterization, with Lemma 2.3.3, will be the basis for the key extension Lemma 5.5.1. A more immediate application will be to the functoriality statement 2.6.3 below.

2.5.1. ([SGA 6] I.2). Let $f : X \to Y$ be a map of schemes. For $E^{\cdot}$ a strict perfect complex on Y, $f^*E^{\cdot}$ is clearly a strict perfect complex on X. This complex represents $Lf^*E^{\cdot}$, as the vector bundles E^i are flat over $\mathcal{O}_Y$ and hence deployed for Lf^*.

In general, $Lf^* : D^-(\mathcal{O}_Y\text{-Mod}) \to D^-(\mathcal{O}_X\text{- Mod})$ sends perfect complexes to perfect complexes. For the question is local on Y, hence reduces to the case where Y is affine and where any perfect complex is quasi-isomorphic to a strict perfect one.

Similarly, $Lf^* = f^*$ preserves strict pseudo-coherent complexes, and it follows that $Lf^* : D^-(\mathcal{O}_Y\text{-Mod}) \to D^-(\mathcal{O}_X\text{Mod})$ preserves pseudo-coherence. If the map f also has finite Tor-dimension, and so induces a functor $Lf^* : D^b(\mathcal{O}_Y\text{-Mod}) \to D^b(\mathcal{O}_X\text{-Mod})$, Lf^* preserves cohomologically bounded pseudo-coherent complexes. (Recall [SGA 6] III 3.1 that f is said to have finite Tor-dimension if $\mathcal{O}_X$ is of finite Tor-dimension as a sheaf of modules over the sheaf of rings $f^{-1}(\mathcal{O}_Y)$ on X.)

If $E^{\cdot}$ and $F^{\cdot}$ are strict pseudo-coherent, $E^{\cdot} \otimes_{\mathcal{O}_X} F^{\cdot} = E^{\cdot} \otimes^L_{\mathcal{O}_X} F^{\cdot}$ is also strict pseudo-coherent. If $E^{\cdot}$ and $F^{\cdot}$ are strict n-pseudo-coherent and strict m-pseudo-coherent respectively, $E^{\cdot} \otimes^L_{\mathcal{O}_X} F^{\cdot}$ may be taken to be isomorphic to the strict $(m+n)$-pseudo-coherent $E^{\cdot} \otimes_{\mathcal{O}_X} F^{\cdot}$ in degrees greater than or equal to $m+n$. (We apply 1.9.5 with $\mathcal{D}$ = flat $\mathcal{O}_X$-modules to replace $E^{\cdot}$ by a quasi-isomorphic flat complex without changing $E^{\cdot}$ in degree $\geq m$, etc.) It follows that if $E^{\cdot}$ and $F^{\cdot}$ are pseudo-coherent, then $E^{\cdot} \otimes^L_{\mathcal{O}_X} F^{\cdot}$ is pseudo-coherent. If $E^{\cdot}$ is perfect, hence of finite Tor-amplitude, and $F^{\cdot}$ is cohomologically bounded and pseudo-coherent, then $E^{\cdot} \otimes^L_{\mathcal{O}_X} F^{\cdot}$ is also cohomologically bounded and pseudo-coherent.

2.5.2. *Definition* ([SGA 6] III). Let $f : X \to Y$ be a map locally of finite type between schemes.

The map f is n-pseudo-coherent at $x \in X$ if there is a *nbd* U of x and an open $V \subseteq Y$ with $f : U \to V$ factoring as $f = gi$, where $i : U \to Z$ is a closed immersion with $i_*\mathcal{O}_U$ n-pseudo-coherent as a complex on Z, and where $g : Z \to V$ is smooth. (The property that $i_*\mathcal{O}_U$ is n-pseudo-coherent is independent of the choice of Z meeting the other conditions by [SGA 6] III 1.1.4. Hence this property depends only on f.)

The map f is n-pseudo-coherent if f is n-pseudo-coherent at x for all points $x \in X$.

The map f is pseudo-coherent if it is n-pseudo-coherent for all integers n.

The map f is perfect if f is pseudo-coherent and of locally finite Tor-dimension.

2.5.3. *Examples* ([SGA 6]).

(a) For Y noetherian, any $f : X \to Y$ locally of finite type is pseudo-coherent.

(b) Any smooth map $f : X \to Y$ is perfect.

(c) Any regular closed immersion ([SGA 6] VII 1.4) $f : X \to Y$ is perfect ([SGA 6] III 1.1.2).

(d) Any locally complete intersection morphism ([SGA 6] VIII 1.1) is perfect.

(e) For Y not noetherian, a closed immersion need not be pseudo-coherent, for $\mathcal{O}_X$ need not be even finitely presented over $\mathcal{O}_Y$.

2.5.4. **Theorem** ([SGA6] III 2.5, 4.8.1). *Let $f : X \to Y$ be a proper map of schemes. Suppose either that f is projective, or that Y is locally noetherian. Suppose that f is a pseudo-coherent (respectively, a perfect) map. Then if $E^\cdot$ is a pseudo-coherent (resp. perfect) complex on X, $Rf_*(E^\cdot)$ is pseudo-coherent (resp. perfect) on Y.*

Proof. The case f projective will follow from the slightly more general results 2.7 below on taking Z there to be a projective space bundle over Y locally.

The case Y locally neotherian follows from the Grothendieck Finiteness Theorem ([EGA] III 3.2) that $R^p f_*$ preserves coherence, the finite cohomological dimension of Rf_* (B.11), the strongly converging spectral sequence $R^p f_*(H^q(E^\cdot)) \Rightarrow H^{p+q}(Rf_*(E^\cdot))$ and criterion 2.2.8 that a complex on a noetherian scheme is pseudo-coherent iff it is cohomologically bounded above with coherent cohomology. This shows $Rf_*(E^\cdot)$ is pseudo-coherent when $E^\cdot$ is. When f and $E^\cdot$ are also perfect, we show that $Rf_*(E^\cdot)$ has locally finite-Tor-amplitude using base-change 2.5.5 locally. Then criterion 2.2.12 shows that $Rf_*(E^\cdot)$ is perfect.

2.5.5. **Theorem** ([SGA 6] III 3.7). *Let Y be a quasi-compact scheme, and let $f : X \to Y$ be a quasi-compact and quasi-separated map. Let $E^\cdot$ be a cohomologically bounded complex of $\mathcal{O}_X$-modules with quasi-coherent cohomology, and let $F^\cdot$ be a cohomologically bounded complex of $\mathcal{O}_Y$-modules with quasi-coherent cohomology. Assume either that $F^\cdot$ has finite Tor-amplitude over $\mathcal{O}_Y$, or else that $E^\cdot$ has finite Tor-amplitude over $\mathcal{O}_X$. Then the canonical map is a quasi-isomorphism in $D(\mathcal{O}_Y-\mathrm{Mod})$*

$$Rf_*(E^\cdot) \otimes^L_{\mathcal{O}_Y} F^\cdot \xrightarrow{\sim} Rf_*\left(E^\cdot \otimes^L_{\mathcal{O}_X} Lf^* F^\cdot\right). \tag{2.5.5.1}$$

Proof. See [SGA 6] III 3.7. We sketch the argument for $F^\cdot$ perfect, the main case used in this paper. The question is local, so we may assume Y is affine. Then $F^\cdot$ is quasi-isomorphic to a strict perfect complex 2.3.1(d). For $F^\cdot$ strict perfect, filtering $F^\cdot$ by $\sigma^n F$ and comparing the long exact sequences of cohomology of the two sides of (2.5.5.1) induced by $0 \to \sigma^{n+1}F \to \sigma^n F \to F^n \to 0$, we reduce to the case where $F^{\cdot}$ is a single vector bundle F^n. Shrinking Y further, we may assume that $F^{\cdot}$ is free, so is a $\oplus^k \mathcal{O}_Y$. Then (2.5.5.1) reduces to the sum of k copies of the identity map of $Rf_*(E^{\cdot})$, and so is clearly an isomorphism.

2.5.6. **Theorem** ([SGA 6] IV 3.1). *Let (2.5.6.1) be a pull-back square of schemes*

$$(2.5.6.1) \qquad \begin{array}{ccc} X & \xleftarrow{g'} & X' \\ {\scriptstyle f}\downarrow & \square & \downarrow{\scriptstyle f'} \\ Y & \xleftarrow[g]{} & Y' \end{array}$$

Suppose Y is quasi-compact, and that f is a quasi-compact and quasi-separated map. Suppose that f and g are Tor-independent over Y so that given $x \in X$, $y' \in Y'$ with $f(x) = y = g(y')$, then for all integers $p \geq 1$ we have

$$(2.5.6.2) \qquad \mathrm{Tor}^p_{\mathcal{O}_{Y,y}}(\mathcal{O}_{X,x},\ \mathcal{O}_{Y',y'}) = 0.$$

Let $E^{\cdot}$ be a cohomologically bounded complex on X, with quasi-coherent cohomology. Suppose either that $E^{\cdot}$ has finite Tor-amplitude over the sheaf of rings $f^{-1}(\mathcal{O}_Y)$ on the space X, or else that the map g has finite Tor-dimension. Then there is a canonical base-change quasi-isomorphism

$$(2.5.6.3) \qquad Lg^* Rf_* E^{\cdot} \xrightarrow{\sim} Rf'_* Lg'^* E^{\cdot}.$$

Proof. [SGA 4] XVII 4.2.12 defines the map, and [SGA 6] IV 3.1 shows it is an isomorphism in the derived category (cf. 3.18. below).

2.5.6.4. An examination of the proof shows that the Tor-independence hypothesis (2.5.6.2) may be weakened to be required only for those x in a closed subspace $Z \subseteq X$ such that $E^{\cdot}$ is acyclic on $X - Z$.

2.5.7. **Proposition** (Trivial Duality). *Let $f : X \to Y$ be a map of schemes. Then for $E^{\cdot}$ in $D^-(\mathcal{O}_Y\text{-Mod})$ and $G^{\cdot}$ in $D^+(\mathcal{O}_X\text{-Mod})$ there is a canonical isomorphism derived from adjointness between f^* and f_**

$$(2.5.7.1) \quad \mathrm{Mor}_{D(\mathcal{O}_X\text{-Mod})}\left(Lf^* E^{\cdot},\ G^{\cdot}\right) \cong \mathrm{Mor}_{D(\mathcal{O}_Y\text{-Mod})}\left(E^{\cdot},\ Rf_* G^{\cdot}\right).$$

Proof. This is [SGA 4] XVII 2.3.7. See also [SGA 6] IV 3, [H] II 5.1, [V] II, Section 3 no. 3. Also see [SGA $4\frac{1}{2}$], Erratum to [SGA 4], to correct an argument preceding all our cited base-change results.

2.6.1. **Lemma**. *Let X be a quasi-compact scheme, $|Y|$ a closed subspace, and $U = X - |Y|$ the complementary open subscheme. Then*

(a) If there is a finitely presented closed immersion $i : Y \to X$ with underlying space $|Y|$, then U is quasi-compact and $j : U \to X$ is a quasi-compact map.

(b) If $Y = \mathrm{Spec}(\mathcal{O}_X/\mathcal{J})$ and $Y' = \mathrm{Spec}(\mathcal{O}_X/\mathcal{I})$ are two finitely presented closed subschemes of X, both with underlying space $|Y|$, then there exists an integer $n \geq 1$ such that $\mathcal{J}^n \subseteq \mathcal{I}$ and $\mathcal{I}^n \subseteq \mathcal{J}$.

(c) If X is also quasi-separated, and if U is quasi-compact, then there exists a finitely presented closed immersion $i : Y \to X$ with underlying space $|Y|$.

Proof. (a) It suffices to show that j is a quasi-compact map. Let $\mathrm{Spec}(A) \subseteq X$ be an affine open, and let $Y \cap \mathrm{Spec}(A) = \mathrm{Spec}(A/J)$. As Y is finitely presented, $J \subseteq A$ is a finitely generated ideal, say $J = (a_1, \ldots, a_n)$. Then $U \cap \mathrm{Spec}(A)$ is quasi-compact as required, for it is covered by the finitely many affine opens $\mathrm{Spec}(A[1/a_i])$ for $i = 1, \ldots, n$.

(b) As the ideals $\mathcal{J}$ and $\mathcal{I}$ are of finite type, and as the support of both $\mathcal{O}/\mathcal{J}$ and $\mathcal{O}/\mathcal{I}$ are $|Y|$, there is a positive integer n so that $\mathcal{J}^n(\mathcal{O}/\mathcal{I}) = 0 = \mathcal{I}^n(\mathcal{O}/\mathcal{J})$ by [EGA] I 6.8.4. Then $\mathcal{J}^n \subseteq \mathcal{I}$ and $\mathcal{J}^n \subseteq \mathcal{J}$.

(c) $|Y|$ is the underlying space of a reduced closed subscheme $Y_{\mathrm{red}} = \mathrm{Spec}(\mathcal{O}_X/\mathcal{K})$. The ideal $\mathcal{K}$ is the direct colimit of its finitely generated subideals $\mathcal{J}_\alpha$, so $\mathcal{K} = \varinjlim \mathcal{J}_\alpha$, by [EGA] I 6.9.9. Each $Y_\alpha = \mathrm{Spec}(\mathcal{O}/\mathcal{J}_\alpha)$ is a finitely presented closed subscheme of X. As $\mathcal{O}/\mathcal{K} = \varinjlim \mathcal{O}/\mathcal{J}_\alpha$, $Y_{\mathrm{red}} = \varprojlim Y_\alpha$, and $U = X - Y_{\mathrm{red}}$ is the direct colimit of the open subschemes $X - Y_\alpha$ of X, $U = \varinjlim (X - Y_\alpha)$. As U is quasi-compact, there is an α such that $U = X - Y_\alpha$. Then Y_α has underlying space $|Y|$ as required.

2.6.2.1. *Definition*. Let $f : X' \to X$ be a map of schemes, with X quasi-compact. Let $|Y| \subseteq X$ be a closed subspace, which is the underlying space of some finitely presented closed immersion $i : Y \to X$ (see 2.6.1(a,c)).

We say f is an isomorphism infinitely near Y if the following two conditions hold:

(a) f is flat over the points of Y; that is, for all $x' \in X'$ with $y = f(x')$ in $|Y| \subseteq X$, $\mathcal{O}_{X',x'}$ is flat over $\mathcal{O}_{X,y}$.

(b) f induces an isomorphism of schemes $Y' = Y \underset{X}{\times} X' \to Y$.

2.6.2.2. **Lemma-Definition**. *In the presence of hypothesis 2.6.2.1(a), the condition 2.6.2.1(b) does not depend on the choice of finitely presented closed subscheme Y with underlying space $|Y|$. In particular, if (b) holds for $Y = \mathrm{Spec}(\mathcal{O}_X/\mathcal{J})$, it also holds for any of the* *infinitesimal thickenings* $Y(n) = \operatorname{Spec}(\mathcal{O}_X/\mathcal{J}^n)$. *We may then say* f *is an isomorphism infinitely near the subspace* $|Y|$.

Proof. For $Y = \operatorname{Spec}(\mathcal{O}_X/\mathcal{J})$, Y' is $\operatorname{Spec}(\mathcal{O}_{X'}/\mathcal{J}') \subseteq X'$ with $\mathcal{J}'$ the ideal generated by $f^{-1}(\mathcal{J})$. As f is flat over the points of $|Y|$, and these are the only points x where $\mathcal{J}_x \neq \mathcal{O}_{X,x}$, we see that $f^*\mathcal{J} \to f^*\mathcal{O}_X = \mathcal{O}_{X'}$, is a monomorphism. So, in fact, $\mathcal{J}' = f^*\mathcal{J}$. Similarly $f^*\mathcal{J}^n \to \mathcal{O}_{X'}$ is a monomorphism, so $\mathcal{J}'^n = f^*\mathcal{J}^n$. It follows that $\mathcal{J}'^k/\mathcal{J}'^{k+n} = f^*(\mathcal{J}^k/\mathcal{J}^{k+n})$, as f^* preserves cokernels. Consider the exact sequence (2.6.2.2.1) of sheaves supported on the space $|Y|$:

$$\text{(2.6.2.2.1)} \qquad 0 \to \mathcal{J}^{k+1}/\mathcal{J}^{k+n} \to \mathcal{J}^k/\mathcal{J}^{k+n} \to \mathcal{J}^k/\mathcal{J}^{k+1} \to 0.$$

As f is flat over the points of $|Y|$, f^* of this sequence is also exact, and is in fact the exact sequence on $|Y'|$ where $\mathcal{J}^m$ is replaced by $\mathcal{J}'^m$ everywhere. The map f induces a map between these two exact sequences. As $f : Y' \to Y$ is an isomorphism of schemes,the induced map $\mathcal{O}/\mathcal{J} \to \mathcal{O}/\mathcal{J}'$, and even $\mathcal{J}^k/\mathcal{J}^{k+1} = \mathcal{J}^k \otimes \mathcal{O}/\mathcal{J} \to \mathcal{J}^k \otimes \mathcal{O}/\mathcal{J}' = \mathcal{J}'^k/\mathcal{J}'^{k+1}$ are isomorphisms of sheaves on the space $|Y| = |Y'|$. By induction on n, using the 5-lemma on the map between the two exact sequences, we get that f induces an isomorphism for all k and $n \geq 1$

$$\text{(2.6.2.2.2)} \qquad \mathcal{J}^k/\mathcal{J}^{k+n} \cong \mathcal{J}'^k/\mathcal{J}'^{k+n}.$$

In particular, for $k = 0$ we get $\mathcal{O}_X/\mathcal{J}^n \cong \mathcal{O}_{X'}/\mathcal{J}'^n$, so that $Y'^{(n)} \to Y^{(n)}$ is an isomorphism of schemes for all $n \geq 1$.

Now for $Z = \operatorname{Spec}(\mathcal{O}_X/\mathcal{I})$ a finitely presented closed subscheme of X with underlying space $|Y|$, there exists an n such that $\mathcal{J}^n \subseteq \mathcal{I}$ by 2.6.1(b). Then the closed immersion $Z \to X$ factors as $Z \to Y^{(n)} \to X$. Thus $Z' = Z \underset{X}{\times} X' \to Z$ is the pullback of the isomorphism $Y'^{(n)} = Y^{(n)} \underset{X}{\times} X' \to Y^{(n)}$, and so $Z' \to Z$ is an isomorphism as required.

2.6.3. **Theorem.** *Let* $f : X' \to X$ *be a quasi-separated map of quasi-compact schemes. Let* $i : Y \to X$ *be a finitely presented closed immersion. Suppose that* f *is an isomorphism infinitely near* Y (2.6.2.1). *Set* $Y' = f^{-1}(Y) = Y \underset{X}{\times} X'$. *Then*

(a) *For* $E^\cdot$ *in* $D^-(\mathcal{O}_X\text{-Mod})$ *with quasi-coherent cohomology and which is acyclic on* $X - Y$, *the canonical map* $E^\cdot \to Rf_*Lf^*E^\cdot$ *is an isomorphism in* $D^-(\mathcal{O}_X\text{-Mod})$.

(b) *For* $E'^\cdot$ *in* $D^-(\mathcal{O}_{X'}\text{-Mod})$ *with quasi-coherent cohomology and which is acyclic on* $X' - Y'$, *the canonical map* $Lf^*Rf_*E'^\cdot \to E'^\cdot$ *is an isomorphism in* $D^-(\mathcal{O}_{X'}\text{-Mod})$.

(c) *If $E'^{\cdot}$ is pseudo-coherent on X' and acyclic on $X'-Y'$, then $Rf_*E'^{\cdot}$ is pseudo-coherent on X.*

(d) *If $E'^{\cdot}$ is perfect on X' and acyclic on $X'-Y'$, then $Rf_*E'^{\cdot}$ is perfect on X.*

Proof. First we note that f is a quasi-compact map ([EGA] I 6.1.10 iii). Thus Rf_* has bounded cohomological dimension on complexes with quasi-coherent cohomology (B.6, B.11), and so Rf_* is defined on $D(\mathcal{O}_{X'}\text{-Mod})$ and $D^-(\mathcal{O}_{X'}\text{-Mod})$, as well as the usual $D^+(\mathcal{O}_{X'}\text{-Mod})$. Using the functorial Godement resolution T to compute sheaf cohomology as in Deligne's treatment in [SGA 4] XVII 4.2, we realize Rf_* as an exact functor defined on the level of complexes, as $Rf_* = f_* \circ T$ (cf. [Th1] Section 1). As R^pf_* preserves direct colimits of modules by B.6, the usual uniformly converging spectral sequence argument à la 2.4.1 shows that Rf_* preserves up to quasi-isomorphism the colimits of directed systems of complexes with quasi-coherent cohomology.

Let $j : U \to X$ be the open complement to Y. Then j is a quasi-compact map by 2.6.1(a), and is also quasi-separated. Hence the discussion of the preceding paragraph applies also to Rj_*. We define local cohomology $R\Gamma_Y$ as the canonical homotopy fibre of the map of complexes $1 \to Rj_*j^*$, as justified by [SGA 4] V 6.5. (More precisely we use the map $1 \to j_*j^* \to j_*Tj^* = Rj_*j^*$ induced by the augmentation $1 \to T$ into the Godement resolution.) Then $R\Gamma_Y$ preserves up to quasi-isomorphism the colimits of directed systems of complexes with quasi-coherent cohomology, as this is true of 1 and Rj_*j^*. Similarly, $R\Gamma_Y$ has finite cohomological dimension on complexes with quasi-coherent cohomology, and preserves quasi-coherence of cohomology.

Now to prove (a). Using the finite cohomological dimension of Rf_*, for any k, $H^k(Rf_*Lf^*E^{\cdot})$ equals $H^k(Rf_*Lf^*\tau^nE^{\cdot})$ for all n sufficiently small. Thus we reduce to the case where $E^{\cdot}$ is cohomologically bounded below as well as above. Now the usual devissage reduces us to the case where $E^{\cdot}$ is a single quasi-coherent module considered as a complex concentrated in one degree. For we induct on n using the homotopy fibre sequence $H^n(E^{\cdot}) \to \tau^nE^{\cdot} \to \tau^{n+1}E^{\cdot}$ and the 5-lemma to reduce to proving the theorem for the $H^n(E^{\cdot})$ as complexes. Now as Rf_* and Lf^* preserve direct colimits, writing the module E as the direct colimit of its submodules of finite type ([EGA] I 6.9.9), we reduce to the case where E is a quasi-coherent module of finite type, and which vanishes off Y. As the defining ideal $\mathcal{J}$ of Y is of finite type, there is a positive integer n such that $\mathcal{J}^nE = 0$ ([EGA] I 6.8.4). By devissage, it suffices to prove that the canonical map $1 \to Rf_*Lf^*$ is a quasi-isomorphism for $\mathcal{J}^kE/\mathcal{J}^{k+1}E$ with $k = 0, 1, \ldots, n-1$. Thus we reduce to the case where E is an $\mathcal{O}_X/\mathcal{J} = \mathcal{O}_Y$ module, $E = i_*\widetilde{E} = Ri_*\widetilde{E}$. Then using

2.5.6, which applies as f is flat over the points of Y (recall 2.5.6.4), we have $Rf_*Lf^*E \simeq Rf_*Lf^*i_*E^\sim \simeq Rf_*i'_*Lf'^*E^\sim \simeq i_*Rf'_*Lf'^*E^\sim$. But as $f' : Y' \to Y$ is an isomorphism, we continue the chain of quasi-isomorphisms: $i_*Rf'_*Lf'^*E^\sim \simeq i_*E^\sim \simeq E$. This proves (a). The proof of (b) requires only a change of notation in this argument (e.g., Lf^*Rf_* in place of Rf_*Lf^*, X' in place of X, etc.).

To prepare to prove (c) and (d) we first note that if $E^\cdot$ is acyclic on $X - Y$, and if $F^\cdot$ is any complex on X with quasi-coherent cohomology, then $R\Gamma_Y F^\cdot \to F^\cdot$ induces an isomorphism:

$$\text{(2.6.3.1)} \qquad \mathrm{Mor}_{D(\mathcal{O}_X\text{-Mod})}\left(E^\cdot,\ R\Gamma_Y F^\cdot\right) \overset{\cong}{\to} \mathrm{Mor}_{D(\mathcal{O}_X\text{-Mod})}\left(E^\cdot,\ F^\cdot\right).$$

For the homotopy fibre sequence $R\Gamma_Y F^\cdot \to F^\cdot \to Rj_*j^*F$ induces a long exact sequence:

$$\text{(2.6.3.2)} \qquad \begin{array}{ccccc} \downarrow & & & & \\ \mathrm{Mor_D}\,(\mathrm{E}\,[1], \mathrm{Rj_*j^*F}\,) & \cong & \mathrm{Mor_{D(U)}}\,(\mathrm{j^*E[1],j^*F}) & = & 0 \\ \partial\downarrow & & & & \\ \mathrm{Mor_D}\,(\mathrm{E}\,, \mathrm{R\Gamma_Y F}\,) & & & & \\ \downarrow & & & & \\ \mathrm{Mor_D}\,(\mathrm{E}\,,\ \mathrm{F}\,) & & & & \\ \downarrow & & & & \\ \mathrm{Mor_D}\,(\mathrm{E}\,,\ \mathrm{Rj_*j^*F}\,) & \cong & \mathrm{Mor_{D(U)}}\,(\mathrm{j^*E,j^*F}) & = & 0 \\ \downarrow & & & & \end{array}$$

The horizontal isomorphisms result from trivial duality 2.5.7. for $j : U \to X$, and the fact that $j^*E^\cdot \simeq 0$ as $E^\cdot$ is acyclic on U. The exactness of (2.6.3.2) yields (2.6.3.1), as required. In particular, (2.6.3.1) holds for $E^\cdot = Rf_*E'^\cdot$ in (c) and (d).

As final preparation, we note that $R\Gamma_Y$ is also represented on the level of complexes as Γ_Y of the Godement resolution on X. This yields $R\Gamma_Y$ as a complex of modules over the localization of $\mathcal{O}_X$ along Y. As f is flat over the points of Y, $f^*R\Gamma_Y$ would then represent $Lf^*R\Gamma_Y$ on the level of complexes. We now switch over to this representative of $R\Gamma_Y$. As a statement in the derived category, (2.6.3.1) remains true.

Now we prove (c) that $Rf_*E'^\cdot$ is pseudo-coherent. As the question is local on X, we may assume X is affine and hence has an ample family of line bundles. We appeal to criterion 2.4.2(g). Let $\{F_\alpha^\cdot\}$ be a direct system of strict perfect complexes on X. Then we have a sequence of isomorphisms:

(2.6.3.3)

$$
\begin{aligned}
&\varinjlim_{\alpha} \operatorname{Mor}_{D(\mathcal{O}_X\text{-Mod})}\left(Rf_*E'^{\cdot},\ \tau^n F_\alpha^{\cdot}\right)\\
&\cong \varinjlim_{\alpha} \operatorname{Mor}_{D(\mathcal{O}_X\text{-Mod})}\left(Rf_*E'^{\cdot},\ R\Gamma_Y\tau^n F_\alpha^{\cdot}\right)\\
&\cong \varinjlim_{\alpha} \operatorname{Mor}_{D(\mathcal{O}_X\text{-Mod})}\left(Rf_*E'^{\cdot},\ Rf_*Lf^*R\Gamma_Y\tau^n F_\alpha^{\cdot}\right)\\
&\cong \varinjlim_{\alpha} \operatorname{Mor}_{D(\mathcal{O}_{X'}\text{-Mod})}\left(Lf^*Rf_*E'^{\cdot},\ Lf^*R\Gamma_Y\tau^n F_\alpha^{\cdot}\right)\\
&\cong \varinjlim_{\alpha} \operatorname{Mor}_{D(\mathcal{O}_{X'}\text{-Mod})}\left(E'^{\cdot},\ Lf^*R\Gamma_Y\tau^n F_\alpha^{\cdot}\right)\\
&\cong \operatorname{Mor}_{D(\mathcal{O}_{X'}\text{-Mod})}\left(E'^{\cdot},\ \varinjlim_{\alpha} Lf^*R\Gamma_Y\tau^n F_\alpha^{\cdot}\right)\\
&\cong \operatorname{Mor}_{D(\mathcal{O}_{X'}\text{-Mod})}\left(E'^{\cdot},\ Lf^*R\Gamma_Y\tau^n \varinjlim F_\alpha^{\cdot}\right)\\
&\cong \operatorname{Mor}_{D(\mathcal{O}_{X'}\text{-Mod})}\left(Lf^*Rf_*E'^{\cdot},\ Lf^*R\Gamma_Y\tau^n \varinjlim F_\alpha^{\cdot}\right)\\
&\cong \operatorname{Mor}_{D(\mathcal{O}_X\text{-Mod})}\left(Rf_*E'^{\cdot},\ Rf_*Lf^*R\Gamma_Y\tau^n \varinjlim F_\alpha^{\cdot}\right)\\
&\cong \operatorname{Mor}_{D(\mathcal{O}_X\text{-Mod})}\left(Rf_*E'^{\cdot},\ R\Gamma_Y\tau^n \varinjlim F_\alpha^{\cdot}\right)\\
&\cong \operatorname{Mor}_{D(\mathcal{O}_X\text{-Mod})}\left(Rf_*E'^{\cdot},\ \tau^n \varinjlim F_\alpha^{\cdot}\right).
\end{aligned}
$$

Here the isomorphisms are successively justified by (2.6.3.1), (a), trivial duality 2.5.7, (b), pseudo-coherence of $E'^{\cdot}$ with the finite Tor-dimension of $Lf^*R\Gamma_Y = f^*R\Gamma_Y$ and 2.4.2(e), the fact that Lf^* and $R\Gamma_Y$ preserve direct colimits of complexes with quasi-coherent cohomology, (b), trivial duality 2.5.7, (a), and (2.6.3.1). By 2.4.2(g), the isomorphism between the first and last terms of (2.6.3.3) shows that $Rf_*E'^{\cdot}$ is pseudo-coherent, proving (c).

The proof of (d) that $Rf_*E'^{\cdot}$ is perfect requires only removal of the truncation τ^n in the proof of (c), and the use of 2.4.3(d) on X and 2.4.1(f) on X' in place of 2.4.2(g) and 2.4.2(e).

2.7. **Proposition** ([SGA 6]). . *Let $f : X \to Y$ be a proper map of schemes. Suppose that locally on Y, the map factors as $f = h \circ i$, with $h : Z \to Y$ a flat and finitely presented map, and $i : X \to Z$ a closed immersion. Then*

(a) *If $i_*\mathcal{O}_X$ is perfect on Z, and $E^{\cdot}$ is a perfect complex on X, then $Rf_*E^{\cdot}$ is perfect on Y.*

(b) *If* $i_*\mathcal{O}_X$ *is pseudo-coherent on* Z, *and if* $h : Z \to Y$ *has an* h*-ample family of line bundles* (2.1.2(f)), *and if* $E^\cdot$ *is a pseudo-coherent complex on* X, *then* $Rf_*E^\cdot$ *is pseudo-coherent on* Y.

Proof. The question is local on Y, so we may assume that $Y = \mathrm{Spec}(A)$ is affine, and that the factorization $f = h \circ i$ exists globally over Y. In case (b), Z will then have an ample family of line bundles 2.1.2(f).

We write $A = \varinjlim A_\alpha$ as a direct colimit of its noetherian subrings of finite type over $\mathbb{Z}$. On passing to a cofinal system of α, there will be flat and finitely presented maps $h_\alpha : Z_\alpha \to Y_\alpha = \mathrm{Spec}(A_\alpha)$ and closed immersions $i_\alpha : X_\alpha \to Z_\alpha$, such that $f_\alpha = h_\alpha \circ i_\alpha$ is proper, $Z_\beta = Z_\alpha \underset{(A_\alpha)}{\times} (A_\beta)$, $Z = \varinjlim Z_\beta$, etc. This all follows by [EGA] IV 8.9.1, 8.10.5, 11.2.6. Let $g_\alpha : Y \to Y_\alpha$, $g'_\alpha : Z \to Z_\alpha$ be the canonical maps.

We use this noetherian approximation to do case (a). The complex $i_*E^\cdot$ is pseudo-coherent on Z by [SGA 6] III 1.1.1. As $E^\cdot$ has finite Tor-amplitude over X and $i_*\mathcal{O}_X$ has finite Tor-dimension over Z, the complex $i_*E^\cdot$ has finite Tor-amplitude over Z ([SGA 6] III 3.7.2). Thus $i_*E^\cdot$ is perfect on Z by 2.2.12. By 3.20 below, whose proof does not depend on this 2.7 (or by [SGA 6] IV 3.2), there is an α in the direct system and a perfect complex $E^\cdot_\alpha$ on Z_α such that $i_*E^\cdot$ is quasi-isomorphic to $Lg'^*_\alpha E^\cdot_\alpha$. As $i_*E^\cdot$ is acyclic on $Z - X$, by taking α larger we may assume that $E^\cdot_\alpha$ is acyclic on $Z_\alpha - X_\alpha$, again by 3.20. (Danger: We do not know that the maps i_α and g'_α are Tor-independent, and so cannot appeal to 2.5.6. Thus we do not claim that we can take $E^\cdot_\alpha$ to be $i_{\alpha *}$ of a perfect complex on X_α.)

As Z_α is noetherian, $E^\cdot_\alpha$ has coherent cohomology sheaves by 2.2.8, which are modules over $\mathcal{O}_{X_\alpha(n)}$ for some infinitesimal thickening $X_\alpha(n)$ of X_α in Z_α ([EGA] I 6.8.4). The maps $f_\alpha(n) = h_\alpha| : X_\alpha(n) \to Y_\alpha$ are proper as $f_{\alpha\mathrm{red}} = f_\alpha(n)_{\mathrm{red}}$ is proper ([EGA] II 5.4.6, I 5.3.1). Thus $Rh_{\alpha *}(E^\cdot_\alpha)$ has coherent cohomology, and in fact is pseudo-coherent by the noetherian case of 2.5.4 proved above.

As $h_\alpha : Z_\alpha \to Y_\alpha$ is flat and $E^\cdot_\alpha$ has finite Tor-amplitude over $\mathcal{O}_{Z_\alpha}$, it follows that $E^\cdot_\alpha$ has finite Tor-amplitude over $f^{-1}(\mathcal{O}_{Y_\alpha})$. Also, $Rh_{\alpha *}$ has finite cohomological dimension by B.11. Then the projection formula 2.5.5 applies to show that $Rh_{\alpha *}(E^\cdot_\alpha)$ has finite Tor-amplitude. Thus $Rh_{\alpha *}(E^\cdot_\alpha)$ is perfect on Y_α by 2.2.12. Then $Lg^*_\alpha Rh_{\alpha *}(E^\cdot_\alpha)$ is perfect on Y. But as h_α is flat, the Base-change Theorem 2.5.6 yields a quasi-isomorphism $Lg^*_\alpha Rh_{\alpha *}(E^\cdot_\alpha) \simeq Rh_*(Lg^*_\alpha E^\cdot_\alpha)$. But this complex is quasi-isomorphic to $Rh_*(i_*E) \simeq Rf_*(E^\cdot)$. Thus $Rf_*(E^\cdot)$ is perfect, proving (a).

To prove (b), it suffices to show for all n that the truncation $\tau^n Rf_*E^\cdot = \tau^n Rh_*(i_*E)$ is n-pseudo-coherent. By B.11, Rh_* has finite cohomological dimension, say, k. Then $\tau^n Rh_*(i_*E)$ is quasi-isomorphic to $\tau^n Rh_*(\tau^{n-k}$

$i_*E^\cdot$). By the hypothesis of pseudo-coherence of $i_*\mathcal{O}_X$ on Z and the devissage of [SGA 6] III 1.1.1, the complex i_*E is pseudo-coherent on Z. The complex $\tau^{n-k}i_*E$ is then $(n-k-1)$-pseudo-coherent by 2.2.6 and 2.2.5, as $i_*E \to \tau^{n-k}i_*E$ is an $(n-k-1)$-quasi-isomorphism. By use of the coherator B.16, we see that the cohomologically bounded $\tau^{n-k}i_*E^\cdot$ is quasi-isomorphic to a complex of quasi-coherent $\mathcal{O}_Z$-modules. (Note Z meets the semi-separated hypothesis of B.16, as Z has an ample family of line bundles.) Then using 2.3.1(c), we see that $\tau^{n-k}i_*E^\cdot$ is quasi-isomorphic to a strict $(n-k-1)$-pseudo-coherent complex $E''^\cdot$. Then $\sigma^{n-k-1}E''$ is strict perfect, and $\tau^{n-k}i_*E$ is quasi-isomorphic to $\tau^{n-k}\sigma^{n-k-1}E''$. So $\tau^n Rf_*E^\cdot$ is quasi-isomorphic to $\tau^n Rh_*(\tau^{n-k}i_*E)$, hence to $\tau^n Rh_*(\tau^{n-k}\sigma^{n-k-1}E'')$, hence to $\tau^n Rh_*(\sigma^{n-k-1}E'')$. But this is n-pseudo-coherent since $Rh_*(\sigma^{n-k-1}E'')$ is perfect by part (a). This concludes the proof of (b).

2.7.1. *Porism.* Let $h : Z \to Y$ be a finitely presented map, $U \subseteq Z$ a quasi-compact open. Suppose $\mathcal{O}_{Z,z}$ is flat over $\mathcal{O}_{Y,y}$ with $y = h(z)$, for all z in $Z - U$. Suppose for some locally finitely presented closed subscheme $X \to Z$ with $Z - X = U$ (2.6.1) that $h|X \to Z$ is proper.

Then for any perfect complex $E^\cdot$ on Z which is acyclic on U, $Rh_*(E^\cdot)$ is perfect on Y.

If there is an h-ample family of line bundles on Z and if $E^\cdot$ is a pseudo-coherent complex on Z which is acyclic on U, then Rh_*E is pseudo-coherent on Y.

Proof. This is the main point of the argument proving 2.7, with attention now paid to the fact that the appeals to 2.5.6 and 2.5.5 really need only the flatness hypothesis in 2.7.1.

2.7.2. *Remark.* If $f = h \circ i$ with $h : Z \to Y$ a smooth map and $i : X \to Z$ a closed immersion, then $i_*\mathcal{O}_X$ is pseudo-coherent on Z iff f is a pseudo-coherent morphism by ([SGA 6] III 1.1.4, 1.2, 1.1). We have that $i_*\mathcal{O}_X$ is perfect on Z iff f is a perfect morphism by ([SGA 6] III 1.1, 3.6, 4.1, or III 4.4).

3. K-theory of schemes: definition, models, functorialities, excision, limits

3.1. *Definition.* For X a scheme $K(X)$ is the K-theory spectrum of the complicial biWaldhausen category (1.2.11) of perfect complexes of globally finite Tor-amplitude (2.2.11), in the abelian category of all $\mathcal{O}_X$-modules. (By the default conventions of 1.2.11, the cofibrations in this biWaldhausen category are the degree-wise split monomorphisms, and the weak equivalences are the quasi-isomorphisms.)

For Y a closed subspace of X, $K(X \text{ on } Y)$ is the K-theory spectrum of the complicial biWaldhausen subcategory of those perfect complexes on X which are acyclic on $X - Y$.

3.1.1. It is clear from the description of K_0 in 1.5.6. that $K_0(X)$ is indeed the Grothendieck group "$K^{\cdot}(X)$" of [SGA 6] IV 2.2.

3.1.2. For X quasi-compact, any complex of locally finite Tor-amplitude, and in particular any perfect complex, automatically has globally finite Tor-amplitude.

3.2. *Definition.* For X a scheme, $K^{\text{naive}}(X)$ is the K-theory spectrum of the complicial biWaldhausen category of strict perfect complexes in the category of $\mathcal{O}_X$-modules. Similarly for $K^{\text{naive}}(X \text{ on } Y)$.

3.2.1. Indeed, $K_0^{\text{naive}}(X)$ is the naive Grothendieck group "$K^{\cdot}(X)_{\text{naif}}$" of [SGA 6] IV 2.2. Soon we will see that $K^{\text{naive}}(X)$ is the Quillen K-theory of X in general, and that $K^{\text{naive}}(X)$ is homotopy equivalent to $K(X)$ when X has an ample family of line bundles. Thus locally K^{naive} and K agree, but it will be K that has good local-to-global properties (cf. Sections 8, 10, esp. 8.5, 8.6).

3.3. *Definition.* For X a scheme, $G(X)$ is the K-theory spectrum of the complicial biWaldhausen category of all pseudo-coherent complexes with globally bounded cohomology in the abelian category of all $\mathcal{O}_X$-modules. For Y a closed subspace of X, $G(X \text{ on } Y)$ is the K-theory spectrum of the subcategory of those complexes acyclic on $X - Y$.

3.3.1. Indeed $G_0(X)$ is the Grothendieck group "$K_{\cdot}(X)$" of [SGA 6] IV 2.2. Quillen [Q1] defined higher K'- or G-theory only for noetherian schemes, and for these his $K'(X)$ is homotopy equivalent to $G(X)$ as we will see.

3.4. The construction of the K-theory spectrum proceeds not from the derived category, but from the underlying complicial biWaldhausen category, and is known to be functorial only in the biWaldhausen category. By 1.9.8, the choice of underlying biWaldhausen category is not critical in that all related choices give the same K-theory. But to exhibit all the required functorialities in K-theory, many different underlying model categories must be employed. Hence we proceed to compile lists of the most useful models.

3.5. **Lemma**. *Let X be a quasi-compact scheme. Consider the following list of complicial biWaldhausen categories* (cf. 1.2.11) *in the abelian category of all $\mathcal{O}_X$-modules (or in cases* 6, 7, 8 *in the abelian* *category of diagrams $A \to B$ of $\mathcal{O}_X$-modules):*

3.5.1. *perfect complexes,*

3.5.2. *perfect strict bounded complexes,*

3.5.3. *perfect bounded above complexes of flat $\mathcal{O}_X$-modules,*

3.5.4. *perfect bounded below complexes of injective $\mathcal{O}_X$-modules,*

3.5.5. *perfect bounded below complexes of flasque $\mathcal{O}_X$-modules,*

3.5.6. *diagrams $E^\cdot \xrightarrow{\sim} F^\cdot$ consisting of a quasi-isomorphism between perfect complexes,*

3.5.7. *diagrams as in 3.5.6, but with $E^\cdot$ degree-wise flat bounded above and $F^\cdot$ degree-wise injective bounded below, perfect complexes,*

3.5.8. *diagrams as in 3.5.6, but with $E^\cdot$ degree-wise flat bounded above and $F^\cdot$ degree-wise flasque bounded below perfect complexes.*

Then the obvious inclusion functors induce homotopy equivalences of all their K-theory spectra, so all are homotopy equivalent to $K(X)$. Similarly there are homotopy equivalences to $K(X \text{ on } Y)$ from the K-theory spectra of the various subcategories of complexes which are also acyclic on $X - Y$.

Moreover, the results will hold for non-quasi-compact X if we add everywhere the extra condition that the perfect complexes have globally finite Tor-amplitude.

Proof. More precisely, we need to show that the inclusion functors of categories 3.5.2 through 3.5.5 into 3.5.1 induce homotopy equivalences on K-theory spectra. We also show that the inclusions of 3.5.8 and 3.5.7 into 3.5.6 induce homotopy equivalences. The two functors from 3.5.6 to 3.5.1 sending $E^\cdot \xrightarrow{\sim} F^\cdot$ to $E^\cdot$ and to $F^\cdot$ respectively both induce homotopic homotopy equivalences, inverse to the homotopy equivalence induced by the functor from 3.5.1 to 3.5.6 sending $E^\cdot$ to the diagram $1 : E^\cdot = E^\cdot$.

The last statement follows immediately from 1.5.4. The inclusion functors will induce homotopy equivalences in K-theory because they will induce equivalences on the derived categories of the complicial biWaldhausen categories, allowing appeal to 1.9.8. (One could also appeal directly to Waldhausen approximation 1.9.1 given 1.9.5 and the facts below.)

By 1.9.7 (or its dual), to prove the equivalence of derived categories it suffices to show for all B in the target category that there is an A in the source category and a quasi-isomorphism $A \xrightarrow{\sim} B$ (resp., $B \xrightarrow{\sim} A$). As the category of $\mathcal{O}_X$-modules has enough flat and enough injective objects (e.g., [H] II Section 1), and as all injectives are flasque, 1.9.5 yields the well-known fact that for any cohomologically bounded complex $B^\cdot$, one has quasi-isomorphisms $A^\cdot \xrightarrow{\sim} B^\cdot \xrightarrow{\sim} C^\cdot$ where $A^\cdot$ is bounded above and degree-wise flat, and $C^\cdot$ is bounded below and degree-wise injective, *a fortiori* degree-wise flasque. This immediately proves the equivalence of derived categories for the inclusions of 3.5.3, 3.5.4, and 3.5.5 into 3.5.1. If one considers an intermediate biWaldhausen category whose objects are diagrams of quasi-isomorphisms $E^{\cdot} \xrightarrow{\sim} F^{\cdot}$ of perfect complexes with $E^{\cdot}$ degree-wise flat and bounded above, one sees similarly that the inclusion of this category into 3.5.6 and the inclusions of 3.5.7 and 3.5.8 into this intermediate category induce equivalences of derived categories.

It remains only to show that the inclusion of 3.5.2 into 3.5.1 induces an equivalence of derived categories. But a perfect complex of globally finite Tor-amplitude is globally cohomologically bounded. Thus such a perfect $E^{\cdot}$ is quasi-isomorphic to the truncations for suitable $n \ll 0$ and $m \gg 0$: $E^{\cdot} \xrightarrow{\sim} \tau^n E^{\cdot} \xleftarrow{\sim} \tau^{\leq m}(\tau^n E^{\cdot})$. As $\tau^n E^{\cdot}$ is strict bounded below and $\tau^{\leq m}(\tau^n E^{\cdot})$ is strict bounded, these quasi-isomorphisms show that the inclusion of 3.5.2 into the intermediate biWaldhausen category of perfect strict bounded below complexes and the further inclusion of this intermediate category into 3.5.1 induce equivalences of derived categories.

Hence all the inclusions induce equivalences of derived categories, and 1.9.8 yields the result. Clearly the proof works if we impose everywhere the extra condition that the complexes are to be acyclic on $X - Y$, yielding the result for $K(X \text{ on } Y)$.

3.6. **Lemma**. *For X either quasi-compact and semi-separated (B.7), or else noetherian, Lemma 3.5 remains true if the following categories are added to the list in 3.5. So all have K-theory spectra homotopy equivalent to $K(X)$;*

3.6.1. *perfect complexes of quasi-coherent $\mathcal{O}_X$-modules,*

3.6.2. *perfect complexes of injective objects in* Qcoh(X).

Proof. The inclusion of 3.6.2 into 3.6.1 induces an equivalence of derived categories as Qcoh(X) has enough injectives (B.3). The inclusion of 3.6.1 into 3.5.1 has an inverse equivalence on the derived category, given by the coherator (B.16). Now we apply 1.9.8.

3.7. **Lemma**. *For X noetherian, Lemmas* 3.5 *and* 3.6 *remain true if the following categories are added to the lists. Hence all have K-theory spectra homotopy equivalent to $K(X)$:*

3.7.1. *perfect complexes of coherent $\mathcal{O}_X$-modules,*

3.7.2. *perfect strict bounded complexes of coherent $\mathcal{O}_X$-modules.*

Similarly for $K(X \text{ on } Y)$ if we add the extra condition that the complexes are acyclic on $X - Y$.

Proof. The inclusion of 3.7.1 into 3.6.1 induces an equivalence of derived categories, as we prove by the Inductive Construction Lemma 1.9.5, whose hypothesis 1.9.5.1 is met by 2.2.8 and [EGA] I 6.9.9. We conclude by the usual 1.9.7 and 1.9.8.

The inclusion of 3.7.2 into 3.7.1 induces an equivalence of derived categories, as we see using truncations just as for the inclusion of 3.5.2 into 3.5.1.

3.8. **Lemma**. *For X with an ample family of line bundles (2.1.1), Lemma 3.5 remains true if the following categories are added to the list in 3.5 and 3.6. In particular, all their K-theory spectra are homotopy equivalent to $K(X)$:*

3.8.1. *perfect complexes of quasi-coherent $\mathcal{O}_X$-modules,*

3.8.2. *perfect bounded above complexes of flat quasi-coherent $\mathcal{O}_X$-modules,*

3.8.3. *strict perfect complexes.*

Similarly for $K(X$ on $Y)$ if we add everywhere the extra condition that the complexes are acyclic on $X - Y$.

Proof. We note by 2.1.1 and B.7 that X is quasi-compact and semi-separated, so that 3.6 applies. We note 3.8.1 = 3.6.1. Finally the inclusion of 3.8.3 into 3.8.2 and into 3.8.1 induces an equivalence of derived categories by 2.3.1(d).

3.9. **Corollary**. *For X a scheme with an ample family of line bundles, there is a natural homotopy equivalence $K^{\text{naive}}(X) \xrightarrow{\sim} K(X)$.*

Proof. Lemma 3.8.3 and Definition 3.2. (cf. [SGA 6] IV 2.9).

3.10. **Proposition**. *For X any scheme, $K^{\text{naive}}(X)$ is naturally homotopy equivalent to Quillen's K-theory spectrum of X ([Q1]).*

For X with an ample family of line bundles, $K(X)$ is naturally homotopy equivalent to the Quillen K-theory spectrum of X.

Proof. The first statement follows from 1.11.7, as strict perfect complexes are precisely bounded complexes in the exact category of algebraic vector bundles on X. The second statement follows by 3.9.

3.11. **Lemma**. *The obvious inclusions of the complicial biWaldhausen categories in the following lists induce homotopy equivalences on K-theory under the conditions preceding each list. In particular, the K-theory spectra are all homotopy equivalent to $G(X)$.*

For X a scheme:

3.11.1. *cohomologically bounded pseudo-coherent complexes of $\mathcal{O}_X$-modules,*

3.11.2. *pseudo-coherent strict bounded complexes of $\mathcal{O}_X$-modules,*

3.11.3. *cohomologically bounded pseudo-coherent complexes of flat $\mathcal{O}_X$-modules,* 3.11.4. *cohomologically bounded pseudo-coherent complexes of injective $\mathcal{O}_X$-modules,*

3.11.5. *cohomologically bounded pseudo-coherent complexes of flasque $\mathcal{O}_X$-modules,*

3.11.6. *diagrams of quasi-isomorphisms $E^\cdot \xrightarrow{\sim} F^\cdot$ of cohomologically bounded pseudo-coherent complexes, with $E^\cdot$ degree-wise flat and $F^\cdot$ degree-wise flasque,*

3.11.7. *as in 3.11.6., except $F^\cdot$ is degree-wise injective instead of merely flasque.*

For X either a quasi-compact and semi-separated scheme, or else noetherian, one may add to the list:

3.11.8. *cohomologically bounded pseudo-coherent complexes of quasi-coherent $\mathcal{O}_X$-modules,*

3.11.9. *cohomologically bounded pseudo-coherent complexes of injectives in* Qcoh(X),

For X with an ample family of line bundles, one may add to the list:

3.11.10. *cohomologically bounded strict pseudo-coherent complexes.*

Proof. Note "cohomologically bounded" means "globally cohomologically bounded," and recall the default conventions of 1.2.11 for cofibrations and weak equivalences.

The proof of 3.11 exactly parallels that of 3.5-3.8, and hence we leave it to the reader.

3.12. **Lemma.** *For X a noetherian scheme, the inclusion of the following biWaldhausen categories into* 3.11.8 *induce homotopy equivalences of their K-theory spectra to $G(X)$.*

3.12.1. *cohomologically bounded complexes of coherent $\mathcal{O}_X$-modules,*

3.12.2. *strict bounded complexes of coherent $\mathcal{O}_X$-modules.*

Proof. One argues as in 3.7, recalling also that coherent complexes are pseudo-coherent by 2.2.8.

3.13. **Corollary.** *For X a noetherian scheme, $G(X)$ is naturally homotopy equivalent to the Quillen G- or K'-spectrum of X defined in* [Q1].

Proof. This follows from 3.12 and 1.11.7.

3.14. $K(X)$ and $K^{\text{naive}}(X)$ are contravariant functors in the scheme X, as a map of schemes $f : X \to X'$ induces a complicial exact (1.2.16) functor $Lf^* = f^*$ between the biWaldhausen categories of perfect bounded above complexes of flat modules (3.5.3), or those of strict perfect complexes. Similarly f induces a map $f^* : K(X' \text{ on } Y') \to K(X \text{ on } f^{-1}(Y'))$.

3.14.1. If $f : X \to X'$ is a map of globally finite Tor-dimension, then $Lf^*E^\cdot$ is cohomologically bounded if $E^\cdot$ is. It follows then f^* is a complicial exact functor between categories of cohomologically bounded pseudo-coherent complexes of flat modules (3.11.3), and so induces a map $f^* : G(X') \to G(X)$. This makes $G(\ )$ a contravariant functor for maps of finite Tor-dimension (cf. [SGA 6] IV 2.12 for G_0, [Q1] Section 7 2.5).

3.15. The functor $(E^\cdot, F^\cdot) \mapsto E^\cdot \underset{\mathcal{O}_X}{\otimes} F^\cdot$ preserves degree-wise split monomorphisms in either variable. This functor represents $E^\cdot \overset{L}{\underset{\mathcal{O}_X}{\otimes}} F^\cdot$ and preserves quasi-isomorphisms if either $E^\cdot$ or $F^\cdot$ runs over a category of bounded above complexes of flat $\mathcal{O}_X$-modules. Thus it is biexact, i.e., exact in each variable, on a pair of biWaldhausen categories if either category consists of flat complexes. It is clear that $E^\cdot \underset{\mathcal{O}_X}{\otimes} F^\cdot$ is strict perfect of both $E^\cdot$ and $F^\cdot$ are strict perfect. Applied locally on X, this shows $E^\cdot \overset{L}{\underset{\mathcal{O}_X}{\otimes}} F^\cdot$ is perfect if both $E^\cdot$ and $F^\cdot$ are. It is easy to see that if $E^\cdot$ has finite Tor-amplitude, then $E^\cdot \overset{L}{\underset{\mathcal{O}_X}{\otimes}} F^\cdot$ has finite Tor-amplitude (respectively, is cohomologically bounded) if $F^\cdot$ has finite Tor-amplitude (resp. is cohomologically bounded). (See [SGA 6] I 5.6, 5.3 if you get stuck.) We have already seen that $E^\cdot \overset{L}{\underset{\mathcal{O}_X}{\otimes}} F^\cdot$ is pseudo-coherent if both $E^\cdot$ is perfect (even pseudo-coherent) and $F^\cdot$ is pseudo-coherent, back in 2.5.1. Thus $\otimes$ induces biexact functors between various biWaldhausen categories 3.5.3, 3.8.3, 3.11.3. As biexact functors induce pairings between K-theory spectra ([W] just after 1.5.3), we get various pairings (cf. [SGA 6] IV 2.7, 2.10):

$$K(X) \wedge K(X) \to K(X) \tag{3.15.1}$$

$$K^{\text{naive}}(X) \wedge K^{\text{naive}}(X) \to K^{\text{naive}}(X) \tag{3.15.2}$$

$$K(X) \wedge G(X) \to G(X) \tag{3.15.3}$$

and even:

$$K(X \text{ on } Y) \wedge K(X \text{ on } Z) \to K(X \text{ on } Y \cap Z) \tag{3.15.4}$$

$$K(X \text{ on } Y) \wedge G(X \text{ on } Z) \to G(X \text{ on } Y \cap Z). \tag{3.15.5}$$

As the tensor product $\otimes$ is associative and commutative up to "coherent natural isomorphism," $K(X)$ and $K^{\text{naive}}(X)$ are in fact "homotopy-everything" ring spectra, and the canonical map $K^{\text{naive}}(X) \to K(X)$ and also $f^* : K(X') \to K(X)$ and $f^* : K^{\text{naive}}(X') \to K^{\text{naive}}(X)$ are maps of such ring spectra. The spectrum $G(X)$ is a module spectrum over $K(X)$, and when it exists, $f^* : G(X') \to G(X)$ is a map of module spectra over $K(X')$ (cf. e.g., [Ma2]).

Moreover $K(X \text{ on } Y)$ has a commutative and associative multiplication up to "coherent homotopy," but fails to have a unit when $X \neq Y$.

There are also external pairings induced by $(E^\cdot, F^\cdot) \to E^\cdot \underset{\mathcal{O}_S}{\otimes} F^\cdot$ for X flat over S and Z over S.

(3.15.6) $$K(X) \wedge K(Z) \to K(X \underset{S}{\times} Z)$$

(3.15.7) $$K(X) \wedge G(Z) \to G(X \underset{S}{\times} Z)$$

See [SGA 6] IV 3.3 for how to go further.

3.16. Let $f : X \to Y$ be a map of schemes. Then f_* is a complicial exact functor in the sense of 1.2.16 between the complicial biWaldhausen categories of all bounded below complexes of flasque $\mathcal{O}$-modules on X and on Y. For flasque modules are deployed for f_*, so f_* represents Rf_* and preserves quasi-isomorphisms on such complexes; and f_* also preserves flasqueness ([SGA 4] V 4.9, and [H] I 5.3 β or [V] II Section 2 no. 2). To conclude that f_* induces maps $f_* : K(X) \to K(Y)$ or $f_* : G(X) \to G(Y)$, we need only find conditions that make Rf_* to preserve the required perfection, pseudo-coherence, and the global bounds on cohomological dimension or Tor-amplitude in the definitions of the biWaldhausen categories 3.5.5 and 3.11.5. Considering B.11, 2.5.4 (= [SGA 6] III 2.5, 4.8.1) and 2.7, we get variously (cf. [SGA 6] IV 2.11, [Th4] 1.13, [Q1] Section 7 2.7):

3.16.1. $G(\ )$ is a covariant functor on the category of noetherian schemes and proper maps.

(Note this improves upon [Q1] 7.2.7, which only made $G(\ )$ a functor up to homotopy and for finite or projective maps. Our method avoids the fuss of Gillet's Chow envelope method [Gil] of proving 3.16.1. Also after the usual rectification to make $f_*g_* = (fg)_*$ on $\mathcal{O}$-modules strictly, instead of up to natural isomorphism, our method yields a strictly functorial $G(\ )$, instead of a functor up to homotopy.)

3.16.2. $G(\ )$ is a covariant functor on the category of quasi-compact schemes and flat proper maps with a relatively ample family of line bundles.

3.16.3. $G(\ )$ is a covariant functor on the category of quasi-compact schemes and pseudo-coherent projective morphisms.

3.16.4. $K(\ )$ is a covariant functor on the category of noetherian schemes, with proper maps of finite Tor-dimension (i.e., perfect proper maps).

3.16.5. $K(\ )$ is a covariant functor on the category of quasi-compact schemes and perfect projective morphisms.

3.16.6. $K(\ )$ is a covariant functor on the category of quasi-compact schemes and flat proper morphisms.

3.16.7. There are analogs of 3.16.2 - 3.16.6 for $G(X$ on $Y)$ and $K(X$ on $Y)$, using 2.6.3 and 2.7.1. We also note that if $Z \subseteq Y \subseteq X$ with Z and Y closed subspaces in X, the exact functor forgetting part of the acyclicity requirement yields a canonical map $K(X$ on $Z) \to K(X$ on $Y)$.

3.17. **Proposition**. Projection Formula (cf. [SGA 6] IV 2.12, [Q1] Section 7 2.10). *Let $f : X \to Y$ be a quasi-compact and quasi-separated map of schemes with Y quasi-compact. Suppose that f is such that Rf_* preserves pseudo-coherence (respectively, preserves perfection), hence induces a map $f_* : G(X) \to G(Y)$, (resp. $f_* : K(X) \to K(Y)$). For example, let f be as in* 3.16.1 - 3.16.3 *(resp.* 3.16.4 - 3.16.6*). Then f_* is a map of module spectra over the ring spectra $K(Y)$. That is, the diagram* (3.17.1) *(resp. the similar diagram where all $G(\ )$'s are replaced by $K(\ )$'s) commutes up to canonically chosen homotopy:*

(3.17.1)
$$\begin{array}{ccccc} & & K(X)\wedge G(X) & \xrightarrow{\otimes} & G(X) \\ & \nearrow^{f^*\wedge 1} & & & \\ K(Y)\wedge G(X) & & & & \downarrow f_* \\ & \searrow_{1\wedge f_*} & & & \\ & & K(Y)\wedge G(Y) & \xrightarrow{\otimes} & G(Y) \end{array}$$

Proof. It is convenient to represent Rf_* on the chain level as $f_* \circ T$ where T is the total complex (totaled via sums, and not products) of the functorial flasque Godement resolution indexed by all the points of X, as in [SGA 4] XVII 4.2. This $Rf_* = f_* \circ T$ is exact on all complexes.

Now consider $F^{\cdot}$ a bounded above perfect complex of flat $\mathcal{O}_Y$-modules, and $E^{\cdot}$ a pseudo-coherent (resp. perfect) complex of $\mathcal{O}_X$-modules. By deployment, we have $F^{\cdot} \overset{L}{\underset{\mathcal{O}_Y}{\otimes}} (\ ) = F^{\cdot} \underset{\mathcal{O}_Y}{\otimes} (\ )$, etc. Then 2.5.5 shows that the canonical map (3.17.2) is a natural quasi-isomorphism:

(3.17.2)

$$F^{\cdot} \overset{L}{\underset{\mathcal{O}_Y}{\otimes}} Rf_*E^{\cdot} = F^{\cdot} \underset{\mathcal{O}_Y}{\otimes} f_*TE^{\cdot} \to f_*\left(f^*F^{\cdot} \underset{\mathcal{O}_Y}{\otimes} TE^{\cdot}\right) \to f_*T\left(f^*F^{\cdot} \underset{\mathcal{O}_X}{\otimes} E^{\cdot}\right)$$
$$\| $$
$$Rf_*\left(f^*F^{\cdot} \overset{L}{\underset{\mathcal{O}_X}{\otimes}} E^{\cdot}\right).$$

As in 1.5.4, this quasi-isomorphism of two functors biexact in $(E^{\cdot}, F^{\cdot})$ yields the desired homotopy between two maps $K(Y) \wedge G(X) \to G(Y)$ (resp., $K(Y) \wedge K(X) \to K(Y)$).

3.18. **Proposition** (cf. [SGA 6] IV 3.1.1, [Q1] Section 7 2.11). *Let* (3.18.1) *be a pullback diagram of quasi-coherent schemes, with f a quasi-separated map.*

(3.18.1)
$$\begin{array}{ccc} X & \xleftarrow{g'} & X' \\ {\scriptstyle f}\downarrow & \square & \downarrow{\scriptstyle f'} \\ Y & \xleftarrow[g]{} & Y' \end{array}$$

Suppose f and g are Tor-independent over Y (2.5.6.2). *Suppose that g has finite Tor-dimension (resp., that f has finite Tor-dimension) and that f and f' are such that Rf_* and Rf'_* preserve pseudo-coherence (resp., that Rf_* and Rf'_* preserve perfection), and so define maps $f_* : G(X) \to G(Y)$ and $f'_* : G(X') \to G(Y')$, (resp., $f_* : K(X) \to K(Y)$ and $f'_* : K(X') \to K(Y')$). For example, we could suppose that f and f' are as in* 3.16.1 - 3.16.3 *(resp.,* 3.16.4 - 3.16.6*).*

Then there is a canonical homotopy between $g^*f_* \simeq f'_*g'^* : G(X) \to G(Y')$ (resp., $g^*f_* \simeq f'_*g'^* : K(X) \to K(Y')$).

Proof. The idea is to use the base change Theorem 2.5.6. We follow Deligne's proof of 2.5.6 in [SGA 4] XVII 4.2, and build yet another model of $G(X)$. Consider $E^{\cdot}$ in the category of bounded above pseudo-coherent complexes of flat $\mathcal{O}_X$-modules. Let $T(E^{\cdot})$ be the total complex, totaled by sums, of the Godement resolution as in [SGA 4] XVII Section 4. Then each stalk $T(E^{\cdot}_x)$ is chain homotopic to $E^{\cdot}_x$ so $g'^*T(E^{\cdot})_x \simeq g'^*E^{\cdot}_x = Lg'^*E^{\cdot}_x$, and $T(E^{\cdot})$ is deployed for Lg'^*. As $T(E^{\cdot})$ is flasque, it is deployed for Rf_* (using B.11 to allow $T^{\cdot}(E)$ to be unbounded below). Using this functor T, the augmentation quasi-isomorphism $E^{\cdot} \simeq T(E^{\cdot})$, 3.11, and the methods of 3.5, we see the complicial biWaldhausen category of cohomologically bounded pseudo-coherent complexes deployed for both Lg'^* and Rf_*, that of cohomologically bounded above pseudo-coherent complexes of flat $\mathcal{O}_X$-modules, and that of cohomologically bounded pseudo-coherent complexes, all have equivalent derived categories. Indeed, after identifying the last two derived categories by their known equivalence, the inverse equivalences to the bideployed derived category are induced by T and the inclusion (cf. [SGA 4] XVII 4.2.10). So by 1.9.8, the bideployed biWaldhausen category has K-theory spectrum homotopy equivalent to $G(X)$.

The techniques of proof of 3.11 and 3.5 show that the derived category of the bideployed biWaldhausen category is equivalent to the derived category of the complicial biWaldhausen category whose objects are data consisting of a cohomologically bounded pseudo-coherent $E^\cdot$ on X which is deployed for both f_* and g'^*, a bounded above degree-wise flat $F^\cdot$ on Y and a quasi-isomorphism $F^\cdot \xrightarrow{\sim} f_*E^\cdot$, and a bounded below degree-wise flasque $G^\cdot$ on X' and a quasi-isomorphism $g'^*E^\cdot \xrightarrow{\sim} G^\cdot$ on X'. The associated abelian category is that of all diagrams $(A \to f_*B,\ g'^*B \to C)$ with A an $\mathcal{O}_Y$-module, B an $\mathcal{O}_X$-module , and C an $\mathcal{O}_{X'}$-module. (Compare [SGA 4] XVII 4.2.12.) The K-theory spectrum of this biWaldhausen category is thus also homotopy equivalent to $G(X)$.

In this model of $G(X)$, g^*f_* is represented by the exact functor sending $(F^\cdot \xrightarrow{\sim} f_*E^\cdot,\ g'^*E^\cdot \xrightarrow{\sim} G^\cdot)$ to $g^*F^\cdot$ (recall, $F^\cdot$ is flat). The map $f'_*g'^*$ is represented by the exact functor sending the object $(F^\cdot \xrightarrow{\sim} f_*E^\cdot,\ g'^*E^\cdot \xrightarrow{\sim} G^\cdot)$ to $f'_*G^\cdot$. There is a natural transformation

$$g^*F^\cdot \to g^*f_*E^\cdot \to g^*f_*g'_*g'^*E^\cdot = g^*g_*f'_*g'^*E^\cdot \to f'_*g'^*E^\cdot \to f'_*G^\cdot.$$

This is the canonical base change map of Deligne ([SGA 4] XVII 4.2.12), and is a quasi-isomorphism by 2.5.6 = [SGA 6] IV 3.1. This natural quasi-isomorphism then induces the desired homotopy $g^*f_* \simeq f'_*g'^*$ of maps on $G(\ )$ by 1.5.4. The proof for $K(\ )$ is essentially the same. Clearly, there are analogs for $K(X \text{ on } Z)$, etc.

3.19. **Proposition** (Excision). *Let $f : X' \to X$ be a map of quasi-compact and quasi-separated schemes. Let $Y \subseteq X$ be a closed subspace such that $X - Y$ is quasi-compact. Set $Y' = f^{-1}(Y) \subseteq X$. Suppose that f is an isomorphism infinitely near Y in the sense of* 2.6.2.2 , 2.6.1. *Then f^* induces homotopy equivalences*

$$(3.19.1)\qquad \begin{aligned} f^* &: K(X \text{ on } Y) \xrightarrow{\sim} K(X' \text{ on } Y') \\ f^* &: G(X \text{ on } Y) \xrightarrow{\sim} G(X' \text{ on } Y'). \end{aligned}$$

Proof. We note that the map f is quasi-separated ([EGA] I 6.1.10). By definition 2.6.2.1, we may choose a scheme structure on Y so that

$i : Y \to X$ is a finitely presented closed immersion with f inducing an isomorphism $f^{-1}(Y) = Y \underset{X}{\times} X' \to Y$. The proposition then results from 2.6.3 which shows Rf_* and Lf^* are inverse up to natural quasi-isomorphism, once we realize Rf_* and Lf^* as exact functors on appropriate model bi-Waldhausen categories. We represent Rf_* by the exact $f_* \circ T$, for T the Godement resolution, and note that Rf_* preserves perfection and pseudo-coherence of complexes acyclic on $X' - Y'$ by 2.6.3, and preserves cohomological boundness by the finite cohomological dimension of Rf_*, B.11. The most appropriate models for $K(X \text{ on } Y)$ and $K(X' \text{ on } Y')$ (resp., $G(X \text{ on } Y)\dots$) are the complicial biWaldhausen category of perfect (resp., cohomologically bounded pseudo-coherent) complexes of $\mathcal{O}_X$-modules which are strictly 0 on $X - Y$. The inclusion functor of this new model into the category of perfect complexes of $\mathcal{O}_X$-modules which are acyclic on $X - Y$ is exact, and $\Gamma_Y \circ T$ provides an exact functor which is inverse to the inclusion up to natural quasi-isomorphism. Thus the new model category indeed has K-theory spectrum homotopy equivalent to $K(X \text{ on } Y)$, (resp. $\dots$).

As f is flat over the points of Y (2.6.2.1), f^* is exact on the category of complexes strictly 0 on $X - Y$. Thus f^* and $f_* \circ T$ induce exact functors on the new models, which are inverse up to natural quasi-isomorphism by 2.6.3. By 1.5.4, this yields the result.

3.19.2. *Examples.* Let A be a noetherian ring, and $I \subseteq A$ an ideal. Then the map to the completion $A \to A\hat{}_I$ induces a homotopy equivalence:

$$K(\mathrm{Spec}(A) \text{ on } \mathrm{Spec}(A/I)) \to K(\mathrm{Spec}(A\hat{}_I) \text{ on } \mathrm{Spec}(A\hat{}_I/IA\hat{}_I))$$

For $A\hat{}_I/IA\hat{}_I = A/I$, and for A noetherian $A \to A\hat{}_I$ is flat over the points of $\mathrm{Spec}(A/I)$, as $A_{(I)}$ is a Zariski ring at (I).

For a general commutative ring A, and $I \subseteq A$ a finitely generated ideal, let A^h_I be the henselization of A along $\mathrm{Spec}(A/I)$. Then the map $A \to A^h_I$ induces a homotopy equivalence $K((A) \text{ on } (A/I)) \xrightarrow{\sim} K((A^h_I) \text{ on } (A^h_I/IA^h_I))$.

If $j : V \to X$ is an open immersion with $Y \subseteq V$, and V is quasi-compact, then $K(X \text{ on } Y) \xrightarrow{\sim} K(V \text{ on } Y)$ is a homotopy equivalence. For this example, it is in fact trivial to prove 2.6.3 directly.

3.20. **Proposition** (cf. [SGA 6] IV 3.2, [Q1] Section 7 2.2). *Let $X = \varprojlim X_\alpha$ be the limit of an inverse system of schemes, where the "bonding" maps $f_{\alpha\beta} : X_\alpha \to X_\beta$ in the system are affine. Suppose that all the X_α are quasi-compact and quasi-separated. Let $Y_\alpha \subseteq X_\alpha$ be a system of closed subspaces with $f^{-1}_{\alpha\beta}(Y_\beta) = Y_\alpha$, and all $X_\alpha - Y_\alpha$ quasi-compact. Then:*

3.20.1. *The derived category of strict perfect (resp., perfect) complexes on X is the direct colimit of the derived categories of strict perfect (resp., perfect) complexes on the X_α, where the maps in the direct system are the $Lf^*_{\alpha\beta}$. Similarly, there is an equivalence of derived categories where one imposes the conditions of acyclicity on $X - Y$ and $X_\alpha - Y_\alpha$ on the complexes in the derived categories.*

3.20.2. *The canonical maps induced by the Lf^*_α are homotopy equivalences:*

$$\varinjlim_\alpha K(X_\alpha) \xrightarrow{\sim} K(X)$$
$$\varinjlim_\alpha K(X_\alpha \text{ on } Y_\alpha) \xrightarrow{\sim} K(X \text{ on } Y).$$

Proof. The construction of $K(\ )$, see 1.5.2 -1.5.3, clearly preserves direct colimits of biWaldhausen categories, and also converts complicial exact functors inducing an equivalence of derived categories into homotopy equivalences of K-theory spectra, by 1.9.8. Hence 3.20.2 follows from 3.20.1.

To prove 3.20.1, we first consider the biWaldhausen category of strict perfect complexes (possibly imposing the condition of acyclicity on $X_\alpha - Y_\alpha$, $X - Y$). As a strict perfect complex is a finite complex of finitely presented modules, it follows from [EGA] IV Section 8 as quoted in C.4 that the biWaldhausen category on X is equivalent to the direct colimit of the biWaldhausen categories on the X_α, and *a fortiori* that the derived categories are equivalent. In more detail, we see that [EGA] IV 8.5.2(ii) applied to the modules in each degree and [EGA] IV 8.5.2(i) applied to the differentials, each applied finitely any times, shows that each strict perfect $E^\cdot$ on X is isomorphic to $f^*_\alpha E^\cdot_\alpha$ for a strict perfect $E^\cdot_\alpha$ on some X_α. Given a morphism $e : E^\cdot \to E'^\cdot$ between strict perfect complexes on X, we apply [EGA] IV 8.5.2(i), first to get maps $E^i_\alpha \to E'^i_\alpha$ defined on some X_α, and then to make them satisfy the identities $\partial e^i = e^{i+1}\partial$, and so to obtain a chain map $e_\alpha : E^\cdot_\alpha \to E'^\cdot_\alpha$ on X_α for α sufficiently large, such that $e = f^*_\alpha(e_\alpha)$. Now if V_α is an affine open in X_α, and $E^\cdot$ is acyclic on $V = V_\alpha \underset{X_\alpha}{\times} X$, then as a complex of projective modules it is chain homotopic to 0 on the affine V. As above, [EGA] IV 8.5.2 shows that this chain nulhomotopy is defined on $E^\cdot_\beta | V_\beta$ for all sufficiently large β, so $E^\cdot_\beta | V_\beta$ is acyclic. Applied to affine covers, this shows that if the strict perfect complex $E^\cdot$ is acyclic on X, or on $X - Y$, then for all sufficiently large β the complex $E^\cdot_\beta$ is acyclic on X_β or $X_\beta - Y_\beta$. Finally e is a quasi-isomorphism iff its mapping cone $\text{cone}(e) = f^*_\alpha\text{cone}(e_\alpha)$ is acyclic on X. By the above, this occurs iff for some β sufficiently large cone (e_β)

is acyclic on X_β, i.e., iff for β sufficiently large e_β is a quasi-isomorphism on X_β. This completes the proof of 3.20.1 for categories of strict perfect complexes.

To prove 3.20.1 for perfect complexes, we work in the biWaldhausen category of perfect strict bounded above complexes of flat modules. This has the correct derived category by 3.5, and all the f^* and f^*_α are exact on this category. By replacing the system of X_α by the cofinal system of all $\alpha \geq \alpha_0$, we may assume that there is a terminal X_0. Then X and all X_α are affine over X_0.

If X_0, and hence X_α and X, has an ample family of line bundles, the derived category of perfect complexes is equivalent to the derived category of strict perfect complexes as in 3.8, and 3.20.1 reduces to the strict case proved above.

We now prove in 3.20.3 - 3.20.6 the result 3.20.1 by induction on the number n of affine open subschemes needed to cover X_0. If $n = 1$, X_0 is affine, hence has an ample family of line bundles, and 3.20.1 is known. To do the induction step, let $n > 1$, and suppose the result is known for schemes covered by fewer affines. Then we write $X_0 = U_0 \cup V_0$ with U_0 open affine, and V_0 open and covered by $n-1$ open affines. Let $U_\alpha = X_\alpha \underset{X_0}{\times} U_0$, etc. Then 3.20.1 is known for $U = \varprojlim U_\alpha$, $V = \varprojlim V_\alpha$, and $U \cap V = \varprojlim U_\alpha \cap V_\alpha$. We note that $U_0 \cap V_0$ is quasi-affine, and so has an ample family of line bundles, so the above indeed yields 3.20.1 for it.

3.20.3. If $E^{\cdot}_\alpha$ is a bounded above flat perfect complex on X_α such that $f^*_\alpha E^{\cdot}_\alpha$ is acyclic on X (or on $X - Y$), then by induction hypothesis for all β sufficiently large $f^*_{\beta\alpha} E^{\cdot}_\alpha = E^{\cdot}_\beta$ has $E^{\cdot}_\beta | U_\beta$ acyclic on U_β (or on $U_\beta - Y_\beta$), and $E^{\cdot}_\beta | V_\beta$ acyclic on V_β (or on $V_\beta - Y_\beta$). As acyclicity is a local question, $E^{\cdot}_\beta$ is then acyclic on X_β (resp., on $X_\beta - Y_\beta$) for all sufficiently large β. We apply this to the mapping cone of a map $e_\alpha : E^{\cdot}_\alpha \to E'^{\cdot}_\alpha$ to show that if $f^*_\alpha e_\alpha$ is a quasi-isomorphism on X, then $e_\beta = f^*_{\beta\alpha} e_\alpha$ is a quasi-isomorphism on X_β for all sufficiently large β.

The next step is to show that any bounded above flat perfect complex on X is quasi-isomorphic to f^*_α of a bounded above flat perfect complex on X_α, for some α. For this, we make use of the patching construction of 3.20.4.

3.20.4. Let U and V form an open cover of a scheme X. Denote the various open immersions as in:

(3.20.4.1)
$$\begin{array}{ccc} U & \xrightarrow{\ j\ } & X \\ {\scriptstyle k'}\uparrow & {\scriptstyle \ell}\nearrow & \uparrow{\scriptstyle k} \\ U \cap V & \xrightarrow[\ j'\ ]{} & V \end{array}$$

Suppose $(E_U^{\cdot}, F_V^{\cdot}, G_{U\cap V}^{\cdot}, \varphi, \psi)$ is a datum consisting of flat perfect complexes $E_U^{\cdot}$, $F_V^{\cdot}$, $G_{U\cup V}^{\cdot}$ on U, V, $U\cup V$ respectively, and of quasi-isomorphisms $\varphi : G_{U\cap V}^{\cdot} \xrightarrow{\sim} k'^{*}E_U^{\cdot}$, $\psi : G_{U\cap V}^{\cdot} \xrightarrow{\sim} j'^{*}F_V^{\cdot}$ on $U\cap V$. Let $j! : \mathcal{O}_U\text{-Mod} \to \mathcal{O}_X\text{-Mod}$ be the left adjoint to j^*. This $j!$ is the extension by 0 functor, is exact, and preserves flatness (e.g., [SGA 4] IV 11.3.3, V 1.3.1). The quasi-isomorphism φ induces a map on X:

$$!\varphi : \ell! G_{U\cap V}^{\cdot} \to \ell! k'^{*} E_U^{\cdot} = j! k'! k'^{*} E_U^{\cdot} \to j! E_U^{\cdot}.$$

The restriction to U, $j^*(!\varphi)$ is isomorphic to the map $k'! G_{U\cap V} \to E_U^{\cdot}$ adjoint to φ. On V, $k^*(!\varphi)$ is the quasi-isomorphism

$$j'!\varphi : j'! G_{U\cap V}^{\cdot} \xrightarrow{\sim} j'! k'^{*} E_U^{\cdot} = k^* j! E_U^{\cdot}.$$

Let $C^{\cdot}(E_U^{\cdot}, F_V^{\cdot}, G_{U\cap V}^{\cdot}, \varphi, \psi)$ be the homotopy pushout (1.1.2) of $!\varphi : \ell! G_{U\cap V}^{\cdot} \to j! E_U^{\cdot}$ and the similar map $!\psi : \ell! G_{U\cap V}^{\cdot} \to k! F_V^{\cdot}$.

$$C^{\cdot}\left(E_U^{\cdot}, F_V^{\cdot}, G_{U\cap V}^{\cdot}, \varphi, \psi\right) = j! E_U^{\cdot} \overset{h}{\underset{\ell! G_{U\cap V}^{\cdot}}{\cup}} k! F_V^{\cdot} \equiv !E \overset{h}{\underset{!G}{\cup}} !F. \tag{3.20.4.2}$$

For any complex $J^{\cdot}$ on X, a choice of maps $E_U^{\cdot} \to j^* J^{\cdot}$ and $F_V^{\cdot} \to k^* J^{\cdot}$, together with a choice of homotopy between the two restrictions of this maps via φ and ψ to maps $G_{U\cap V}^{\cdot} \to \ell^* J^{\cdot}$, determine maps $j! E_U^{\cdot} \to J^{\cdot}$, $k! F_V^{\cdot} \to J^{\cdot}$, and a homotopy of maps $\ell! G_{U\cap V}^{\cdot} \to J^{\cdot}$, and hence (1.1.2) determine a map $C^{\cdot}(E, F, G, \varphi, \psi) \to J$.

The complex $C^{\cdot}(E, F, G, \varphi, \psi)$ is flat, as the $j!$, $k!$, $\ell!$ preserve flatness, as does the construction of the homotopy pushout (1.1.2.1).

As $j'!(\varphi) = k^*(!\varphi) : k^* !G \to k^*(!E)$ is a quasi-isomorphism on V, the corresponding canonical map into the homotopy pushout

$$F_V^{\cdot} \xrightarrow{\sim} F_V^{\cdot} \overset{h}{\underset{k^*(!G^{\cdot})}{\cup}} k^*(!E^{\cdot}) = k^* C^{\cdot}(E, F, G, \varphi, \psi) = k^* C^{\cdot}$$

is a quasi-isomorphism on V. Hence $k^* C^{\cdot} = C^{\cdot}|V$ is perfect. Similarly $C^{\cdot}|U = j^* C^{\cdot}$ is quasi-isomorphic to $E_U^{\cdot}$, and so perfect. Hence $C^{\cdot}(E, F, G, \varphi, \psi)$ is locally perfect, hence perfect.

Suppose $J^{\cdot}$ is a perfect complex on X, from which we obtain a datum $(E_U^{\cdot}, F_V^{\cdot}, G_{U\cap V}^{\cdot}, \varphi, \psi)$ by $E_U^{\cdot} = j^* J^{\cdot} = J^{\cdot}|U$, $F_V^{\cdot} = k^* J^{\cdot} = J^{\cdot}|V$, $G_{U\cap V}^{\cdot} = \ell^* J^{\cdot} = J^{\cdot}|U \cup V$, and $\varphi = 1$, $\psi = 1$. Then the adjunction maps $j! E_U^{\cdot} = j! j^* J^{\cdot} \to J^{\cdot}$, $k! F_V^{\cdot} = k! k^* J^{\cdot} \to J^{\cdot}$, and the 0 homotopy between identical maps $\ell! G_{U\cap V}^{\cdot} = \ell! \ell^* J^{\cdot} \to J^{\cdot}$ induce a map $C^{\cdot}(j^* J, k^* J, \ell^* J, 1, 1,) \to J^{\cdot}$ which is a quasi-isomorphism, as one checks locally on U and V. (Indeed we have already seen this, as $C^{\cdot}$ here is the mapping cone of the left map in the exact sequence (2.4.1.10) with $E^{\cdot} = J^{\cdot}$.)

3.20.5. Now let $J^\cdot$ be a bounded above flat perfect complex on X. We want to show that it is quasi-isomorphic to f_α^* of a bounded above flat perfect complex on some X_α. By induction hypothesis on U and V, for α sufficiently large, there are flat bounded above perfect complexes $E^\cdot_{U_\alpha}$ on U_α, $F^\cdot_{V_\alpha}$ on V_α, and isomorphisms in the derived category $f_\alpha^* E^\cdot_{U_\alpha} \sim J^\cdot|U$, $f_\alpha^* F^\cdot_{V_\alpha} \sim J^\cdot|V$. By the calculus of fractions we represent these by strict quasi-isomorphisms of flat bounded above perfect complexes $f_\alpha^* E^\cdot_{U_\alpha} \xleftarrow{\sim} A^\cdot \xrightarrow{\sim} J^\cdot|U$, $f_\alpha^* F^\cdot_{V\alpha} \xleftarrow{\sim} B^\cdot \xrightarrow{\sim} J^\cdot|V$. In the derived category on $U \cap V$, we have a composite isomorphism $f_\alpha^* E^\cdot_{U_\alpha}|U \cap V \sim J^\cdot \sim f_\alpha^* F^\cdot_{V\alpha}|U \cap V$. By 3.20.1 for the quasi-affine $U_\alpha \cap V_\alpha$, we see that for α sufficiently large that this isomorphism is f_α^* of an isomorphism on $U_\alpha \cap V_\alpha$, represented in the calculus of fractions by strict quasi-isomorphisms $E^\cdot_{U_\alpha}|U_\alpha \cap V_\alpha \xleftarrow{\sim} G^\cdot_{U_\alpha \cap V_\alpha} \xrightarrow{\sim} F^\cdot_{V_\alpha}|U_\alpha \cap V_\alpha$. On $U \cap V$, the criterion of the calculus of fractions for equivalence of representations of maps in the derived category gives a chain homotopy commutative diagram:

$$
(3.20.5.1)\qquad
\begin{array}{ccccc}
J^\cdot|U\cap V & \xleftarrow{\;1\;} & J^\cdot|U\cap V & \xrightarrow{\;1\;} & J^\cdot|U\cap V \\
{\scriptstyle \sim|U\cap V}\uparrow & & \uparrow{\scriptstyle \sim} & & \uparrow{\scriptstyle \sim|U\cap V} \\
A^\cdot|U\cap V & \xleftarrow{\;\sim\;} & D^\cdot & \xrightarrow{\;\sim\;} & B^\cdot|U\cap V \\
{\scriptstyle \sim|U\cap V}\downarrow & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim|U\cap V} \\
f_\alpha^* E^\cdot_\alpha|U_\alpha\cap V_\alpha & \underset{f_\alpha^*\varphi}{\xleftarrow{\;\sim\;}} & f_\alpha^* G^\cdot_{U_\alpha\cap V_\alpha} & \underset{f_\alpha^*\psi}{\xrightarrow{\;\sim\;}} & f_\alpha^* F^\cdot_{V_\alpha}|U_\alpha\cap V_\alpha
\end{array}
$$

We make choices of homotopies in this diagram, and appeal to the mapping properties of the construction of 3.20.4. This yields quasi-isomorphisms of perfect complexes

$$
J^\cdot \xleftarrow{\sim} !J^\cdot|U \underset{!J^\cdot U\cap V}{\overset{h}{\cup}} !J^\cdot|V \xleftarrow{\sim} !A^\cdot \underset{!D^\cdot}{\overset{h}{\cup}} !B^\cdot \xrightarrow{\sim} f_\alpha^*\left(!E^\cdot_\alpha \underset{!G^\cdot_\alpha}{\cup} !F^\cdot_\alpha \equiv C^\cdot_\alpha\right).
$$

Thus $J^\cdot$ is quasi-isomorphic to $f_\alpha^* C^\cdot_\alpha$ for a bounded above flat perfect complex $C^\cdot_\alpha$ on some X_α, as required. If $J^\cdot|X - Y$ is to be acyclic, $C^\cdot_\beta = f_{\beta\alpha}^* C^\cdot_\alpha$ will be acyclic on $X_\beta - Y_\beta$ for some β sufficiently large.

3.20.6. To complete the induction step, it remains only to show that if $E^\cdot_\alpha$ and $E'^\cdot_\alpha$ are bounded above flat perfect complexes on X_α, then with $E^\cdot_\beta = f_{\beta\alpha}^* E^\cdot_\alpha$ and $E^\cdot = f_\alpha^* E^\cdot_\alpha$, etc., that the canonical map is an isomorphism:

$$\varinjlim_{\beta} \ \mathrm{Mor}_{D(X_\beta)}\left(E^{\cdot}_\beta,\ E'^{\cdot}_\beta\right) \xrightarrow{\cong} \mathrm{Mor}_{D(X)}\left(E^{\cdot}, E'^{\cdot}\right). \tag{3.20.6.1}$$

By the induction hypothesis, the corresponding maps to (3.20.6.1) for U, V and for the quasi-affine $U \cap V$ are isomorphisms. The result follows for X by the 5-lemma applied to the map (3.20.6.1) between the Mayer-Vietoris exact sequences (2.4.1.8).

This completes the proof of the induction step, hence of 3.20.1 and of the proposition.

3.21. **Theorem** (Poincaré duality) (cf. [SGA 6] IV 2.5, [Q1] Section 7-1). *Let X be a quasi-compact scheme. Suppose for every local ring $\mathcal{O}_{X,x}$ of X, that every finitely presented $\mathcal{O}_{X,x}$-module has finite Tor-dimension over $\mathcal{O}_{X,x}$. (In fact, it suffices to suppose that every pseudo-coherent $\mathcal{O}_{X,x}$-module has finite Tor-dimension over $\mathcal{O}_{X,x}$.) (Note that any regular noetherian scheme meets these hypotheses.) Then the canonical map is a homotopy equivalence:*

$$K(X) \xrightarrow{\sim} G(X).$$

Proof. This follows from the definitions once we show that any cohomologically bounded pseudo-coherent complex $E^{\cdot}$ on X is perfect. As X is quasi-compact, there is no need to worry about global bounds, and the question is local. So we take a point $x \in X$, and restrict to a small affine *nbd* U of x. By 2.3.1(e), $E^{\cdot}|U$ is quasi-isomorphic to a strict pseudo-coherent complex, so we may assume that $E^{\cdot}$ is strict pseudo-coherent. As $E^{\cdot}$ is cohomologically bounded, there is an integer k such that $H^n(E^{\cdot}) = 0$ for $n \leq k$. Then $Z^n E^{\cdot} = B^n E^{\cdot}$ for $n \leq k$. For $n \leq k$, $E^{n-2} \to E^{n-1} \to Z^n E \to 0$ is then exact, so that $Z^n E$ is finitely presented. (In fact $Z^n E^{\cdot}$ is resolved by the exact complex of algebraic vector bundles $\sigma^{\leq n-1}E^{\cdot}$, and so is a pseudo-coherent module.) Then the stalk $Z^k E^{\cdot}_x$ has finite Tor-dimension, say p. By descending induction, using the facts that E^n_x is free and that $0 \to Z^{n-1}E^{\cdot}_x \to E^{n-1}_x \to Z^n E^{\cdot}_x \to 0$ is exact for $n \leq k$, we get that $Z^{k-p}E^{\cdot}_x$ is flat and finitely presented over $\mathcal{O}_{X,x}$, and hence free. Then the finitely presented $Z^{k-p}E^{\cdot}$ is free over some smaller open *nbd* U of x. Thus $\tau^{k-p}E^{\cdot}$ is strict perfect on U, and is also quasi-isomorphic to $E^{\cdot}|U$. So $E^{\cdot}$ is perfect, as required.

3.22. To prepare the key Localization Theorem 5.1, 7.4, we must consistently use K-theory with supports, $K(X \text{ on } Y)$. We remark that the absolute case $K(X)$ is the special case $K(X) = K(X \text{ on } X)$. Also $K(X \text{ on } \phi) = 0$ for ϕ empty.

4. Projective space bundle theorem

4.1. **Theorem** (Projective space bundle Theorem). *Let X be a quasi-compact and quasi-separated scheme. Let Y be a closed subspace such that $X - Y$ is quasi-compact. Let $\mathcal{E}$ be an algebraic vector bundle of rank r over X, and let $\pi : \mathbf{P}\mathcal{E}_X \to X$ be the associated projective space bundle. Then there are natural homotopy equivalences*

$$\prod^{r} K(X) \xrightarrow{\sim} K(\mathbf{P}\mathcal{E}_X) \tag{4.1.1}$$

$$\prod^{r} K(X \text{ on } Y) \xrightarrow{\sim} K(\mathbf{P}\mathcal{E}_X \text{ on } \mathbf{P}\mathcal{E}_Y)\,. \tag{4.1.2}$$

These equivalences are given by the formula

$$(x_0, x_1, \ldots, x_{r-1}) \mapsto \sum_{i=0}^{r-1} \pi^*(x_i) \otimes [\mathcal{O}_{\mathbf{P}\mathcal{E}}(-i)]\,. \tag{4.1.3}$$

Proof. 4.3 - 4.12 below.

4.2 Theorem 4.1 for K_0 is proved in Berthelot's exposé, [SGA 6] VI, and for K_0^{naive} goes back to Grothendieck's early work on Riemann-Roch and Chern classes. Modifying these arguments to make them functorial on the Q-category, Quillen ([Q1] Section 8) proved (4.1.1) for K^{naive}. Below, we will modify Quillen's argument to make it work for K.

Logically, the reader should now proceed to Appendix C before returning to 4.3.

4.3. We first reduce to the case where X is noetherian. By C.9, X is the inverse limit $\varprojlim X_\alpha$ of an inverse system of schemes in which all the bounding maps $X_\beta \to X_\alpha$ are affine, and in which all the X_α are finitely presented over Spec($\mathbb{Z}$), and hence noetherian.

As $U = X - Y$ is quasi-compact, and X is quasi-separated, the open immersion $j : U \to X$ is finitely presented ([EGA] I 6.1.10(iii), 6.3.8(i)). Of course $\mathcal{E}$ is a finitely presented $\mathcal{O}_X$-module. Then by restricting to a cofinal system of α, we may assume there are quasi-compact opens $U_\alpha \subseteq X_\alpha$ and vector bundles $\mathcal{E}_\alpha$ on X_α such that $f_{\beta\alpha}^{-1}(U_\alpha) = U_\beta$, $f_{\beta\alpha}^*(\mathcal{E}_\alpha) = \mathcal{E}_\beta$, $U = \varprojlim U_\alpha$, $\mathbf{P}\mathcal{E} = \varprojlim \mathbf{P}\mathcal{E}_\alpha$ is the pullback of $\pi_\alpha : \mathbf{P}\mathcal{E}_\alpha \to X_\alpha$ along $X \to X_\alpha$, etc., ([EGA] IV Section 8, as quoted in C.3, C.4).

We set $Y_\alpha = X_\alpha - U_\alpha$. Then by 3.20 we obtain the diagram (4.3.1) in which the indicated maps are homotopy equivalences:

$$
(4.3.1) \qquad
\begin{array}{ccc}
\varinjlim_{\alpha} \prod^r K(X_\alpha \text{ on } Y_\alpha) & \xrightarrow{\sim} & \prod^r K(X \text{ on } Y) \\
\downarrow & & \downarrow \\
\varinjlim_{\alpha} K(\mathbf{P}\mathcal{E}_{X_\alpha} \text{ on } \mathbf{P}\mathcal{E}_{Y_\alpha}) & \xrightarrow{\sim} & K(\mathbf{P}\mathcal{E}_X \text{ on } \mathbf{P}\mathcal{E}_Y)
\end{array}
$$

Thus it suffices to prove the theorem for the noetherian schemes X_α.

4.4. We assume X is noetherian for the rest of Section 4. We need to recall some standard facts about the cohomology of coherent sheaves on $\mathbf{P}\mathcal{E}$, due to Serre and Grothendieck, and to recall Mumford's notion of a "regular" coherent sheaf, to set up Quillen's argument.

4.5. *Recollection.* (a) For all integers q, $R^q\pi_*$ preserves quasi-coherence and coherence.

(b) For $q \geq r = \operatorname{rank} \mathcal{E}$, and $\mathcal{F}$ any quasi-coherent sheaf on $\mathbf{P}\mathcal{E}$, $R^q\pi_*\mathcal{F} = 0$.

(c) For $\mathcal{F}$ coherent on $\mathbf{P}\mathcal{E}$, there is an integer $n_0(\mathcal{F}) = n_0$ such that for all $n \geq n_0$ and all $q \geq 1$, $R^q\pi_*(\mathcal{F} \otimes \mathcal{O}_{\mathbf{P}\mathcal{E}}(n)) = 0$.

(d) For $\mathcal{F}$ quasi-coherent on $\mathbf{P}\mathcal{E}$ and for $\mathcal{M}$ flat and quasi-coherent on X there is a canonical isomorphism

$$R^q\pi_*(\mathcal{F} \otimes \pi^*\mathcal{M}) \cong R^q\pi_*(\mathcal{F}) \otimes \mathcal{M}.$$

(e) For all integers n, there are natural isomorphisms

$$R^q\pi_*(\mathcal{O}_{\mathbf{P}\mathcal{E}}(n)) = \begin{cases} 0 & q \neq 0,\ r-1 \\ S^n\mathcal{E} & q = 0 \\ (S^{-r-n}\mathcal{E})^\vee \otimes (\wedge^r\mathcal{E})^\vee & q = r-1 \end{cases}$$

where $S^k\mathcal{E}$ is the k-th symmetric power of $\mathcal{E}$, considered to be 0 for $k \leq -1$, $\wedge^r\mathcal{E}$ is the maximal exterior power of $\mathcal{E}$, and $(\ )^\vee$ sends a vector bundle to its dual, $(\ )^\vee = \operatorname{Hom}(\ ,\mathcal{O}_X)$.

(f) For all quasi-coherent sheaves $\mathcal{M}$ on X, there is a natural isomorphism

$$R^q\pi_*(\mathcal{O}_{\mathbf{P}\mathcal{E}}(n) \otimes \pi^*\mathcal{M}) \cong R^q\pi_*(\mathcal{O}_{\mathbf{P}\mathcal{E}}(n)) \otimes \mathcal{M}.$$

Proof. Of course, (a), (b), and (c) are very well-known. ([EGA] III 1.4.10, 2.2.2, 2.2.1). The formula (e) results from a standard Čech cohomology computation [EGA] III 2.1.15, 2.1.16 or [Q1] Section 8.1.1(c).

(Note both these citations give the formula (e) with different typographical errors!) We see from (e) that $R^q\pi_*\mathcal{O}(n)$ is a flat $\mathcal{O}_X$-module. Now (d) and (f) are recovered from the quasi-isomorphism (2.5.5.1), as this flatness of the cohomology groups forces the Tor-spectral sequences of Künneth to degenerate (cf. [EGA] III Section 6, Section 7).

4.6. On $\mathbf{P}\mathcal{E}$, the canonical map

$$(\pi^*\mathcal{E}) \otimes \mathcal{O}(-1) = (\pi^*\pi_*\mathcal{O}(1)) \otimes \mathcal{O}(-1) \twoheadrightarrow \mathcal{O}(1) \otimes \mathcal{O}(-1) = \mathcal{O}_{\mathbf{P}\mathcal{E}}$$

is an epimorphism by [EGA] II 4.1.6. Locally on $\mathbf{P}\mathcal{E}$, the vector bundle $\pi^*\mathcal{E} \otimes \mathcal{O}(-1)$ is free, hence locally is a sum of line bundles $\oplus^i \mathcal{L}_i$. As then $\oplus^i \mathcal{L}_i \twoheadrightarrow \mathcal{O}$ is epimorphic, for each point p some $\mathcal{L}_i$ has image not contained in the maximal ideal of the local ring $\mathcal{O}_p$. For this $\mathcal{L}_i$, $\mathcal{L}_{ip} \to \mathcal{O}_p$ is an epimorphism of rank 1 free modules over the local ring $\mathcal{O}_p$, and so is an isomorphism.

The Koszul complex of $\pi^*\mathcal{E} \otimes \mathcal{O}(-1) \to \mathcal{O}_{\mathbf{P}\mathcal{E}}$ is thus locally isomorphic to a tensor product of complexes $\otimes^r(\mathcal{L}_i \to \mathcal{O})$, where at each point p one of the complexes $\mathcal{L}_i \to \mathcal{O}$ is acyclic. Thus the Kozsul complex of $\pi^*\mathcal{E}\otimes\mathcal{O}(-1) \to \mathcal{O}$ is acyclic (cf. [EGA] III 1.1, [SGA 6] VII 1). Expanding out the Koszul complex yields the well-known long exact sequence of algebraic vector bundles on $\mathbf{P}\mathcal{E}$ ([SGA 6] VI 1.11, [Q1] Section 8):

$$(4.6.1) \quad 0 \to \pi^*\left(\overset{r}{\wedge}\mathcal{E}\right) \otimes \mathcal{O}(-r) \to \pi^*\left(\overset{r-1}{\wedge}\mathcal{E}\right) \otimes \mathcal{O}(1-r) \to \dots$$
$$\dots \to \pi^*\left(\overset{2}{\wedge}\mathcal{E}\right) \otimes \mathcal{O}(-2) \to \pi^*\mathcal{E} \otimes \mathcal{O}(-1) \to \mathcal{O} \to 0.$$

Dually, there is the exact sequence:

$$(4.6.2) \quad 0 \to \mathcal{O} \to (\pi^*\mathcal{E}^\vee) \otimes \mathcal{O}(1) \to \pi^*\left(\overset{2}{\wedge}\mathcal{E}^\vee\right) \otimes \mathcal{O}(2) \to \dots$$
$$\dots \to \pi^*\left(\overset{r}{\wedge}(\mathcal{E}^\vee)\right) \otimes \mathcal{O}(r) \to 0.$$

4.7.0 Let m be an integer. A quasi-coherent sheaf $\mathcal{F}$ on $\mathbf{P}\mathcal{E}$ is said to be m-regular in the sense of Mumford if $R^q\pi_*(\mathcal{F}(m-q)) = 0$ for all $q \geq 1$ (cf. [Q1] Section 8, [SGA 6] XIII Section 1, [Mum] 14). We note that if $\mathcal{F}$ is m-regular, then $\mathcal{F}(n)$ is $(m-n)$-regular.

If $\mathcal{F}$ is a coherent sheaf on $\mathbf{P}\mathcal{E}$, there exists an integer m_0 such that $\mathcal{F}$ is n-regular for all $n \geq m_0$. This follows from 4.5(b) and (c) on taking $m_0 = n_0 + r - 1$.

4.7.1. **Lemma** [Q1], [SGA 6], [Mum]). *Let* $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ *be an exact sequence of quasi-coherent sheaves on* $\mathbf{P}\mathcal{E}$. *Then*

(a) If $\mathcal{F}$ *and* $\mathcal{H}$ *are n-regular,* $\mathcal{G}$ *is n-regular.*

(b) If $\mathcal{G}$ *is n-regular and* $\mathcal{F}$ *is* $(n+1)$*-regular, then* $\mathcal{H}$ *is n-regular.*

(c) If $\mathcal{G}$ *is* $(n+1)$*-regular and* $\mathcal{H}$ *is n-regular and if* $\pi_*\mathcal{G}(n) \to \pi_*\mathcal{H}(n)$ *is an epimorphism, then* $\mathcal{F}$ *is* $(n+1)$*-regular.*

Proof. All these results follow from the long exact sequence in cohomology:

$$\cdots \to R^{q-1}\pi_*\mathcal{G}(n-q) \to R^{q-1}\pi_*\mathcal{H}(n-q) \xrightarrow{\partial} R^q\pi_*\mathcal{F}(n-q) \to R^q\pi_*\mathcal{G}(n-q) \\ \to R^q\pi_*\mathcal{H}(n-q) \xrightarrow{\partial} R^{q+1}\pi_*\mathcal{F}(n-q) \to \cdots.$$

4.7.2. **Lemma** ([SGA 6] XIII 1.3, [Q1] 8.1.3, 8.1.7). *If* $\mathcal{F}$ *is m-regular on* $\mathbf{P}\mathcal{E}$, *then for all* $k \geq m$ *we have that:*

(a) $\mathcal{F}$ *is k-regular.*

(b) *The product map* $\pi_*(\mathcal{F}(k)) \otimes \mathcal{E} = \pi_*(\mathcal{F}(k)) \otimes \pi_*\mathcal{O}(1) \to \pi_*\mathcal{F}(k+1)$ *is an epimorphism on* X.

(c) $\pi^*\pi_*\mathcal{F}(k) \to \mathcal{F}(k)$ *is an epimorphism on* $\mathbf{P}\mathcal{E}$.

Proof. To prove (a), we induct on $k-m$. We must show that if $\mathcal{F}$ is k-regular, then it is $(k+1)$-regular. We tensor the Koszul exact sequence 4.6.1 with $\mathcal{F}(k)$, to obtain an exact sequence

(4.7.2.1)
$$0 \to \left(\pi^* \overset{r}{\wedge} \mathcal{E}\right) \otimes \mathcal{F}(k-r) \to \cdots \to (\pi^*\mathcal{E}) \otimes \mathcal{F}(k-1) \to \mathcal{F}(k) \to 0.$$

This sequence breaks up into short exact sequences (4.7.2.2), where the Z_p are the kernels of the maps in (4.7.2.1).

$$0 \to Z_p \to \left(\pi^* \overset{p}{\wedge} \mathcal{E}\right) \otimes \mathcal{F}(k-p) \to Z_{p-1} \to 0. \tag{4.7.2.2}$$

The sheaf $(\pi^* \overset{p}{\wedge} \mathcal{E}) \otimes \mathcal{F}$ is k-regular by 4.5. Hence $(\pi^* \wedge^p \mathcal{E}) \otimes \mathcal{F}(k-p)$ is p-regular. By descending induction on p, starting from the $(r+1)$-regular $0 = Z_r$, and applying 4.7.1(b) to (4.7.2.2), we see that Z_{p-1} is p-regular. In particular, $Z_0 = \mathcal{F}(k)$ is 1-regular; i.e., $\mathcal{F}$ is $(k+1)$-regular, as required to prove (a).

We also have obtained that Z_1 is 2-regular,, so $R^1\pi_*(Z_1(1)) = 0$. The long exact cohomology sequence induced by (4.7.2.2) for $p = 1$ then shows that $\pi_*(\pi^*\mathcal{E} \otimes \mathcal{F}(k-1+1)) \to \pi_*(Z_0(1) = \mathcal{F}(k+1))$ is an epimorphism for all $k \geq m$. This proves (b).

From (b), we get that (4.7.2.3) is an epimorphism of graded quasi-coherent $\mathcal{O}_X$-modules for $k \geq m$:

$$S^{\cdot}\mathcal{E} \underset{\mathcal{O}_X}{\otimes} \pi_*\mathcal{F}(k) \to \coprod_{n\geq 0} \pi_*\mathcal{F}(n+k). \tag{4.7.2.3}$$

The result (c) now follows on applying the functor Proj to (4.7.2.3) to show that the corresponding canonical map $\pi^*\pi_*\mathcal{F}(k) \to \mathcal{F}(k)$ is an epimorphism on $\mathbf{P}\mathcal{E}$ ([EGA] II 3.2, 3.3, 3.4.4).

4.8. Next we recall Quillen's functorial resolution for quasi-coherent 0-regular sheaves on $\mathbf{P}\mathcal{E}$, [Q1] 8.1.11.

For any quasi-coherent $\mathcal{F}$ on $\mathbf{P}\mathcal{E}$, we inductively define quasi-coherent sheaves $T_n\mathcal{F}$ on X and $Z_n\mathcal{F}$ on $\mathbf{P}\mathcal{E}$. We start with $Z_{-1}\mathcal{F} = \mathcal{F}$. Let $T_n\mathcal{F} = \pi_*((Z_{n-1}\mathcal{F})(n))$, and let $Z_n\mathcal{F}$ be the kernel of the product map $\mathcal{O}(-n) \otimes \pi^*T_n\mathcal{F} \to Z_{n-1}\mathcal{F}$, inductively defining these for $n \geq 0$.

Clearly Z_n and T_n are additive functors, and preserve coherence.

If $\mathcal{F}$ is 0-regular, then by induction on n we see that $Z_{n-1}(\mathcal{F})(n)$ is 0-regular. This is clear for $n = 0$ and $Z_{-1}\mathcal{F} = \mathcal{F}$. The induction step results from 4.7.1(c) applied to the exact sequence

$$0 \to (Z_n\mathcal{F})(n) \to \pi^*T_n\mathcal{F} \to (Z_{n-1}\mathcal{F})(n) \to 0. \tag{4.8.1}$$

Note that $\pi^*T_n\mathcal{F} = \pi^*\pi_*((Z_{n-1}\mathcal{F})(n)) \to (Z_{n-1}\mathcal{F})(n)$ is an epimorphism by 4.7.2(c), and (4.8.1) is exact in the other places by definition of $Z_n\mathcal{F}$. We also note that $\pi_*\pi^*T_n\mathcal{F} \to \pi_*((Z_{n-1}\mathcal{F})(n))$ is not only epimorphic as required by 4.7.1(c), but is actually an isomorphism as we see from the definition of $T_n\mathcal{F}$ and the fact that $\pi_*\pi^* = 1$ for $\pi : \mathbf{P}\mathcal{E} \to X$. From this remark and (4.8.1), we also see that $\pi_*(Z_n(\mathcal{F})(n)) = 0$.

As $R^q\pi_* = 0$ for $q \geq 1$ on the category of 0-regular sheaves, the functor π_* is exact on this exact subcategory of quasi-coherent sheaves. By induction on n, we then see that $Z_{n-1}(\mathcal{F})$ and $T_n\mathcal{F} = \pi_*((Z_{n-1}\mathcal{F})(n))$ are exact functors on the exact category of 0-regular sheaves.

Next we note that $Z_{r-1}\mathcal{F} = 0$ for $r = \text{rank } \mathcal{E}$ and $\mathcal{F}$ 0-regular. For (4.8.1) yields a long exact sequence in cohomology for any $n \geq 0$:

$$\begin{array}{r} \ldots \to R^{q-1}\pi_*\left((Z_{n+q-1}\mathcal{F})(n)\right) \xrightarrow{\partial} R^q\pi_*\left((Z_{n+q}\mathcal{F})(n)\right) \\ \downarrow \\ R^q\pi_*\left(\mathcal{O}(-q) \otimes \pi^*T_{n+q}\mathcal{F}\right) \to \ldots. \end{array} \tag{4.8.2}$$

Using the facts that $\pi_*((Z_n\mathcal{F})(n)) = 0$ and the formula of 4.5, we obtain from (4.8.2) by ascending induction on q that $R^q\pi_*((Z_{n+q}\mathcal{F})(n)) = 0$. As

$R^q\pi_* = 0$ for $q \geq r$, this shows that $(Z_{r-1}\mathcal{F})(r-1)$ is 0-regular. Then $\pi_*(Z_{r-1}\mathcal{F})(r-1) = 0$ combined with 4.7.2(c) shows that $Z_{r-1}(\mathcal{F})(r-1) = 0$ as required.

Now we tensor (4.8.1) with $\mathcal{O}(-n)$, and splice together into a long exact sequence, to obtain Quillen's functorial resolution of a 0-regular sheaf.

4.8.4. **Lemma**. *On the exact category of 0-regular coherent sheaves on* $\mathbf{P}\mathcal{E}$, *there are exact functors* T_i, $i = 0, 1, 2, \ldots, r-1$, *to the category of coherent sheaves on* X, *and a functorial exact sequence on* $\mathbf{P}\mathcal{E}$:

$$0 \to \mathcal{O}(-r+1) \otimes \pi^* T_{r-1}(\mathcal{F}) \to \cdots \to \mathcal{O} \otimes \pi^* T_0(\mathcal{F}) \to \mathcal{F} \to 0.$$

4.8.4. These T_i extend to functors on the category of strict bounded complexes of 0-regular coherent sheaves on $\mathbf{P}\mathcal{E}$, by $T_i(E^\cdot)^k = T_i E^k$. These extended T_i preserve mapping cones of complexes. Suppose $F^\cdot$ is an acyclic complex, which is a bounded complex of 0-regular coherent sheaves. By increasing induction on n starting from $n \ll 0$ where $F^n = 0$, using the exact sequence $0 \to Z^n F \to F^n \to B^{n+1} F \to 0$, the isomorphisms $Z^{n+1}F^\cdot \cong B^{n+1}F^\cdot$ and Lemma 4.7.1(b) and 4.7.2(a), we see that $Z^{n+1}F^\cdot = B^{n+1}F^\cdot$ is 0-regular and coherent for all n. It follows that $0 \to T_i(Z^n F^\cdot) \to T_i F^n \to T_i(B^{n+1}F^\cdot) \to 0$ is exact, so $T_i(B^n F^\cdot) = B^n(T_i F^\cdot)$, $T_i(Z^n F^\cdot) = Z^n(T_i F^\cdot)$, and so $Z^n(T_i F^\cdot) = B^n(T_i F^\cdot)$ for all n. Hence $T_i F^\cdot$ is acyclic. Applying this to mapping cones $F^\cdot$, we see that each functor T_i preserves quasi-isomorphisms between strict bounded complexes of 0-regular coherent sheaves. Thus the T_i will be exact functors between the biWaldhausen categories that we will soon introduce in 4.9.

Also, if $F^\cdot$ is acyclic on $\mathbf{P}\mathcal{E}_X - \mathbf{P}\mathcal{E}_Y = \mathbf{P}\mathcal{E}_X - \pi^{-1}(Y)$, the above argument shows that $T_i F^\cdot$ is acyclic on $X - Y$.

4.8.5. Tensoring the exact sequence of 4.8.3 with $\mathcal{O}(k)$ for $0 \leq k \leq r-1$, applying π_* and considering 4.5 and 4.7, we obtain for any strict bounded complex of 0-regular coherent sheaves $F^\cdot$ on $\mathbf{P}\mathcal{E}$, an exact sequence of complexes on X

$$\begin{aligned} \ldots \to 0 \to T_k(F^\cdot) \to \mathcal{E} \otimes T_{k-1}(F^\cdot) \to S^2\mathcal{E} \otimes T_{k-2}(F^\cdot) \to \ldots \\ \to \cdots \to S^k\mathcal{E} \otimes T_0(F^\cdot) \to \pi_* F^\cdot \to 0. \end{aligned} \tag{4.8.5.1}$$

Also, $\pi_* F^\cdot$ represents $R\pi_* F^\cdot$ as $R^q\pi_* F^i = 0$ for all $q \geq 1$. Suppose now that $F^\cdot$ is perfect. Then $\pi_* F^\cdot = T_0(F^\cdot)$ is perfect by 2.7 for the proper flat $\pi : \mathbf{P}\mathcal{E} \to X$. By ascending induction on k, using (4.8.5.1) and 2.2.13(b), it follows that $T_k(F^\cdot)$ is perfect.

4.9. Consider the following complicial biWaldhausen categories (1.2.11) (with associated abelian categories the categories of all quasi-coherent sheaves).

A: perfect strict bounded complexes of 0-regular coherent sheaves on $\mathbf{P}\mathcal{E}_X$ (resp., such as are also acyclic on $\mathbf{P}\mathcal{E}_X - \mathbf{P}\mathcal{E}_Y$).

B: perfect strict bounded complexes of coherent sheaves on $\mathbf{P}\mathcal{E}_X$ (resp., such as are also acyclic on $\mathbf{P}\mathcal{E}_X - \mathbf{P}\mathcal{E}_Y$).

C: perfect strict bounded complexes of coherent sheaves on X (resp., such as are also acyclic on $X - Y$).

From 4.8.3, 4.8.4, and 4.8.5, we get:

There is an obvious exact inclusion $I : \boldsymbol{A} \to \boldsymbol{B}$.

There are exact functors $T_k : \boldsymbol{A} \to \boldsymbol{C}$ for $k = 0, 1, \ldots, r-1$.

There are exact functors $\mathcal{O}(-k) \otimes \pi^*(\ ) : \boldsymbol{C} \to \boldsymbol{B}$ for all k.

There is a natural quasi-isomorphism (4.9.1) in $\boldsymbol{B}$ for $A^\cdot$ in $\boldsymbol{A}$

(4.9.1)
$$I(A^\cdot) \xleftarrow{\sim} \text{Total complex}\left[\mathcal{O}(-r+1) \otimes \pi^* T_{r-1} A^\cdot \to \cdots \to \mathcal{O} \otimes \pi^* T_0 A^\cdot\right].$$

As X and $\mathbf{P}\mathcal{E}_X$ are noetherian, we see from 3.7 that $K(\boldsymbol{B})$ is $K(\mathbf{P}\mathcal{E}_X)$ (resp., $K(\mathbf{P}\mathcal{E}_X$ on $\mathbf{P}\mathcal{E}_Y)$) and that $K(\boldsymbol{C})$ is $K(X)$ (resp., $K(X$ on $Y))$. Thus the proof of Theorem 4.1 is reduced to showing that the exact functor

$$\bigoplus_{k=0}^{r-1} \mathcal{O}(-k) \otimes \pi^*(\ ) : \prod^r \boldsymbol{C} \to \boldsymbol{B} \tag{4.9.2}$$

induces a homotopy equivalence on K-theory spectra. We show the map on K-theory is both a split epimorphism and a split monomorphism in the homotopy category of spectra.

4.10. First we show that $\prod^r K(\boldsymbol{C}) \to K(\boldsymbol{B})$ is split mono up to homotopy. The formulae of 4.5 show that $R\pi_*(\mathcal{O}(n-k) \otimes \pi^* E^\cdot) = 0$ for $0 \le n < k \le r-1$, and that $R\pi_*(\mathcal{O} \otimes \pi^* E^\cdot) = R\pi_* \pi^* E^\cdot = E^\cdot$, for $E^\cdot$ any complex of quasi-coherent sheaves on X.

Consider the map $K(\boldsymbol{B}) \to \prod^r K(\boldsymbol{C})$ induced by $F^\cdot \mapsto (R\pi_* F^\cdot, R\pi_*(F^\cdot(1)), \ldots, R\pi_*(F^\cdot(r-1)))$. Composing this map with the $\prod^r K(\boldsymbol{C}) \to K(\boldsymbol{B})$ induced by (4.9.2), we get an endomorphism of $\prod^r K(\boldsymbol{C})$. This endomorphism is represented in the homotopy category of spectra by an $r \times r$ matrix of maps $K(\boldsymbol{C}) \to K(\boldsymbol{C})$. The calculation of the preceding paragraph shows that this matrix has 0's above the diagonal and has 1's along the diagonal. Thus the matrix is invertible, and the composite endomorphism of $\prod^r K(\boldsymbol{C})$ is a homotopy equivalence. This shows that $\prod^r K(\boldsymbol{C}) \to K(\boldsymbol{B})$ is a split mono, as required.

4.11. To show that $\prod^r K(\boldsymbol{C}) \to K(\boldsymbol{B})$ is a split epi, we use the T_i. By 4.9.1, the inclusion $I : K(\boldsymbol{A}) \to K(\boldsymbol{B})$ is homotopic to the map induced by the total complex of the $\mathcal{O}(-k) \otimes \pi^* T_k$. Filtering this total complex so that the $\mathcal{O}(-k) \otimes \pi^* T_k$ are the filtration quotients, and appealing to the Additivity Theorem (1.7.3, 1.7.4), we see that the map induced by this total complex is homotopic to $\Sigma(-1)^k \mathcal{O}(-k) \otimes \pi^* T_k$, where the sum is over $k = 0, 1, \ldots, r-1$. Thus up to homotopy the map $I : K(\boldsymbol{A}) \to K(\boldsymbol{B})$ is $\Sigma(-1)^k \mathcal{O}(-k) \otimes \pi^* T_k$ and thus factors through the map (4.9.2) $\prod^r K(\boldsymbol{C}) \to K(\boldsymbol{B})$ via the map $K(\boldsymbol{A}) \to \prod^r K(\boldsymbol{C})$ given by $(K(T_0), -K(T_1), \ldots, (-1)^{r-1} K(T_{r-1}))$.

Thus it suffices to show that $I : K(\boldsymbol{A}) \to K(\boldsymbol{B})$ is a homotopy equivalence. This will follow from the approximation theorem in the form 1.9.8, set up by the dual of 1.9.7, once we show that for every $B^\cdot$ in $\boldsymbol{B}$ there is an $A^\cdot$ in $\boldsymbol{A}$ and a quasi-isomorphism $B^\cdot \xrightarrow{\sim} A^\cdot$. So let $B^\cdot$ be a bounded complex of coherent sheaves on $\mathbf{P}\mathcal{E}_X$. By 4.7.0, there is an n such that every B^i is n-regular. If $n \leq 0$, $B^\cdot$ is in $\boldsymbol{A}$, as every B^i is 0-regular by 4.7.2(a). We now proceed by descending induction on n, for $n > 0$. To do the induction step, suppose the result is known for complexes of $(n-1)$-regular sheaves. For $k \geq 1$ all $B^i(k)$ are $(n-1)$-regular by 4.7.0 and 4.7.2. Tensoring the (locally split) exact Koszul sequence (4.6.2) with $B^\cdot$ yields an exact sequence of complexes. We reinterpret this as a quasi-isomorphism of $B^\cdot = B^\cdot \otimes \mathcal{O}$ into the total complex of the rest of the sequence

$$B^\cdot \to \text{ Total complex } \left[\pi^* \mathcal{E}^\vee \otimes B^\cdot(1) \to \cdots \to \pi^* \left(\overset{r}{\wedge} \mathcal{E}^\vee\right) \otimes B^\cdot(r)\right] \equiv B'^\cdot.$$

The total complex $B'^\cdot$ consists of $(n-1)$-regular sheaves. By induction hypothesis, there is then a quasi-isomorphism $B'^\cdot \xrightarrow{\sim} A^\cdot$ and so $B \xrightarrow{\sim} B'^\cdot \xrightarrow{\sim} A^\cdot$ with $A^\cdot$ in $\boldsymbol{A}$, as required.

This completes the proof that $\prod^r K(\boldsymbol{C}) \to K(\boldsymbol{B})$ is split epi up to homotopy.

4.12. We have shown that $\prod^r K(\boldsymbol{C}) \to K(\boldsymbol{B})$ is both split mono and split epi in the homotopy category of spectra. It follows that this map is a homotopy equivalence, with homotopy inverse given by either of the splitting maps.

This completes the proof of 4.1.

5. Extension of perfect complexes, and the proto-localization theorem

5.1. **Theorem** (Proto-localization, cf. 7.4). *Let X be a quasi-compact and quasi-separated scheme. Let $U \subseteq X$ be a quasi-compact open subscheme, and set $Y = X - U$, a closed subspace of X. Let Z be a closed subspace of X with $X - Z$ quasi-compact. Then aside from possible failure of surjectivity for $K_0(X) \to K_0(U)$ and $K_0(X$ on $Z) \to K_0(U$ on $U \cap Z)$, the usual maps give homotopy fibre sequences*

$$K(X \text{ on } Y) \to K(X) \to K(U) \tag{5.1.1}$$

$$K(X \text{ on } Y \cap Z) \to K(X \text{ on } Z) \to K(U \text{ on } U \cap Z). \tag{5.1.2}$$

That is, (5.1.1) *becomes a homotopy fibre sequence of spectra after $K(U)$ is replaced by the covering spectrum $K(U)^\sim$ with $\pi_i K(U)^\sim = K_i(U)$ for $i > 0$, and $\pi_0 K(U)^\sim =$ image $K_0(X) \to K_0(U)$. Similarly for* (5.1.2).

Proof. The proof will occupy 5.2 - 5.6. First, we make some remarks.

5.1.3. Later in Section 6, we will use 5.1 and 4.1 to define non-connective deloopings $K^B(X$ on $Y)$, etc., with $\pi_n K^B(X$ on $Y) = K_n(X$ on $Y)$ for $n \geq 0$, but possibly $\neq 0$ for $n < 0$. In 7.4, we will show that K^B analogs of (5.1.1) and (5.1.2) are homotopy fibre sequences without any covering spectrum fudge. This will be the mature localization theorem.

5.1.4. The fibre terms $K(X$ on $Y)$, $K(X$ on $Y \cap Z)$ have explicit descriptions in terms of complexes, and satisfy excision 3.19. This makes Theorem 5.1 very useful. Exercise 5.7 will give an alternate description of the fibre term comparable to the traditional kind of fibre terms in the very special cases where some form of localization theorem has been previously established.

5.1.5. Unlike our results in Sections 1 - 4, which have been at most minor improvements on the work of Grothendieck, Illusie, Berthelot, Quillen, and Waldhausen, this result is a revolutionary advance. Quillen proved a localization theorem for the G-theory of noetherian schemes [Q1], which is the most important tool in that subject. For K-theory Quillen proved [Gr1] a localization homotopy fibre sequence similar to (5.1.1) only in the case where U is affine, and where Y is a divisor defined by a section s of a line bundle $\mathcal{L}$ which is a monomorphism $s : \mathcal{O} \rightarrowtail \mathcal{L}$, i.e., such that the local equation $t = 0$ of Y has t a non-zero-divisor in $\mathcal{O}_X$. These restrictions greatly hinder applications of the result even to the K-theory of rings, and have obstructed the development of K-theory to any level approaching that of G-theory. For noteworthy previous attempts to break out of these restrictions, see the work of Gersten [Ge] Sections 5, 6, 7; Levine [L1]; and Weibel [We4], [We5].

5.2. We begin the proof of 5.1. Let $K(X \text{ on } Z \text{ for } U)$ be the K-theory spectrum of the complicial biWaldhausen category of those perfect complexes on X which are acyclic on $X - Z$, but where now the weak equivalences are the maps of complexes on X which are quasi-isomorphisms when restricted to U. The open immersion $j : U \to X$ induces an exact functor j^*, and so a map $j^* : K(X \text{ on } Z \text{ for } U) \to K(U \text{ on } Z \cap U)$.

The Waldhausen Localization Theorem 1.8.2 immediately gives a homotopy fibre sequence (5.2.1), after we note that a complex acyclic on $X - Z$ and quasi-isomorphic to 0 on $U = X - Y$ is acyclic on $(X - Z) \cup (X - Y) = X - Z \cap Y$.

$$K(X \text{ on } Z \cap Y) \to K(X \text{ on } Z) \to K(X \text{ on } Z \text{ for } U). \tag{5.2.1}$$

This reduces the proof of (5.1.2) to showing that $j^* : K(X \text{ on } Z \text{ for } U) \to K(U \text{ on } U \cap Z)$ induces an isomorphism on homotopy groups π_i for $i > 0$, and induces a monomorphism on π_0. Cofinality 1.10.1 reduces this to showing that j^* is a homotopy equivalence of $K(X \text{ on } Z \text{ for } U)$ to the K-theory spectrum of the biWaldhausen category of those perfect complexes on U which are acyclic on $U - U \cap Z$, and which have Euler characteristic in the image of $K_0(X \text{ on } Z) \to K_0(U \text{ on } U \cap Z)$. By the Approximation Theorem in form 1.9.8, this in turn reduces to showing that j^* induces an equivalence of the derived categories of the two complicial biWaldhausen categories. This equivalence follows from the results 5.2.2., 5.2.3, and 5.2.4, below.

5.2.2. **Key Proposition.** *Adopt the hypotheses and notations of 5.1. Then:*

(a) A perfect complex $F^\cdot$ on U is isomorphic in the derived category $D(\mathcal{O}_U\text{-Mod})$ to the restriction $j^*E^\cdot$ of some perfect complex $E^\cdot$ on X, if and only if the class $[F^\cdot]$ in $K_0(U)$ is in the image of $K_0(X)$.

(b) More generally, for a perfect complex $F^\cdot$ on U which is acyclic on $U - U \cap Z$, there exists a perfect complex $E^\cdot$ on X which is acyclic on $X - Z$ and an isomorphism between $F^\cdot$ and $j^*E^\cdot$ in the derived category $D(\mathcal{O}_U\text{-Mod})$ if and only if the class $[F^\cdot]$ in $K_0(U \text{ on } U \cap Z)$ is in the image of $K_0(X \text{ on } Z)$.

5.2.3. **Proposition.** *Adopt the hypotheses and notations of* 5.1. *Then:*

(a) For any two perfect complexes $E^\cdot$ and $E'^\cdot$ on X, and for any map $b : j^*E^\cdot \to j^*E'^\cdot$ in the derived category on U, $D(\mathcal{O}_U\text{-Mod})$, there is a

perfect complex E''' on X and maps $a : E''' \to E^{\cdot}$, $a' : E''' \to E'^{\cdot}$ in the derived category on X, $D(\mathcal{O}_X$-Mod), such that j^*a is an isomorphism in $D(\mathcal{O}_U$-Mod) and $b \cdot j^*a = j^*a'$.

(b) Moreover, if in (a) *both $E^{\cdot}$ and $E'^{\cdot}$ are acyclic on $X - Z$, then E''' may also be taken to be acyclic on $X - Z$.*

5.2.4. **Proposition.** *Adopt the hypotheses and notations of* 5.1. *Then:*

(a) Let $E^{\cdot}$ and $E'^{\cdot}$ be two perfect complexes on X. Suppose that a, $b : E^{\cdot} \to E'^{\cdot}$ are two maps in the derived category on X, $D(\mathcal{O}_X$-Mod), such that $j^*a = j^*b$ in $D(\mathcal{O}_U$-Mod) on U. Then there is a perfect complex E''' on X, and a map $c : E''' \to E^{\cdot}$ in $D(\mathcal{O}_X$-Mod), such that $ac = bc$, and such that $j^*(c)$ is an isomorphism in $D(\mathcal{O}_U$-Mod) on U.

(b) Moreover, if in (a) *$E^{\cdot}$ and $E'^{\cdot}$ are acyclic on $X - Z$, then E''' may also be taken to be acyclic on $X - Z$.*

5.2.5. These three propositions will be proved in 5.2.6 - 5.6. This will complete the proof of 5.1.

5.2.6. We begin by showing that 5.2.4 in fact follows from 5.2.3. First we note that to prove 5.2.4 it suffices to show that if $j^*(a - b) = 0$ then there is a $c : E''' \to E^{\cdot}$ as in 5.2.4 with $(a - b)c = 0$. Thus we reduce 5.2.4 to the special case where $b = 0$.

In this case, let $F^{\cdot}$ be the homotopy fibre of $a : E^{\cdot} \to E'^{\cdot}$ with $f : F^{\cdot} \to E^{\cdot}$ the canonical map. Then $j^*F^{\cdot} \to j^*E^{\cdot} \to j^*E'^{\cdot}$ is a homotopy fibre sequence on U. As $j^*a = 0$ by hypothesis, the long exact sequence of Mor($j^*E[*]$,) resulting from this fibre sequence shows that there is a map $g : j^*E^{\cdot} \to j^*F^{\cdot}$ in $D(\mathcal{O}_U$-Mod) such that $j^*f \cdot g = 1$. Then granting 5.2.3, there is a perfect E''' on X and maps $d : E''' \to F^{\cdot}$, $d' : E''' \to E^{\cdot}$ such that $j^*(d')$ is an isomorphism in $D(\mathcal{O}_U$-Mod) and $g \cdot j^*(d') = j^*(d)$. If $E^{\cdot}$ and $E'^{\cdot}$, hence also $F^{\cdot}$, are acyclic on $X - Z$, then E''' can be taken to be acyclic on $X - Z$ by 5.2.3(b). Now $f \cdot d : E''' \to E^{\cdot}$ has $a \cdot f \cdot d = 0$ as $a \cdot f = 0$. Also $j^*(f \cdot d) = j^*f \cdot g \cdot j^*(d') = j^*(d')$ is an isomorphism in $D(\mathcal{O}_U$-Mod). Thus E''' and $f \cdot d$ satisfy the conclusion of 5.2.4.

This proves that 5.2.4(a) follows from 5.2.3(a) and 5.2.4(b), from 5.2.3(b).

5.3. To prove Proposition 5.2.2 and 5.2.3, we begin by reducing them to the case where X is noetherian of finite Krull dimension. (This will be convenient for studying extensions of morphisms in the derived categories as the coherator and injectives in Qcoh(X) work well for such X, (cf. Appendix B)).

For X quasi-compact and quasi-separated as in 5.1, by C.9 with $\Lambda = \mathbb{Z}$ we have $X = \varprojlim X_\alpha$ for $\{X_\alpha\}$ an inverse system of schemes of finite type over Spec($\mathbb{Z}$), in which the bonding maps $X_\beta \to X_\alpha$ are affine. Then the

X_α are noetherian schemes of finite Krull dimension. As $X - Y = U$ and $X - Z$ are quasi-compact open, by C.2 we may pass to a cofinal subsystem to get opens $X_\alpha - Y_\alpha = U_\alpha$, $X_\alpha - Z_\alpha$, with $\varprojlim U_\alpha = U$, etc. Now 3.20.1 shows that the various derived categories of perfect complexes on X, on U, on X and acyclic on U, etc., are the direct colimits of the corresponding systems of derived categories of perfect complexes on the X_α, on the U_α, on the X_α and acyclic on U_α. Also $K_0(X) = \varinjlim K_0(X_\alpha)$, by 3.20.2. So it will suffice to prove 5.2.2 and 5.2.3 for each of the noetherian X_α. If X had an ample of line bundles, we may assume the X_α do, by C.9.

5.4. We next turn to the case where X has an ample family of line bundles, and study the extension of morphisms, i.e., 5.2.3.

5.4.1. **Lemma.** *Let X be quasi-compact and quasi-separated Let $\mathcal{L}$ be a line bundle on X, $s \in \Gamma(X, \mathcal{L})$ a global section, and let $U = X_s$ be the non-vanishing locus, with $j : U \to X$ the open immersion.*

Let $E^\cdot$ be a strict perfect complex on X, and $F^\cdot$ a complex of quasi-coherent $\mathcal{O}_X$-modules. Then

(a) For any strict map of the restrictions of the complexes to U, $f : j^*E^\cdot \to j^*F^\cdot$, there exists an integer $k > 0$ and a strict map of complexes on X, $\tilde{f} : E^\cdot \otimes \mathcal{L}^{-k} \to F^\cdot$, such that $j^*\tilde{f} = f \cdot s^k$.

(b) Given any two strict maps of the complexes on X, $\tilde{f}_1$, $\tilde{f}_2 : E^\cdot \to F^\cdot$, such that $j^*\tilde{f}_1 = j^*\tilde{f}_2$ on U, there is an $n > 0$ such that $s^n\tilde{f}_1 = s^n\tilde{f}_2 : E^\cdot \otimes \mathcal{L}^{-n} \to F^\cdot$.

(c) Given any two strict maps $\tilde{f}_1$, $\tilde{f}_2 : E^\cdot \to F^\cdot$ on X, such that $j^*\tilde{f}_1 \simeq j^*\tilde{f}_2$ are chain homotopic on U, there is an $m > 0$ such that $s^m\tilde{f}_1$ and $s^m\tilde{f}_2$ are chain homotopic as maps $E^\cdot \otimes \mathcal{L}^{-m} \to F^\cdot$ on X.

(Note in (a) and (b), equality of maps means strict equality in the category of chain complexes, not in the derived category.)

Proof. Under the adjointness of j^* and j_* on categories of complexes, a map $f : j^*E^\cdot \to j^*F^\cdot$ corresponds to a map $E^\cdot \to j_*j^*F^\cdot$. But $j_*j^*F^\cdot$ is $F^\cdot[1/s]$, i.e., is the direct colimit

$$(5.4.1.1) \quad F^\cdot[1/s] = \varinjlim \left(F^\cdot \xrightarrow{s} F^\cdot \otimes \mathcal{L} \xrightarrow{s} F^\cdot \otimes \mathcal{L}^2 \xrightarrow{s} F^\cdot \otimes \mathcal{L}^3 \to \dots\right).$$

Indeed, there is an obvious map of this colimit into $j_*j^*F^\cdot$, which is easily seen to be an isomorphism by looking at open affines in X.

The complex $E^\cdot$ is finitely presented, as it is even a finite complex of vector bundles. Hence the mapping complex $\mathrm{Hom}^\cdot(E^\cdot, \ )$ preserves direct colimits (cf. (2.4.1.4)). Thus we have isomorphisms of mapping complexes:

(5.4.1.2)

$$\begin{aligned}\mathrm{Hom}^{\cdot}(j^*E^{\cdot}, j^*F^{\cdot}) &\cong \mathrm{Hom}^{\cdot}(E^{\cdot}, j_*j^*F^{\cdot}) \cong \mathrm{Hom}^{\cdot}\left(E^{\cdot}, \varinjlim F^{\cdot} \otimes \mathcal{L}^k\right) \\ &\cong \varinjlim_k \mathrm{Hom}^{\cdot}(E^{\cdot}, F^{\cdot} \otimes \mathcal{L}^k) \cong \varinjlim_k \mathrm{Hom}^{\cdot}(E^{\cdot} \otimes \mathcal{L}^{-k}, F^{\cdot}).\end{aligned}$$

As the cycle group $Z^0\mathrm{Hom}^{\cdot}$ is the group of chain maps of complexes and the cohomology group $H^0\mathrm{Hom}^{\cdot}$ is the group of chain homotopy classes of maps, applying these functors to (5.4.1.2) yields 5.4.1(a), (b), and (c). Compare [EGA] I 6.8.

5.4.2. **Proposition.** *Let X be noetherian, and have an ample family of line bundles. Let $j : U \to X$ be an open immersion.*

Let $E^{\cdot}$ be a perfect complex on X, and $F^{\cdot}$ a complex on X with quasi-coherent cohomology and which is cohomologically bounded below (i.e., $F^{\cdot} \in D^+(\mathcal{O}_X\text{-Mod})_{qc}$).

Let $a : j^*E^{\cdot} \to j^*F^{\cdot}$ be a map in the derived category of U, $D(\mathcal{O}_U\text{-Mod})$.

Then there is a perfect complex $E'^{\cdot}$ on X, a map $b : E'^{\cdot} \to F^{\cdot}$ in the derived category of X, and a map $c : E'^{\cdot} \to E^{\cdot}$ in the derived category of X such that $j^*(c)$ is an isomorphism in the derived category of U and such that $a \cdot j^*(c) = j^*(b)$ there. Moreover, if $E^{\cdot}$ is acyclic off a closed subspace $Z \subseteq X$, $E'^{\cdot}$ may be chosen to be acyclic there.

Proof. We note that the open U is quasi-compact, as X is noetherian. As X has an ample family of line bundles, there is a finite set of line bundles $\mathcal{L}_i$, $i = 1, 2, \ldots n$, and sections $s_i \in \Gamma(X, \mathcal{L}_i)$ such that $X_{s_i} \subseteq U$ and $U = \cup_{i=1}^n X_{s_i}$. This follows from 2.1.1(b), letting the $\mathcal{L}_i$ be tensor powers of line bundles in the ample family. Note $X_{s_i} = U_{s_i}$.

As X has an ample family, we may choose a strict perfect representative $E^{\cdot}$ of the quasi-isomorphism class of the original $E^{\cdot}$, by 2.3.1(d). As X is noetherian, the coherator B.16 allows us to choose a representative $F^{\cdot}$ of its quasi-isomorphism class which is a complex of quasi-coherent $\mathcal{O}_X$-modules. As $F^{\cdot}$ is cohomologically bounded below, we may then replace it by a quasi-isomorphic complex of injective objects in the category of quasi-coherent $\mathcal{O}_X$-modules (B.3). As X is noetherian these injectives in $\mathrm{Qcoh}(X)$ are still injective in the category of all $\mathcal{O}_X$-modules, and j^*F is a complex of injectives in $\mathrm{Qcoh}(U)$, as we see by B.4 and B.5. Henceforth, we use these representatives for $E^{\cdot}$ and $F^{\cdot}$.

As $j^*F^{\cdot}$ is a complex of injectives, the map in the derived category from $j^*E^{\cdot}$ to $j^*F^{\cdot}$ is represented by a strict map of complexes $a : j^*E^{\cdot} \to j^*F^{\cdot}$.

By 5.4.1(a), there is a positive integer k such that for $i = 1, 2, \ldots, n$, the map $s_i^k a | U_{s_i} = X_{s_i}$ extends to a strict map of complexes on X, b_i : $E^{\cdot} \otimes \mathcal{L}_i^{-k} \to F^{\cdot}$. On $X_{s_i} \cap X_{s_j} = X_{s_i s_j}$, we have $s_j^k b_i = s_j^k s_i^k a = s_i^k b_j$.

Hence by 5.4.1(b) there is an $m > 0$ such that $s_i^m s_j^m s_j^k b_i = s_i^m s_j^m s_i^k b_j$ on X. Taking m large enough to work for all pairs (i,j), we get $s_j^{m+k}(s_i^m b_i) = s_i^{m+k}(s_j^m b_j)$ on X for all (i,j). Then replacing b_i by $s_i^m b_i$ and k by $m+k$, we may assume that $s_j^k b_i = s_i^k b_j$ on X. We have then $s_i^k a = b_i|U$.

Consider now the Koszul complex of $s_1^k, \ldots, s_n^k$, that is the tensor product of the complexes $\mathcal{L}_i^{-k} \to \mathcal{O}_X$

(5.4.2.1)

$$K\left(s_1^k, \ldots, s_n^k\right) = \bigotimes_{i=1}^{n} \left(\mathcal{L}_i^{-k} \xrightarrow{s_i^k} \mathcal{O}_X\right)$$
$$\|$$
$$\overset{n}{\wedge}\left(\bigoplus_{i=1}^{n} \mathcal{L}_i^{-k}\right) \to \overset{n-1}{\wedge}\left(\bigoplus_{i=1}^{n} \mathcal{L}_i^{-k}\right) \to \ldots \to \overset{1}{\wedge}\left(\bigoplus_{i=1}^{n} \mathcal{L}_i^{-k}\right) \to \mathcal{O}_X$$

Let K^+ be the part of the complex outside of $\mathcal{O}_X$, that is, the part which consists of the $\wedge^p(\oplus \mathcal{L}_i^{-k})$ for $p \geq 1$. Let $K^+ \to \mathcal{O}_X$ be the obvious map, so that $K(s_1^k, \ldots, s_n^k)$ is the mapping cone of $K^+ \to \mathcal{O}_X$. As s_i^k is an isomorphism on X_{s_i}, the complex $K(s_1^k, \ldots, s_n^k)$ is acyclic there because $\mathcal{L}_i^{-k} \to \mathcal{O}_X$ is acyclic there, and so $K^+ \to \mathcal{O}_X$ is a quasi-isomorphism there. Thus $K^+ \to \mathcal{O}_X$ is a quasi-isomorphism and $K(s_1^k, \ldots, s_n^k)$ is acyclic on $U = \cup X_{s_i}$.

Let $E'^{\cdot}$ be $K^+ \otimes E^{\cdot}$, and let $c : E'^{\cdot} \to E^{\cdot}$ be the map induced by tensoring $K^+ \to \mathcal{O}_X$ with $E^{\cdot}$. Then as $E^{\cdot}$ is strict perfect, hence flat, $j^*(c)$ is a quasi-isomorphism on U. As K^+ is strict perfect, hence flat, $K^+ \otimes E^{\cdot}$ is acyclic on $X - Z$ if $E^{\cdot}$ is acyclic there. Also $K^+ \otimes E^{\cdot}$ is strict perfect.

The map $b : E'^{\cdot} = K^+ \otimes E^{\cdot} \to F^{\cdot}$ will be the map induced on total complexes by the map of bicomplexes (5.4.2.2) induced by the $b_i : E^{\cdot} \otimes \mathcal{L}_i^{-k} \to F^{\cdot}$

(5.4.2.2)

$$\begin{array}{ccccccc}
\overset{n}{\wedge}\left(\overset{n}{\oplus}\mathcal{L}_i^{-k}\right) \otimes E^{\cdot} & \longrightarrow \ldots \longrightarrow & \overset{2}{\wedge}\left(\overset{n}{\oplus}\mathcal{L}_i^{-k}\right) \otimes E^{\cdot} & \xrightarrow{\delta} & \left(\overset{n}{\oplus}\mathcal{L}_i^{-k}\right) \otimes E^{\cdot} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \Sigma b_i} \\
0 & \longrightarrow \ldots \longrightarrow & 0 & \longrightarrow & F^{\cdot}
\end{array}$$

To check this is a map of bicomplexes, we need to see that $\Sigma b_i \circ \delta = 0$ on $\wedge^2(\oplus \mathcal{L}_i^{-k}) \otimes E^{\cdot} = \oplus_{i<j} \mathcal{L}_i^{-k} \otimes \mathcal{L}_j^{-k} \otimes E^{\cdot}$. But on the factor $\mathcal{L}_i^{-k} \otimes \mathcal{L}_j^{-k} \otimes E^{\cdot}$, $(\Sigma b_i) \circ \delta = s_i^k b_j - s_j^k b_i = 0$.

We check that $j^* b = a \cdot j^* c$ by restricting to the summands $\mathcal{L}_i^{-k} \otimes E^{\cdot}$ of $(\overset{n}{\oplus}\mathcal{L}_i^{-k}) \otimes E^{\cdot}$ where the equation reduces to the valid $j^* b_i = a \cdot j^* s_i^k$.

Thus taking this $E'^{\cdot}$, (b) and (c) we have proved 5.4.2.

5.4.3. Then specialized to the case where $F^{\cdot}$ is perfect, 5.4.2 yields 5.2.3 and hence 5.2.4 for X noetherian with an ample family of line bundles.

5.4.4. By the reduction of 5.3, we conclude that 5.2.3 and 5.2.4 hold whenever X is a scheme with an ample family of line bundles (hence quasi-compact and quasi-separated (2.1.1)).

5.5. We now study the extension of perfect complexes when X has an ample family of line bundles. We first consider the case $X = Z$ of unrestrained support of 5.2.2(a).

5.5.1. **Lemma.** *Let X be a scheme with an ample family of line bundles, a fortiori a quasi-compact and quasi-separated scheme. Let $j : U \to X$ be an open immersion with U quasi-compact. Then for every perfect complex $F^{\cdot}$ on U, there exists a perfect complex $E^{\cdot}$ on X such that $F^{\cdot}$ is isomorphic to a summand of $j^*E^{\cdot}$ in the derived category $D(\mathcal{O}_U\text{-}Mod)$.*

Proof. Consider $Rj_*F^{\cdot}$ on X. This complex is cohomologically bounded below with quasi-coherent cohomology (B.6), and so by 2.3.3 is quasi-isomorphic to a colimit of a directed system of strict perfect complexes $E^{\cdot}_\alpha$,

$$\varinjlim_{\alpha} E^{\cdot}_\alpha \simeq Rj_*F^{\cdot}. \tag{5.5.1.1}$$

We consider the induced isomorphism in $D^+(\mathcal{O}_U\text{-Mod})$

$$\varinjlim_{\alpha} j^*E^{\cdot}_\alpha = j^*\left(\varinjlim E^{\cdot}_\alpha\right) \simeq j^*Rj_*(F^{\cdot}) \simeq F^{\cdot}. \tag{5.5.1.2}$$

By 2.4.1(f), the map (5.5.1.3) is an isomorphism

$$\varinjlim_{\alpha} \ \mathrm{Mor}_{D(U)}\left(F^{\cdot},\ j^*E^{\cdot}_\alpha\right) \cong \ \mathrm{Mor}_{D(U)}\left(F^{\cdot},\ \varinjlim j^*E^{\cdot}_\alpha\right). \tag{5.5.1.3}$$

Thus in $D(\mathcal{O}_U\text{-Mod})$ the inverse isomorphism to (5.5.1.2) must factor through some $j^*E^{\cdot}_\alpha$. Thus $F^{\cdot}$ is a summand of $j^*E^{\cdot}_\alpha$ in $D(\mathcal{O}_U\text{-Mod})$, proving the lemma.

5.5.2. The idea of 5.5.1 is that perfect complexes are finitely presented objects in the derived category 2.4.4, and so we may adapt Grothendieck's method of extending finitely presented sheaves ([EGA] I 6.9.1), as suggested by the Trobaugh simulacrum. While this adaptation does not allow us to extend all perfect complexes, it does lead quickly to the determination of which perfect complexes do extend.

Despite the flagrant triviality of the proof of 5.5.1, this result is the key point in the paper.

5.5.3. **Lemma**. *Let X have an ample family of line bundles. Let $j : U \to X$ be an open immersion with U quasi-compact. Suppose $F_1^{\cdot} \to F_2^{\cdot} \to F_3^{\cdot}$ is a homotopy fibre sequence in $D(\mathcal{O}_U\text{-Mod})$, i.e., two sides of a distinguished triangle. Suppose the three $F_i^{\cdot}$ are perfect complexes on U, and that two of the three are isomorphic in $D(\mathcal{O}_U\text{-Mod})$ to the restrictions of perfect complexes on X. Then the third is also isomorphic in $D(\mathcal{O}_U\text{-Mod})$ to the restriction of a perfect complex X.*

Proof. By "rotating the triangle," we see that $F_2^{\cdot} \to F_3^{\cdot} \to F_1^{\cdot}[1]$ and $F_3^{\cdot} \to F_1^{\cdot}[1] \to F_2^{\cdot}[1]$ are also homotopy fibre sequences of perfect complexes. Thus we reduce to the case where $F_1^{\cdot}$ and $F_2^{\cdot}$ are quasi-isomorphic to j^* of perfect complexes $E_1^{\cdot}$ and $E_2^{\cdot}$ on X. By 5.4.4, after replacing E_1 with a new perfect complex whose $j^*E_1^{\cdot}$ is quasi-isomorphic to the old $j^*E_1^{\cdot} \simeq F_1^{\cdot}$, we may assume that $F_1 \to F_2$ is $j^*(e)$ of a map $E_1 \to E_2$ in $D(\mathcal{O}_X\text{-Mod})$. Then the mapping cone cone(e) is perfect on X, and there are isomorphisms in $D(\mathcal{O}_U\text{-Mod})$, $j^*(\text{cone}(e)) \simeq \text{cone}(j^*e) \simeq \text{cone}(F_1 \to F_2) \simeq F_3$, as required.

5.5.4. **Proposition**. *Let X be a scheme with an ample family of line bundles. Let $j : U \to X$ be an open immersion with U quasi-compact.*

Then a perfect complex $F^{\cdot}$ on U is quasi-isomorphic to the restriction to U of some perfect complex on X if and only if the class $[F^{\cdot}]$ in $K_0(U)$ is in the image of $j^* : K_0(X) \to K_0(U)$.

Proof. This will follow from 5.5.1 and 5.5.3 by a cofinality trick of Grayson, (cf. [Gr3] Section 1).

Let π be presented as the free abelian monoid generated by the quasi-isomorphism classes $\langle F \rangle$ of perfect complexes on U, modulo the relations

(5.5.4.1) $$\langle F_1 \rangle + \langle F_2 \rangle = \langle F_1 \oplus F_2 \rangle$$

(5.5.4.2) $$\langle F \rangle = 0 \text{ if } F \simeq j^*E^{\cdot} \text{ for some } E^{\cdot} \text{ perfect on } X.$$

By 5.5.1, for each F there is an F' such that $F \oplus F'$ is quasi-isomorphic to the restriction of a perfect complex on X. Then $\langle F \rangle + \langle F' \rangle = \langle F \oplus F' \rangle = 0$ and π is a group.

Suppose $\langle G \rangle = 0$ in π. This means that there is an F such that $G \oplus F$ is quasi-isomorphic to $H \oplus F$ for some H quasi-isomorphic to the restriction of a perfect complex on X. Let F' be an "inverse to F" as above. Then $G \oplus F \oplus F' \simeq H \oplus F \oplus F'$, and both H and $F \oplus F'$, hence $G \oplus F \oplus F' \simeq H \oplus (F \oplus F')$ extend to perfect complexes on X. Then by 5.5.3 applied to $F \oplus F' \to G \oplus F \oplus F' \to G$, we see that G extends to a perfect complex on X. Thus $\langle G \rangle = 0$ in π iff G is quasi-isomorphic to $j^*E^{\cdot}$ for some perfect complex $E^{\cdot}$ on X.

Hence it remains only to show that π is isomorphic to $K_0(U)/\text{ im } K_0(X)$. Comparing the presentation of π by (5.5.4.1) and (5.5.4.2) with the presentation of $K_0(U)/\text{ im } K_0(X)$ resulting from 1.5.6, we see that it suffices to show that if $F_1 \to F_2 \to F_3$ is a homotopy fibre sequence of perfect complexes on U, then $\langle F_2\rangle = \langle F_1\rangle + \langle F_3\rangle$ in π.

Let F_1', F_3' be such that $F_1 \oplus F_1'$ and $F_3 \oplus F_3'$ extend to perfect complexes on X. Thus $\langle F_1'\rangle = -\langle F_1\rangle$ and $\langle F_3'\rangle = -\langle F_3\rangle$ in π. There is a homotopy fibre sequence $F_1 \oplus F_1' \to F_2 \oplus F_1' \oplus F_3' \to F_3 \oplus F_3'$, obtained by adding $F_1' \to F_1' \to 0$ and $0 \to F_3' \to F_3'$ to the given $F_1 \to F_2 \to F_3$. By 5.5.3, we see that $F_2 \oplus F_1' \oplus F_3'$ extends to a perfect complex on X, as $F_1 \oplus F_1'$ and $F_3 \oplus F_3'$ do. Hence in π, $0 = \langle F_2 \oplus F_1' \oplus F_3''\rangle = \langle F_2\rangle + \langle F_1'\rangle + \langle F_3'\rangle = \langle F_2\rangle - \langle F_1\rangle - \langle F_3\rangle$, as required.

5.5.5. **Proposition.** *Let X have an ample family of line bundles. Let $j : U \to X$ be an open immersion with U quasi-compact. Let $Z \subseteq X$ be a closed subspace with $X - Z$ quasi-compact.*

Then for a perfect complex $F^\cdot$ on U which is acyclic on $U - U \cap Z$, there exists a perfect complex $E^\cdot$ on X which is acyclic on $X - Z$ and is such that j^*E is isomorphic to $F^\cdot$ in $D(\mathcal{O}_U\text{-Mod})$, if and only if the class $[F^\cdot]$ in $K_0(U \text{ on } U \cap Z)$ is in the image of $K_0(X \text{ on } Z)$.

Proof. The "only if" part is trivial.

To prove the "if" direction, we suppose that $[F^\cdot]$ is the image of a class in $K_0(X \text{ on } Z)$. Let $F'^\cdot = k!F^\cdot$ be the extension of $F^\cdot$ by 0 along the open immersion $k : U \to U \cup X - Z$. Recall that the functor $k!$ is exact and is left adjoint to the exact k^* ([SGA 4] IV 11.3.1). As $F^\cdot$ is acyclic on $U - U \cap Z$, $F'^\cdot$ is acyclic, hence perfect on $X - Z$. The restriction $k^*F'^\cdot$ of $F'^\cdot$ to U is isomorphic to $F^\cdot$, hence $F'^\cdot$ is perfect on U. Thus $F'^\cdot$ is perfect on $U \cup X - Z$.

Now we consider the commutative diagram of K_0's.

(5.5.5.1)
$$
\begin{array}{ccc}
K_0(X) & \longrightarrow & K_0(U \cup (X-Z)) \\
\uparrow & & \uparrow \\
K_0(X \text{ on } Z) & \longrightarrow & K_0(U \cup (X-Z) \text{ on } U \cap Z) \\
& \searrow & \downarrow k^* \cong \\
& & K_0(U \text{ on } U \cap Z)
\end{array}
$$

The map k^* is an isomorphism $K_0(U \cup (X-Z) \text{ on } U \cap Z) \cong K_0(U \text{ on } U \cap Z)$ by excision 3.19, as U is an open *nbd* of $U \cap Z$ in $U \cup (X - Z)$. The class $[F'^\cdot]$ in $K_0(U \cup (X-Z) \text{ on } Z)$ goes to the class $[F^\cdot]$ in $K_0(U \text{ on } U \cap Z)$ under k^*. Then the hypothesis implies that $[F'^\cdot]$ is the image of a class in $K_0(X \text{ on } Z)$. It follows that the class $[F'^\cdot]$ in $K_0(U \cup X - Z)$ is the image of a class in $K_0(X)$.

Then by 5.5.4 there is a perfect complex $E^\cdot$ on X such that $E^\cdot|U \cup (X - Z)$ is quasi-isomorphic to $F'^\cdot$. Then the restriction to U, $j^*E^\cdot$ is quasi-isomorphic to $F'^\cdot|U \simeq F^\cdot$, and $E^\cdot|X - Z \simeq F'^\cdot|X - Z$ is acyclic. This proves the proposition.

5.5.6. **Corollary.** *If X has an ample family of line bundles (so a fortiori, is quasi-compact and quasi-separated), then Proposition 5.2.2, 5.2.3, 5.2.4, and Theorem 5.1 are true for X.*

Proof. 5.5.5, 5.5.4, 5.4.4, 5.2.

5.6. We now proceed to remove the hypothesis of an ample family of line bundles, using the techniques of 3.20.4-6.

5.6.1. **Lemma.** *Let X be a quasi-compact and quasi-separated scheme. Let $j : U \to X$ be an open immersion with U quasi-compact. Let Z be a closed subspace of X with $X - Z$ quasi-compact.*

Let V be an open subscheme of X, such that V has an ample family of line bundles. Suppose $X = U \cup V$. Then

(a) *Suppose $F^\cdot$ is a perfect complex on U, and that $F^\cdot$ is acyclic on $U - U \cap Z$. Suppose the class $[F^\cdot|U \cap V]$ in $K_0(U \cap V$ on $Z \cap U \cap V)$ is in the image of $K_0(V$ on $V \cap Z)$. Then there is a perfect complex $E^\cdot$ on X, such that $E^\cdot$ is acyclic on $X - Z$ and $j^*E^\cdot$ is quasi-isomorphic to $F^\cdot$ on U.*

(b) *Suppose $E_1^\cdot$ and $E_2^\cdot$ are perfect complexes on X which are acyclic on $X - Z$. Suppose $a : j^*E_1^\cdot \to j^*E_2^\cdot$ is a map in the derived category on U, $D(\mathcal{O}_U$-Mod). Then there is a perfect complex $E'^\cdot$ on X which is acyclic on $X - Z$, and maps $c : E'^\cdot \to E_1^\cdot$, $b : E'^\cdot \to E_2^\cdot$ in $D(\mathcal{O}_X$-Mod) such that $j^*(c)$ is an isomorphism in $D(\mathcal{O}_U$-Mod) and $a \cdot j^*(c) = b$ there.*

(c) *Moreover, in* (b) *we may choose $E'^\cdot$ so that $[E'^\cdot] = [E_1^\cdot]$ in $K_0(X$ on $Z)$.*

(d) *The conclusion* (5.1.2) *of Theorem* 5.1 *is valid for this X, U, and Z. In particular, there is an induced exact sequence of homotopy groups for $Y = X - U$:*

$$(5.6.1.1) \quad \cdots \to K_0(X \text{ on } Z \cap Y) \to K_0(X \text{ on } Z) \to K_0(U \text{ on } Z \cap U).$$

Proof. First we note that both V and its open subscheme $U \cap V$ are quasi-compact, quasi-separated, and have an ample family of line bundles.

By 5.5.5 applied to the open immersion $j' : U \cap V \to V$, there is a perfect complex $F_V^\cdot$ on V, acyclic on $V - V \cap Z$, and a quasi-isomorphism of $F_V^\cdot|U \cap V$ with $F^\cdot|U \cap V$. By the calculus of fractions, this quasi-isomorphism is represented by data consisting of a complex $G^\cdot$ on $U \cap V$ and strict quasi-isomorphisms

(5.6.1.2) $$F_V^{\cdot}|U \cap V \overset{\sim}{\leftarrow} G^{\cdot} \overset{\sim}{\rightarrow} F^{\cdot}|U \cap V.$$

In the notation of 3.20.4, let $E^{\cdot}$ be the complex on X given by

(5.6.1.3) $$E^{\cdot} = !F_V^{\cdot} \overset{h}{\underset{!G^{\cdot}}{\cup}} !F^{\cdot} = k!F_V^{\cdot} \overset{h}{\underset{\ell !G^{\cdot}}{\cup}} j!F^{\cdot}.$$

Then, as in 3.20.4, $E^{\cdot}|U = j^*E^{\cdot}$ is quasi-isomorphic to $F^{\cdot}$, and $E^{\cdot}|V = k^*E^{\cdot}$ is quasi-isomorphic to $F_V^{\cdot}$. As $X = U \cup V$, this shows $E^{\cdot}$ is perfect on X and acyclic on $X - Z$. This proves (a).

To prove (b), we consider $a|U \cap V : E_1^{\cdot}|U \cap V \to E_2^{\cdot}|U \cap V$ in $D(\mathcal{O}_{U \cap V}$-Mod). By 5.4.4 applied to $U \cap V \to V$, there is a perfect complex $E_V'^{\cdot}$ on V, which is acyclic on $V - V \cap Z$, and there are maps in $D(\mathcal{O}_V$-Mod) $b_V : E_V'^{\cdot} \to E_2|V$, $c_V : E_V'^{\cdot} \to E_1|V$ such that $c_V|U \cap V$ is an isomorphism in $D(\mathcal{O}_{U\cap V}$-Mod) and $a \cdot c_V|U \cap V = b_V|U \cap V$ there. We choose representatives of the quasi-isomorphism classes $E_1^{\cdot}$ and $E_2^{\cdot}$ among complexes of injective $\mathcal{O}_X$-modules. Then b_V, c_V, and a are represented by strict maps of complexes, and $a \cdot c_V|U \cap V$ is chain homotopic to $b_V|U \cap V$.

Now in the notation of 3.20.4 we set

(5.6.1.4)
$$E'^{\cdot} = !E_V'^{\cdot} \underset{!E_V'^{\cdot}|U\cap V}{\cup} !E_1^{\cdot}|U = C^{\cdot}\left(E_V'^{\cdot},\ E_V'^{\cdot}|U \cap V,\ E_1^{\cdot}|U,\ 1,\ c_V|U \cap V\right).$$

Then $E'^{\cdot}|U$ is quasi-isomorphic to $E_1^{\cdot}|U$ and $E'^{\cdot}|V$ is quasi-isomorphic to $E_V'^{\cdot}$. Thus $E'^{\cdot}$ is perfect and acyclic on $X - Z$.

Let $c : E'^{\cdot} = C^{\cdot}(E_V'^{\cdot}, E_V'^{\cdot}|U \cap V, E_1^{\cdot}|U, 1, c_V|U \cap V) \to E_1^{\cdot}$ be the map induced by $1 : E_1^{\cdot}U \to E_1^{\cdot}|U$ and $c_V : E_V'^{\cdot} \to E_1'^{\cdot}|V$ according to the mapping property of 3.20.4. Then $j^*(c) = c|U$ is a quasi-isomorphism, and in fact is inverse to the canonical quasi-isomorphism $E_1^{\cdot}|U \to E'^{*}|U$.

Let $b : E'^{\cdot} \to E_2^{\cdot}$ be the map induced by $b_V : E_V'^{\cdot} \to E_2^{\cdot}|V$, $a : E_1^{\cdot}|U \to E_2^{\cdot}|U$, and a choice of chain homotopy between $b_V|U \cap V$ and $a \cdot c_V|U \cap V$. Then it is easy to see that $a \cdot c|U = b|U$ in $D(\mathcal{O}_U$-Mod), and are even equal up to chain homotopy of strict maps of complexes, as $E'^{\cdot}|U$ deformation retracts to the summand $E_1^{\cdot}|U$ on which $a \cdot c = b$ reduces to $a \cdot 1 = a$.

This completes the proof of (b).

To prove (c), it suffices to find a new perfect complex on X, which is acyclic on U, and whose class in $K_0(X \text{ on } Z)$ is $[E_1^{\cdot}] - [E'^{\cdot}]$. For then we may replace the old complex $E'^{\cdot}$ by its direct sum with this new perfect complex, and extend the maps b and c to be 0 on this new summand. But clearly the mapping cone of $c : E'^{\cdot} \to E_1$ meets the requirements to be the new summand.

Now (d) follows as 5.6.1(a), (b) proves 5.2.2 and 5.2.3 for this X, U, and Z; and then 5.2 shows that this implies 5.1 for this X, U, and Z.

5.6.2. **Lemma.** *Let X be quasi-compact and quasi-separated. Let $j : U \to X$ be an open immersion with U quasi-compact. Set $Y = X - U$. Let Z be a closed subspace of X such that $X - Z$ is quasi-compact.*

Then 5.2.2, 5.2.3 and 5.1 are true for X, U, Z. That is

(a) *If $F^\cdot$ is a perfect complex on U, acyclic on $U - U \cap Z$, and with its class $[F^\cdot]$ in the image of $K_0(X \text{ on } Z) \to K_0(U \text{ on } U \cap Z)$, then there is a perfect complex $E^\cdot$ on X which is acyclic on $X - Z$ and with $j^* E^\cdot$ quasi-isomorphic to $F^\cdot$ on U.*

(b) *If $E_1^\cdot$ and $E_2^\cdot$ are perfect complexes on X which are acyclic on $X - Z$, and if $a : j^* E_1^\cdot \to j^* E_2^\cdot$ is a map in $D(\mathcal{O}_U\text{-Mod})$, then there is a perfect complex $E'^\cdot$ on X, which is acyclic on $X - Z$, and has $[E'^\cdot] = [E_1^\cdot]$ in $K_0(X \text{ on } Z)$, and there exist maps $b : E'^\cdot \to E_2^\cdot$ and $c : E'^\cdot \to E_1^\cdot$ in $D(\mathcal{O}_X\text{-Mod})$ such that $j^*(c)$ is an isomorphism in $D(\mathcal{O}_U\text{-Mod})$ and $a \cdot j^*(c) = j^*(b)$ there.*

(c) *There is a homotopy fibre sequence as in 5.1. In particular, there is an induced exact sequence*

$$(5.6.2.1) \quad \ldots \to K_0(X \text{ on } Z \cap Y) \to K_0(X \text{ on } Z) \to K_0(U \text{ on } U \cap Z).$$

Proof. There exist a finite set $\{V_1, \ldots, V_n\}$ of open affine subschemes of X such that $X = U \cup V_1 \cup V_2 \cdots \cup V_n$. We prove the results by induction on the number n of affines in the set.

For $n = 1$, the result follows from 5.6.1, as the the affine V_1 has an ample family of line bundles. (Note the hypothesis of 5.6.1(a) is indeed weaker than that of 5.6.2(a).)

To do the induction step, we suppose the results are known for schemes with a set of less than n affine V's. Set $W = U \cap V_1 \cup \cdots \cup V_{n-1}$. Then $X = W \cup V_n$, and the results hold for W.

To do the induction step for (a), we note by induction hypothesis that there is a perfect complex $F^{\sim\cdot}$ on W, acyclic on $W - Z \cap W$, and such that $F^{\sim\cdot}|U$ is quasi-isomorphic to $F^\cdot$. By hypothesis and 1.5.7, there is a perfect complex $H^\cdot$ on X, acyclic on $X - Z$, and such that $[H^\cdot|U] = [F^\cdot]$ in $K_0(U \text{ on } U \cap Z)$. Then $[H^\cdot|W] - [F^{\sim\cdot}]$ in $K_0(W \text{ on } W \cap Z)$ goes to 0 in $K_0(U \text{ on } U \cap Z)$. By 5.6.2(c) for $U \to W$, known by the induction hypothesis, and 1.5.7, there is a $[G^\cdot]$ in $K_0(W \text{ on } W \cap Y \cap Z)$ such that $[F^{\sim\cdot} \oplus G^\cdot] = [H^\cdot|W]$ in $K_0(W \text{ on } W \cap Z)$. This $[G^\cdot]$ is the class of a perfect complex $G^\cdot$ on W which is acyclic on $U \cup (W - Z)$. Then $F^{\sim\cdot} \oplus G^\cdot|U$ is quasi-isomorphic to $F^{\sim\cdot} \oplus 0|U \simeq F^\cdot$. Replacing the old $F^{\sim\cdot}$ by $F^{\sim\cdot} \oplus G^\cdot$, we may assume that $[F^{\sim\cdot}]$ is in the image of $K_0(X \text{ on } Z) \to K_0(W \text{ on } W \cap Z)$. But then $[F^{\sim\cdot}|V_n \cap W]$ is in the image of $K_0(V_n \text{ on } Z \cap V_n) \to K_0(W \cap V_n \text{ on } Z \cap W \cap V_n)$. We now appeal to 5.6.1(a) with $V = V_n$, $U = W$ to get a perfect $E^\cdot$ on X, acyclic on $X - Z$, and a quasi-isomorphism $E^{\cdot}|W \simeq F^{\sim\cdot}$. Thus there is a quasi-isomorphism $E^{\cdot}|U \simeq F^{\sim\cdot}|U \simeq F^{\cdot}$. This proves (a).

To prove (b), we note by the induction hypothesis that there is a perfect complex $E^{\sim\cdot}$ on W, acyclic on $W - W \cap Z$, and maps $b^{\sim} : E^{\sim\cdot} \to E_2^{\cdot}|W$, $c^{\sim} : E^{\sim\cdot} \to E_1^{\cdot}|W$ in $D(\mathcal{O}_W\text{-Mod})$ such that $c^{\sim}|U$ is an isomorphism in $D(\mathcal{O}_U\text{-Mod})$ and $a \cdot c^{\sim}|U = b^{\sim}|U$ there. Also we can arrange that $[E^{\sim\cdot}] = [E_1^{\cdot}|W]$ in $K_0(W \text{ on } W \cap Z)$. Because of the last condition, 5.6.1(a) shows that $E^{\sim\cdot}$ on W extends to a perfect complex on X which is acyclic on $X - Z$. Henceforth, we denote this perfect complex by $E^{\sim\cdot}$, and the old $E^{\sim\cdot}$ on W by $E^{\sim\cdot}|W$. Now by 5.6.1(d), we may arrange that $[E^{\sim\cdot}] = [E_1^{\cdot}]$ in $K_0(X \text{ on } Z)$, adding to $E^{\sim\cdot}$ a perfect complex acyclic on $W \cup (X - Z)$ if necessary.

Applying 5.6.1(b) twice on $X = W \cup V_n$, we get perfect complexes $G^{\cdot}$ and $H^{\cdot}$ on X, acyclic on $X - Z$, and maps $G^{\cdot} \to E^{\sim\cdot}$, $H^{\cdot} \to E^{\sim\cdot}$ in $D(\mathcal{O}_X\text{-Mod})$ that are quasi-isomorphisms on W, and also maps $G^{\cdot} \to E_1^{\cdot}$, $H^{\cdot} \to E_2^{\cdot}$ in $D(\mathcal{O}_X\text{-Mod})$, forming diagrams (5.6.2.2) in $D(\mathcal{O}_X\text{-Mod})$ and (5.6.2.3) in $D(\mathcal{O}_W\text{-Mod})$

$$\begin{array}{ccccc} E'^{\cdot} & \dashrightarrow & H^{\cdot} & \longrightarrow & E_2^{\cdot} \\ \downarrow & & \downarrow & & \\ G^{\cdot} & \longrightarrow & E^{\sim\cdot} & & \\ \downarrow & & & & \\ E_1^{\cdot} & & & & \end{array} \tag{5.6.2.2}$$

$$\begin{array}{ccccc} E'^{\cdot}|W & \overset{\sim}{\dashrightarrow} & H^{\cdot}|W & \longrightarrow & E_2^{\cdot}|W \\ \sim\downarrow & & \downarrow\sim & \nearrow b^{\sim} & \\ G^{\cdot}|W & \overset{\sim}{\longrightarrow} & E^{\sim\cdot}|W & & \\ \downarrow & \swarrow c^{\sim} & & & \\ E_1^{\cdot}|W & & & & \end{array} \tag{5.6.2.3}$$

By 5.6.1(c) we may assume that $[G^{\cdot}] = [H^{\cdot}] = [E^{\sim\cdot}] = [E_1^{\cdot}]$ in $K_0(X$ on $Z)$. We choose representatives of the quasi-isomorphism classes of $E_1^{\cdot}$, $E^{\sim\cdot}$, $E_2^{\cdot}$ among complexes of injective $\mathcal{O}_X$- modules. Then the diagrams (5.6.2.2) and (5.6.2.3) exist as chain homotopy commutative diagrams of strict maps of chain complexes.

Let $E'^{\cdot}$ be the canonical homotopy pullback of $G^{\cdot}$ and $H^{\cdot}$ over $E^{\sim\cdot}$ (1.1.2), so $E'^{\cdot} \to G^{\cdot} \oplus H^{\cdot} \to E^{\sim\cdot}$ is a homotopy fibre sequence. Then $E'^{\cdot}$ is perfect on X and acyclic on $X - Z$. In $K_0(X \text{ on } Z)$, $[E'^{\cdot}] = -[E^{\sim\cdot}] + [G^{\cdot}] + [H^{\cdot}] = -[E^{\sim\cdot}] + [E^{\sim\cdot}] + [E^{\sim\cdot}] = [E^{\sim\cdot}] = [E_1^{\cdot}]$.

Let $c : E'^{\cdot} \to G^{\cdot} \to E_1^{\cdot}$ and $b : E'^{\cdot} \to H^{\cdot} \to E_2^{\cdot}$ be the compositions of the canonical projections of $E'^{\cdot}$ onto $G^{\cdot}$ and $H^{\cdot}$ with the maps $G^{\cdot} \to E_1^{\cdot}$

and $H^\cdot \to E_2^\cdot$. On $U \subseteq W$, the restriction of diagram (5.6.2.3) shows that $c|U :\xrightarrow{\sim} E'^\cdot|U \xrightarrow{\sim} G^\cdot|U \to E_1^\cdot \xrightarrow{\sim} |U$ is a quasi-isomorphism, and that $a \cdot c|U : E'^\cdot|U \to E_1^\cdot|U \to E_2^\cdot|U$ is chain homotopic to $b : E'^\cdot|U \to E_2^\cdot|U$ as required. This proves (b).

Statement (c) for X, U, Z follows from (a) and (b) by 5.2. This completes the induction step and proves 5.6.2.

5.6.3. With 5.6.2, the proofs of 5.2.2, 5.2.3, 5.2.4, and Theorem 5.1 are complete.

5.7. *Exercise* (Optional). Let X be a scheme with an ample family of line bundles. Let $i : Y \to X$ be a regular closed immersion ([SGA 6] VII Section 1) defined by ideal $\mathcal{J}$. Suppose Y has codimension k in X.

Then show that $K(X \text{ on } Y)$ is homotopy equivalent to the Quillen K-theory of the exact category of pseudo-coherent $\mathcal{O}_X$-modules supported on the subspace Y and of Tor-dimension $\le k$ on X.

(a) Begin by noting that $\mathcal{O}_X/\mathcal{J}$ is pseudo-coherent and of Tor-dimension $\le k$ by the Koszul resolution. As $\mathcal{J}^n/\mathcal{J}^{n+1}$ is locally a sum of copies of $\mathcal{O}_X/\mathcal{J}$, $\mathcal{J}^n/\mathcal{J}^{n+1}$ is pseudo-coherent of Tor-dimension $\le k$ ([SGA 6] VII 1.3 iii). By induction, using the exact sequence $0 \to \mathcal{J}^{n+1}/\mathcal{J}^{n+p} \to \mathcal{J}^n/\mathcal{J}^{n+p} \to \mathcal{J}^n/\mathcal{J}^{n+1} \to 0$, show that all $\mathcal{J}^n/\mathcal{J}^{n+p}$ and in particular, all $\mathcal{O}_X/\mathcal{J}^p$ are pseudo-coherent of Tor-dimension $\le k$ (cf. 2.2.13).

(b) Using the functor $R(\text{Qcoh})\, \underline{\Gamma}_Y = \varinjlim \underline{\text{Ext}}^\cdot(\mathcal{O}_X/\mathcal{J}^p, \ )$, calculated using injective resolutions in $\text{Qcoh}(X)$, construct a map between appropriate models of $K(X \text{ on } Y)$ (3.6.2, 3.6.1) and show $K(X \text{ on } Y)$ is homotopy equivalent to the K-theory spectrum of the complicial biWaldhausen category of perfect complexes of quasi-coherent modules that vanish on $X - Y$.

(c) Let $\mathcal{A}$ be the abelian category of quasi-coherent modules that vanish on $X - Y$. Note every submodule of finite type of an object of $\mathcal{A}$ is annihilated by all $\mathcal{J}^p$ for p sufficiently large. Let $\mathcal{D}$ be the additive category generated by all $\mathcal{L}_\alpha^m \otimes \mathcal{O}_X/\mathcal{J}^p$ with $m \in \mathbb{Z}$, $p \ge 1$, and $\mathcal{L}_\alpha$ a line bundle in the ample family. All objects of $\mathcal{D}$ are pseudo-coherent of Tor-dimension $\le k$ over $\mathcal{O}_X$. The inclusion $\mathcal{D} \to \mathcal{A}$ satisfies the hypotheses of 1.9.5. Hence $K(X \text{ on } Y)$ is homotopy equivalent to the K-theory spectrum of the complicial biWaldhausen category of perfect complexes of pseudo-coherent modules of Tor-dimension $\le k$ supported on the subspace Y.

(d) Now appeal to 1.11.7, (or to [W] 1.7.1) to conclude that $K(X \text{ on } Y)$ is homotopy equivalent to the K-theory spectrum of the exact category as claimed.

(e) An $\mathcal{O}_X$-module of Tor-dim $\le k$ is pseudo-coherent iff it has a resolution by vector bundles of length $\le k$ iff it is k-pseudo-coherent. Thus for $k = 1$ with $Y \to X$ a regularly immersed divisor, the exact category is that of finitely presented $\mathcal{O}_X$-modules supported on Y and of Tor-dim ≤ 1. For X with an ample family of line bundles, prove the conjecture of Gersten [Ge1] Section 7, and recover the localization theorem of [Gr1]. Recover the results of [L1].

(f) If U is affine, and open in X, any strict perfect complex on U is trivially a summand of a strict perfect complex of free $\mathcal{O}_U$-modules. If also $U = X_s$ for $s : \mathcal{O} \to \mathcal{L}$, use [EGA] I 6.8.1 to show that any strict bounded complex of free modules on U extends to a strict perfect complex on X. This yields trivially for such U the strict perfect analog of 5.5.1. Combining this with 5.4.1, prove the analog of 5.1 for K^{naive} of such U. Now assuming also that $s : \mathcal{O}_X \rightarrowtail \mathcal{L}$ is a monomorphism, use the ideas of (a) - (d) to recover Quillen's localization theorem of [Gr1] in general, without assuming that X has an ample family of line bundles.

(g) Contemplate Deligne's counterexample in [Ge1] Section 7 to an attempt to generalize the identification of the fibre of $K(X) \to K(X-Y)$ to $K(\ )$ of an exact category as above when $Y \to X$ is not a regular immersion.

6. Bass fundamental theorem and negative K-groups, K^B

6.0. To control the failure of surjectivity of $K_0(X) \to K_0(U)$ and $K_0(X \text{ on } Z) \to K_0(U \text{ on } U \cap Z)$ in the proto-localization Theorem 5.1, one wants to find a non-connective spectrum K^B with K as its -1-connective cover, so that $K^B(X \text{ on } Y) \to K^B(X) \to K^B(U)$ is a homotopy fibre sequence in 5.1 without fudging, and so that the resulting long exact sequence of homotopy groups extends through the K_n^B for $n < 0$. This is done by combining Sections 4 and 5 with ideas of Bass [B] (cf. also Carter [Ca].)

For once it is notationally easier to work first on the level of abelian-group valued functors, and then to produce a spectrum level version.

6.1. **Theorem** (Bass fundamental proto-theorem) (cf. [B] XII 7; [Gr1]; 6.6 below). *Let X be quasi-compact and quasi-separated. Set $X[T] = X \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[T]$. Let Z be a closed subspace of X with $X - Z$ quasi-compact. Then*

(a) *For $n \geq 1$, there is an exact sequence*

(6.1.1)

$$0 \to K_n(X \text{ on } Z) \xrightarrow{(p_1^*, -p_2^*)} K_n(X[T] \text{ on } Z[T]) \oplus K_n\left(X[T^{-1}] \text{ on } Z[T^{-1}]\right)$$
$$\xrightarrow{(j_1^*, j_2^*)} K_n\left(X[T,T^{-1}] \text{ on } Z[T,T^{-1}]\right) \xrightarrow{\partial_T} K_{n-1}(X \text{ on } Z) \to 0.$$

Here p_1^*, p_2^* are induced by the projections $(X[T]) \to X$, etc. and j_1^*, j_2^* are induced by the obvious open immersions $(X[T,T^{-1}]) \to (X[T])$, etc.

The sum of these exact sequences for $n = 1, 2, 3, \dots$ is an exact sequence of graded $K_*(X)$-modules.

(b) For $n \geq 0$, $\partial_T : K_{n+1}(X[T,T^{-1}] \text{ on } Z[T,T^{-1}]) \to K_n(X \text{ on } Z)$ is naturally split by a map h_T of $K_*(X)$ modules. Indeed, cup product with $T \in K_1(\mathbb{Z}[T,T^{-1}])$ splits ∂_T up to a natural automorphism of $K_n(X \text{ on } Z)$.

(c) There is an exact sequence for $n = 0$

$$0 \to K_0(X \text{ on } Z) \xrightarrow{(p_1^*,p_2^*)} K_0(X[T] \text{ on } Z[T]) \oplus K_0\left(X[T^{-1}] \text{ on } Z[T^{-1}]\right)$$
$$\downarrow (j_1^*, j_2^*)$$
$$K_0\left(X[T,T^{-1}] \text{ on } Z[T,T^{-1}]\right)$$

Proof. Consider $\mathbb{P}^1_X$. By 4.1, there is an isomorphism $K_*(\mathbb{P}^1_X \text{ on } \mathbb{P}^1_Z) \cong K_*(X \text{ on } Z) \oplus K_*(X \text{ on } Z)$, where the two summands are $K_*(X \text{ on } Z)[\mathcal{O}]$ and $K_*(X \text{ on } Z)[\mathcal{O}(-1)]$ with respect to the external product $K(X \text{ on } Z) \wedge K(\mathbb{P}^1_{\mathbb{Z}}) \to K(\mathbb{P}^1_X \text{ on } \mathbb{P}^1_Z)$ and with $[\mathcal{O}]$, $[\mathcal{O}(-1)]$ in $K_0(\mathbb{P}^1_{\mathbb{Z}})$. We prefer now to shift to a direct sum decomposition of $K_*(\mathbb{P}^1_X \text{ on } \mathbb{P}^1_Z)$ with basis $\{[\mathcal{O}],\ [\mathcal{O}] - [\mathcal{O}(-1)]\}$.

We consider the cover of $\mathbb{P}^1_X$ by opens $X[T]$ and $X[T^{-1}]$, intersecting in $X[T,T^{-1}]$. The proto-localization Theorem 5.1 shows that the columns in (6.1.2) are homotopy fibre sequences. Here the $K(\ )^\sim$ are covering spectra of the $K(\ )$ to change π_0 suitably, as in 5.1.

$$\begin{array}{ccc} K(\mathbb{P}^1_X \text{ on } (T=0) \cap \mathbb{P}^1_Z) & \xrightarrow{\sim} & K(X[T] \text{ on } (T=0) \cap Z[T]) \\ \downarrow & & \downarrow \\ K(\mathbb{P}^1_X \text{ on } \mathbb{P}^1_Z) & \xrightarrow{k_1^*} & K(X[T] \text{ on } Z[T]) \\ \downarrow k_2^* & & \downarrow j_2^* \\ K(X[T^{-1}] \text{ on } Z[T^{-1}])^\sim & \xrightarrow{j_1^*} & K(X[T,T^{-1}] \text{ on } Z[T,T^{-1}])^\sim \end{array} \tag{6.1.2}$$

As $X[T]$ is an open *nbd* of the locus $(T = 0)$ in $\mathbb{P}^1_X$, excision 3.19 shows that the top horizontal map in (6.1.2) is a homotopy equivalence. Thus the bottom square of (6.1.2) is homotopy cartesian. Thus it yields a long exact Mayer-Vietoris sequence on homotopy groups. Recalling that $\pi_i K(\ )^\sim = \pi_i K(\ )$ for $i > 0$, we see this long exact sequence is:

(6.1.3)

$$\begin{array}{c}
\downarrow \\
K_{n+1}(X[T,T^{-1}] \text{ on } Z[T,T^{-1}]) \\
\downarrow \partial_T \\
K_n(\mathbb{P}^1_X \text{ on } \mathbb{P}^1_Z) \\
\downarrow (k_1^*, -k_2^*) \\
K_n(X[T] \text{ on } Z[T]) \oplus K_n(X[T^{-1}] \text{ on } Z[T^{-1}]) \\
\downarrow (j_1^*, j_2^*) \\
K_n(X[T,T^{-1}] \text{ on } Z[T,T^{-1}]) \\
\downarrow \partial_T \\
\cdots \\
\downarrow \partial_T \\
K_0(\mathbb{P}^1_X \text{ on } \mathbb{P}^1_Z) \\
\downarrow (k_1^*, -k_2^*) \\
K_0(X[T] \text{ on } Z[T]) \oplus K_0(X[T^{-1}] \text{ on } Z[T^{-1}])^\sim
\end{array}$$

Here $K_0(X[T^{-1}] \text{ on } Z[T^{-1}])^\sim$ is some subgroup of $K_0(X[T^{-1}]$ on $Z[T^{-1}])$, namely the image under k_2^* of $K_0(\mathbb{P}^1_X \text{ on } \mathbb{P}^1_Z)$.

Now for $k = k_1$ and k_2, $k^*(\mathcal{O}_{\mathbb{P}}(i)) = \mathcal{O}$. Hence $k^*([\mathcal{O}]) = [\mathcal{O}] = 1$, and $k^*([\mathcal{O}]-[\mathcal{O}(-1)]) = [\mathcal{O}]-[\mathcal{O}] = 0$ in $K_0(\mathbb{Z}[T])$ or $K_0(\mathbb{Z}[T^{-1}])$. Thus on the summand $K_n(X \text{ on } Z)[\mathcal{O}]$ of $K_n(\mathbb{P}^1_X \text{ on } \mathbb{P}^1_Z)$, k_1^* is the map p_1^* induced by the canonical $p_1 : X[T] \to X$. The map p_1 has a section $T = 0$, so p_1^* is a split monomorphism. Similarly, on this summand k_2^* is p_2^*, which is a split monomorphism. On the summand $K_n(X \text{ on } Z)([\mathcal{O}] - [\mathcal{O}(-1)])$, k_1^* and k_2^* are 0. Hence in (6.1.3) the boundary map ∂_T is onto this summand. Thus the long exact sequence (6.1.3) breaks up into short exact sequences, yielding 6.1(a).

To prove (b), it suffices to show that the natural map $\partial_T \cdot (T \cup p^*(\ ))$: $K_n(X \text{ on } Z) \to K_{n+1}(X[T,T^{-1}] \text{ on } Z[T,T^{-1}]) \to K_n(X \text{ on } Z)$ is an automorphism of K_n for $n \geq 0$. For we then define h_T to be $(T \cup p^*(\ ))$ composed with the inverse of this automorphism. By the 5-lemma applied to diagram (6.1.4) where the rows are the exact localization sequences 5.1 for X and $X - Z$ and for $X[T,T^{-1}]$ and $(X - Z)[T,T^{-1}]$, it suffices to prove the map is an automorphism in the absolute case $K_*(S)$ for $S = X$ and for $S = X - Z$. (In diagram (6.1.4) we abbreviate $X[T,T^{-1}]$ as $X[T^{\pm}]$, etc.)

(6.1.4)

$$\begin{array}{ccccccc}
\cdots \longrightarrow & K_{n+1}(X) & \longrightarrow & K_{n+1}(X-Z) & \longrightarrow \\
& \downarrow{\scriptstyle \cup T} & & \downarrow{\scriptstyle \cup T} & \\
\cdots \longrightarrow & K_{n+2}(X[T^{\pm}]) & \longrightarrow & K_{n+2}((X-Z)[X[T^{\pm}]) & \longrightarrow \\
& \downarrow{\scriptstyle \partial_T} & & \downarrow{\scriptstyle \partial_T} & \\
\cdots \longrightarrow & K_{n+1}(X) & \longrightarrow & K_{n+1}(X-Z) & \longrightarrow
\end{array}$$

$$\begin{array}{ccccc}
K_n(X \text{ on } Z) & \longrightarrow & K_n(X) & \longrightarrow \cdots \\
\downarrow{\scriptstyle \cup T} & & \downarrow{\scriptstyle \cup T} & \\
K_{n+1}(X[X[T^{\pm}] \text{ on } Z[T^{\pm}]) & \longrightarrow & K_{n+1}(X[T^{\pm}]) & \longrightarrow \cdots \\
\downarrow{\scriptstyle \partial_T} & & \downarrow{\scriptstyle \partial_T} & \\
K_n(X \text{ on } Z) & \longrightarrow & K_n(X) & \longrightarrow \cdots
\end{array}$$

(We see that the squares of (6.1.4) involving ∂_T commute, as they are derived in a canonical way from a commutative 3×3 diagram of homotopy fibre sequences of spectra.)

To prove that our map is an automorphism in the absolute case $K_*(S)$, it suffices to prove that $\partial_T T = \pm 1 = \pm[\mathcal{O}]$ in $K_0(S)$, as this is a generator of $K_*(S)$ as a free $K_*(S)$ module, and $\partial_T \cdot (T \cup \ ) = \partial_T T \cup (\ )$ is a map of $K_*(S)$ modules. By naturality in S, it suffices to prove this for $S = \operatorname{Spec}(\mathbb{Z})$, i.e., that $\partial_T : K_1(\mathbb{Z}[T, T^{-1}]) \to K_0(\mathbb{Z})$ sends T to ± 1. This is known classically (cf. [B], [Gr1]). (Briefly one has $K_0(\mathbb{Z}) = \mathbb{Z}$, $K_1(\mathbb{Z}[T, T^{-1}]) =$ units in $\mathbb{Z}[T, T^{-1}] \cong \mathbb{Z} \oplus \mathbb{Z}/2$ generated by T and -1, and that ∂_T is onto (examine (6.1.1) and note that $K(\mathbb{Z}) \simeq K(\mathbb{Z}[T]) \simeq K(\mathbb{Z}[T^{-1}])$ as $\mathbb{Z}$ is regular noetherian). So the torsion element $\partial_T(-1)$ of $\mathbb{Z}$ must be 0, and $\partial_T T$ must be a generator ± 1.)

(A careful calculation of $\partial_T(T)$ by building categorical models of everything via [Th3], and considering our choice of signs in forming a Mayer-Vietoris sequence from a homotopy cartesian square yields that in fact $\partial_T T = 1$. In fact, $\partial \cdot (T \cup \ ) = 1$ on $K(X \text{ on } Z)$.) This proves (b).

To prove (c), we consider the diagram:

(6.1.5)

$$\begin{array}{ccccccc}
0 & \to & K_0(X \text{ on } Z) & \to & K_0(X[T] \text{ on } Z[T]) \oplus K_0(X[T^{-1}] \text{ on } Z[T^{-1}]) & \to & K_0(X[T^{\pm}] \text{ on } Z[T^{\pm}]) \\
& & \downarrow h_S & & \downarrow h_S \oplus h_S & & \downarrow h_S \\
0 & \to & K_1(X[S^{\pm}] \text{ on } Z[S^{\pm}]) & \to & \begin{array}{c} K_1(X[S^{\pm}][T] \text{ on } Z[S^{\pm}][T]) \\ \oplus \\ K_1(X[S^{\pm}][T^{-1}] \text{ on } Z[S^{\pm}][T^{-1}]) \end{array} & \to & K_1(X[S^{\pm}][T^{\pm}] \text{ on } Z[S^{\pm}][T^{\pm}]) \\
& & \downarrow \partial_S & & \downarrow \partial_S \oplus \partial_S & & \downarrow \partial_S \\
0 & \to & K_0(X \text{ on } Z) & \to & K_0(X[T] \text{ on } Z[T]) \oplus K_0(X[T^{-1}] \text{ on } Z[T^{-1}]) & \to & K_0(X[T^{\pm}] \text{ on } Z[T^{\pm}])
\end{array}$$

The middle row is exact by (a). By (b), the composite $\partial_S \cdot h_S$ is 1, so the bottom row is a retract of the middle row. Hence the bottom row is also exact, as required.

6.2. **Lemma** (Bass construction) (cf. [B] XII 7). *By descending induction on $k = 1, 0, -1, -2, -3, \ldots$ one may define contravariant abelian group valued functors B_k on the category of quasi-compact and quasi-separated schemes X with a chosen closed subspace Z such that $X - Z$ is quasi-compact. One may also define natural transformations*

$$d_{kT} : B_{k+1}\left(X\left[T, T^{-1}\right] \text{ on } Z\left[T, T^{-1}\right]\right) \to B_k(X \text{ on } Z)$$
$$h_{kT} : B_k(X \text{ on } Z) \to B_{k+1}\left(X\left[T, T^{-1}\right] \text{ on } Z\left[T, T^{-1}\right]\right).$$

These functors and natural transformations are uniquely characterized by the following properties that they satisfy:

(a) *$B_1(X$ on $Z) = K_1(X$ on $Z)$, $B_0(X$ on $Z) = K_0(X$ on $Z)$, and $d_1 = \partial_1$, h_1 are the maps of* 6.1.

(b) *The sequence induced by the natural maps as in* 6.1(a) *is exact:*

$$\begin{aligned}
0 \to B_{k+1}&(X \text{ on } Z) \\
&\to B_{k+1}(X[T] \text{ on } Z[T]) \oplus B_{k+1}\left(X\left[T^{-1}\right] \text{ on } Z\left[T^{-1}\right]\right) \\
&\to B_{k+1}\left(X\left[T, T^{-1}\right] \text{ on } Z\left[T, T^{-1}\right]\right) \xrightarrow{d_{kT}} B_k(X \text{ on } Z) \to 0.
\end{aligned}$$

(c) *$d_{kT} \cdot d_{(k+1)S} = d_{kS} \cdot d_{(k+1)T} : B_{k+2}(X[T, T^{-1}][S, S^{-1}]$ on $Z[T, T^{-1}][S, S^{-1}]) \to B_k(X$ on $Z)$ provided that $k \leq -1$.*

(d) *$d_k \cdot h_k = 1$ on $B_k(X$ on $Z)$.*

(e) *The sequence induced by the natural maps as in* 6.1(a) *is exact*

$$\begin{aligned}
0 \to B_k(X \text{ on } Z) \to B_k(X[T] \text{ on } Z[T]) &\oplus B_k\left(X\left[T^{-1}\right] \text{ on } Z\left[T^{-1}\right]\right) \\
&\to B_k\left(X\left[T, T^{-1}\right] \text{ on } Z\left[T, T^{-1}\right]\right).
\end{aligned}$$

Proof. We start by abbreviating $X[T,T^{-1}]$ as $X[T^{\pm}]$. We will give maps and diagrams for the absolute case $B_k(X)$. To obtain the correct diagrams for the case with supports, the reader should replace every $B_k(X[T^{-1}][S^{\pm}])$ by $B_k(X[T^{-1}][S^{\pm}]$ on $Z[T^{-1}][S^{\pm}])$.

We prove statements (b) - (e) by descending induction on k, starting with $k=0$. The statements (b) - (e) hold for $k=0$ by 6.1, on taking (a) as a definition. Note (c) says nothing in this case.

To do the induction step, we assume that $k \leq 0$, and that the B_k have been constructed and satisfy (a) - (e). Then for $k-1$ we define $B_{k-1}(X)$ to be the cokernel of the map $B_k(X[T]) \oplus B_k(X[T^{-1}]) \to B_k(X[T^{\pm}])$, and $d_{(k-1)T} : B_k(X[T^{\pm}]) \to B_{k-1}(X)$ to be the cokernel map. Now (b) for $k-1$ results from this definition and (e) for k.

Consider the diagram (6.2.1)

(6.2.1)

$$
\begin{array}{ccccc}
\begin{array}{c} B_{k+1}(X[T][S^{-1}]) \oplus B_{k+1}(X[T^{-1}][S^{-1}]) \\ \oplus B_{k+1}(X[T][S]) \oplus B_{k+1}(X[T^{-1}][S]) \end{array} & \to & \begin{array}{c} B_{k+1}(X[T^{\pm}][S^{-1}]) \\ \oplus\, B_{k+1}(X[T^{\pm}][S]) \end{array} & \to & \begin{array}{c} B_k(X[S^{-1}]) \\ B_k(X[S]) \end{array} \\
\downarrow & & \downarrow & & \downarrow \\
B_{k+1}(X[T][S^{\pm}]) \oplus B_{k+1}(X[T^{-1}][S^{\pm}]) & \to & B_{k+1}(X[T^{\pm}][S^{\pm}]) & \xrightarrow{d_T} & B_k(X[S^{\pm}]) \\
\downarrow{\scriptstyle d_S \oplus d_S} & & \downarrow{\scriptstyle d_S} & & \downarrow{\scriptstyle d_S} \\
B_k(X[T]) \oplus B_k(X[T^{-1}]) & \to & B_k(X[T^{\pm}]) & \xrightarrow{d_T} & B_{k-1}(X)
\end{array}
$$

All the small squares of (6.2.1) except possibly the lower right one commute by naturality. All rows and columns are cokernel sequences. Since colimits commute and the d_T, d_S into $B_{k-1}(X)$ are defined to be the canonical cokernel map, this shows that the lower right square commutes. This proves (c) for $k-1$. (As the reader expects, we will identify $B_k(X)$ with $K_k^B(X) = \pi_k K^B(X)$ for $k<0$, and d_T will become a boundary map in a long exact sequence of homotopy subgroups coming from a homotopy fibre sequence. So the reader might have expected to see $\partial_T \partial_S = -\partial_S \partial_T$ instead of $d_T d_S = d_S d_T$. But the natural identification will be $B_k(X) = \pi_0 \Sigma^k K^B(X)$, and the degree k shift changes the sign conventions.)

Now to construct $h_{(k-1)T}$ and prove (d) for $k-1$, we consider diagram (6.2.2) which commutes by naturality and (c).

(6.2.2)

$$\begin{array}{ccccccc}
B_k(X[S]) \oplus B_k(X[S^{-1}]) & \to & B_k(X[S^{\pm}]) & \xrightarrow{d_S} & B_{k-1}(X) & \to & 0 \\
\downarrow{\scriptstyle h_{kT} \oplus h_{kT}} & & \downarrow{\scriptstyle h_{kT}} & & \vdots & & \\
B_{k+1}(X[S][T^{\pm}]) \oplus B_{k+1}(X[S^{-1}][T^{\pm}]) & \to & B_{k+1}(X[S^{\pm}][T^{\pm}]) & \xrightarrow{d_S} & B_k(X[T^{\pm}]) & \to & 0 \\
\downarrow{\scriptstyle d_{kT} \oplus d_{kT}} & & \downarrow{\scriptstyle d_{kT}} & & \downarrow{\scriptstyle d_{(k-1)T}} & & \\
B_k(X[S]) \oplus B_k(X[S^{-1}]) & \to & B_k(X[S^{\pm}]) & \xrightarrow{d_S} & B_{k-1}(X) & \to & 0
\end{array}$$

The rows are exact by definition of B_{k-1}. We define $h_{(k-1)T} : B_{k-1}(X) \to B_k(X[T^{\pm}])$ to be the map induced on the cokernels by h_{kT} in (6.2.2). As $d_{kT} h_{kT} = 1$ by (d) for k, it follows that $d_{(k-1)T} \cdot h_{(k-1)T} = 1$, proving (d) for $k-1$. Note h_T is "$T \cup$" composed with an automorphism on $\oplus_{n \geq k} B_n(X)$.

It remains to prove (e) for $k-1$. This follows from (d) $d_S h_S = 1$ and the diagram (6.2.3)

(6.2.3)

$$\begin{array}{ccccccc}
0 & \to & B_{k-1}(X) & \to & B_{k-1}(X[T]) \oplus B_{k-1}(X[T^{-1}]) & \to & B_{k-1}(X[T^{\pm}]) \\
& & \downarrow{\scriptstyle h_S} & & \downarrow{\scriptstyle h_S \oplus h_S} & & \downarrow{\scriptstyle h_S} \\
0 & \to & B_k(X[S^{\pm}]) & \to & B_k(X[S^{\pm}][T]) \oplus B_k(X[S^{\pm}][T^{-1}]) & \to & B_k(X[S^{\pm}][T^{\pm}]) \\
& & \downarrow{\scriptstyle d_S} & & \downarrow{\scriptstyle d_S \oplus d_S} & & \downarrow{\scriptstyle d_S} \\
0 & \to & B_{k-1}(X) & \to & B_{k-1}(X[T]) \oplus B_{k-1}(X[T^{-1}]) & \to & B_{k-1}(X[T^{\pm}])
\end{array}$$

This diagram exhibits (e) for $B_{k-1}(X)$ as a retract of the exact sequence (e) for $B_k(X[S^{\pm}])$. This proves (e) for $k-1$.

This completes the induction step, and hence the proof of the theorem.

6.3. **Lemma** (Bass Spectral Lemma). *There exist for $k = 0, -1, -2, -3, \cdots$ contravariant functors F^k from the category of (X, Z) as in 6.2 to the category of spectra, such that:*

(a) $F^0(X \text{ on } Z) = K(X \text{ on } Z)$.

(b) *There is a natural homotopy fibre sequence*

$$F^k(X[T] \text{ on } Z[T]) \underset{F^k(X \text{ on } Z)}{\overset{h}{\cup}} F^k\left(X\left[T^{-1}\right] \text{ on } Z\left[T^{-1}\right]\right)$$
$$\downarrow b$$
$$F^k\left(X\left[T^{\pm}\right] \text{ on } Z\left[T^{\pm}\right]\right)$$
$$\downarrow d$$
$$\Sigma F^{k-1}(X \text{ on } Z).$$

(In particular, we have a natural nullhomotopy of $d \cdot b$.) The map b from the homotopy pushout is induced by the obvious open immersions $(X[T^{\pm}]) \to (X[T])$, $(X[T^{-1}])$, and the trivial constant homotopy of the two equal maps $F^k(X \text{ on } Z) \to F^k(X[T^{\pm}] \text{ on } Z[T^{\pm}])$.

(c) *The fibre sequence in* (b) *is a fibre sequence of $K(X)$ modules.*

(d) *The homotopy groups of F^k are given by*

$$\pi_n F^k(X \text{ on } Z) = \begin{cases} K_n(X \text{ on } Z) & n \geq 0 \\ B_n(X \text{ on } Z) & 0 \geq n \geq k \\ 0 & k-1 \geq n. \end{cases}$$

The map $\pi_n(b)$ in (b) *is a monomorphism for all n.*

(e) *The map $f : F^k \to F^{k-1}$, given by*

$$F^k(X \text{ on } Z)$$
$$\downarrow T\cup$$
$$\Omega F^k(X\left[T^{\pm}\right] \text{ on } Z[T^{\pm}])$$
$$\downarrow \Omega d$$
$$\Omega\Sigma F^{k-1}(X \text{ on } Z) \simeq F^{k-1}(X \text{ on } Z)$$

induces an isomorphism on homotopy π_n for $n \geq k$.

Proof. The proof is by descending induction on k. $F^0 = K$ is defined by (a). Inductively define $F^{k-1}(X \text{ on } Z)$ as $\Omega = \Sigma^{-1}$ of the canonical mapping cone of the map b of F^k in (b). As the map b is strictly natural in X, and is a strict $K(X)$ module map, its mapping cone is functorial in X and is a $K(X)$ module. Hence so is F^{k-1}. This constructs $F^{k-1}(X \text{ on } Z)$ and proves statements (b) and (c) by induction.

As $F^k(X \text{ on } Z) \to F^k(X[T] \text{ on } Z[T])$ is canonically split by the map induced by the 0-section map of schemes $X \to X[T] = X \times A^1$, the Mayer-Vietoris sequence for π_* of the homotopy pushout in (b) breaks up into split short exact sequences, of which we will print only the absolute case, by the convention of the proof of 6.2

$$(6.3.1)\quad 0 \to \pi_n F^k(X) \to \pi_n F^k(X[T]) \oplus \pi_n F^k\left(X\left[T^{-1}\right]\right) \to \pi_n \left(F^k(X[T]) \overset{h}{\underset{F^k(X)}{\cup}} F^k\left(X\left[T^{-1}\right]\right)\right) \to 0.$$

We substitute (6.3.1) into the long exact sequence of π_* induced by the homotopy fibre sequence of (b). This breaks up this long exact sequence into short exact sequences like the top two rows of (6.3.2), which we print only in the absolute case:

(6.3.2)

$$\begin{array}{ccccc}
0 & \longrightarrow & \pi_n \left(F^k(X[T]) \overset{h}{\underset{F^k(X)}{\cup}} F^k\left(X\left[T^{-1}\right]\right)\right) & \longrightarrow \\
 & & \wr\| & \\
0 & \longrightarrow & \operatorname{coker}\left(\pi_n F^k(X) \to \pi_n F^k(X[T]) \oplus \pi_n F^k(X[T^{-1}])\right) & \longrightarrow \\
 & & \wr\| & \\
0 & \longrightarrow & \operatorname{coker}\left(B_n(X) \to B_n(X[T]) \oplus B_n(X[T^{-1}])\right) & \longrightarrow
\end{array}$$

$$\begin{array}{ccccc}
\pi_n F^k\left(X\left[T^{\pm}\right]\right) & \longrightarrow & \pi_n \Sigma F^{k-1}(X) & \longrightarrow & 0 \\
\| & & \wr\| & & \\
\pi_n F^k\left(X\left[T^{\pm}\right]\right) & \longrightarrow & \pi_{n-1} F^{k-1}(X) & \longrightarrow & 0 \\
\wr\| & & \wr\| & & \\
B_n\left(X\left[T^{\pm}\right]\right) & \xrightarrow{d_T} & B_{n-1}(X) & \longrightarrow & 0
\end{array}$$

By descending induction on k, with an inner descending induction on n for $0 \geq n \geq k$, we compare these short exact sequences for $n \geq k$ to the exact sequence of B_n in the bottom of (6.3.2) that results from the long exact sequence 6.2(b). The induction starts at $k = 0$ by 6.2(a) and 6.3(a), and proves (d) and also that the exact sequences in (6.3.2) for $\pi_n F^k$ and for B_n correspond under the isomorphisms of (d) for $n \geq k$.

To prove (e), we first note that "$T\cup$" : $F^k(X \text{ on } Z) \to \Omega F^k(X[T^{\pm}]$ on $Z[T^{\pm}])$ factors as the composite of the map p^* induced by the projection $(X[T^{\pm}]) \to X$ and the cup product on $\pi_* F^k(X[T^{\pm}]$ on $Z[T^{\pm}])$ with $T \in K_1(X[T^{\pm}])$. Thus we see that the homotopy fibre sequence of (b) is natural with respect to this map. Now comparing the map induced by "$T \cup$ " in the top of (6.3.2) with a corresponding map "$T \cup$ " on the B_n, and with the inductive construction of the h_T in (6.2.2) and 6.1(b), we see that "$T \cup$ " is identified by the isomorphisms of (d) to h_T composed with some natural automorphism of $B_n(X \text{ on } Z)$. Also the "d" on $\pi_*(F)$ is identified to the "d" on B up to sign. Then $\pi_n f = \pi_n(d \cdot (T\cup\ ))$ is identified to $d_T \cdot h_T$ composed with a natural automorphism of $B_n(X \text{ on } Z)$. Thus (e) follows from 6.2(d). This completes the proof.

6.4. *Definition.* Let $K^B(X \text{ on } Z)$ be the homotopy colimit of the solid arrow diagram (6.4.1), where the F^k are the $F^k(X \text{ on } Z)$ of 6.3, and the maps $F^k \to \Omega\Sigma F^{k-1}$ are the $\Omega d \cdot (T\cup\ )$ of 6.3(e), corresponding to the maps f of 6.3 under the homotopy equivalence $F^{k-1} \xrightarrow{\sim} \Omega\Sigma F^{k-1}$.

(6.4.1)

$$
\begin{array}{ccccccc}
F^0 & \longrightarrow & \Omega\Sigma F^{-1} & \dashrightarrow & F'^{-2} & \dashrightarrow & F'^{-3} & \dashrightarrow & \cdots \\
 & & \uparrow\sim & & \vdots & & \uparrow & & \\
 & & F^{-1} & \longrightarrow & \Omega\Sigma F^{-2} & & | & & \\
 & & & & \uparrow\sim & & | & & \\
 & & & & F^{-2} & \longrightarrow & \Omega\Sigma F{-3} & & \\
 & & & & & & \uparrow & & \\
 & & & & & & F^{-3} & \longrightarrow & \cdots
\end{array}
$$

Equivalently, let $F'^{-2}, F'^{-3}, \ldots$ be the canonical homotopy pushouts of the indicated squares. Let $\Omega\Sigma F^{-1} = F'^{-1}$. Then let $K^B(X \text{ on } Z)$ be the colimit of the direct system $F^0 \to F'^{-1} \to F'^{-2} \to F'^{-3} \to \ldots$.

As the maps $F^k \to F'^k$ are homotopy equivalences, $K^B(X \text{ on } Z)$ is homotopy equivalent to the direct colimit of $F^0 \to F^{-1} \to F^{-2} \to \ldots$ where the bounding maps are the maps f of 6.3(e). Thus we have $\pi_n K^B(X \text{ on } Z) = K_n(X \text{ on } Z)$ for $n \geq 0$, and $= B_n(X \text{ on } Z)$ for $n \leq 0$.

6.5. So defined, $K^B(X \text{ on } Z)$ is a contravariant functor in X, just like $K(X \text{ on } Z)$. It has the same covariant functoriality with respect to flat proper, perfect projective, and (for noetherian schemes) perfect proper maps as does $K(X \text{ on } Z)$, (cf. 3.16). It is also covariant with respect to enlarging Z, as in 3.16.7. The canonical homotopy $g^* f_* \simeq f'_* g'^*$ of 3.18 extends from K to K^B.

There is a canonical map $K(X \text{ on } Z) \to K^B(X \text{ on } Z)$, natural with respect to both contravariant and covariant functorialities, and which induces isomorphisms $K_n(X \text{ on } Z) \cong K_n^B(X \text{ on } Z)$ for $n \geq 0$.

$K^B(X \text{ on } Z)$ is a module spectrum over $K(X)$. The projection formula 3.17 holds in that $f_* : K^B(X) \to K^B(Y)$ will be a map of $K(Y)$ modules under the hypotheses of 3.17.

All these remarks follow by passing the relevant properties of K through the inductive construction of 6.3 and the direct colimit of 6.4. (If the reader wishes to verify exactly the compatibilities of various systems of homotopies involved, he should build symmetric monoidal category models of everything starting from the simplicial symmetric monoidal $\boldsymbol{w}S.\boldsymbol{A}$ and using [Th3], realize the involved homotopies as symmetric monoidal natural transformations, and calculate compatibilities using the calculus of 2-categories as in [Th3]).

6.6. **Theorem** (Bass Fundamental Theorem) (cf. [B] XII Section 7; [Gr1]). *Let X be a quasi-compact and quasi-separated scheme, and let $Z \subseteq X$ be a closed subscheme with $X - Z$ quasi-compact. Then*

(a) *The natural map $K(X \text{ on } Z) \to K^B(X \text{ on } Z)$ induces isomorphism on π_n for $n \geq 0$: $K_n(X \text{ on } Z) \cong K_n^B(X \text{ on } Z)$, $K_n(X) \cong K_n^B(X)$.*

(b) *For all integers $n \in \mathbb{Z}$, there are natural exact sequences:*

$$0 \to K_n^B(X \text{ on } Z) \to K_n^B(X[T] \text{ on } Z[T]) \oplus K_n^B(X[T^{-1}] \text{ on } Z[T^{-1}]) \to K_n^B(X[T^{-1}] \text{ on } Z[T^{-1}]) \xrightarrow{\partial} K_{n-1}^B(X \text{ on } Z) \to 0$$

$$0 \to K_n^B(X) \to K_n^B(X[T]) \oplus K_n^B(X[T^{-1}]) \to K_n^B(X[T,T^{-1}]) \xrightarrow{\partial} K_{n-1}^B(X) \to 0.$$

(c) *There is a homotopy fibre sequence*

$$K^B(X[T] \text{ on } Z[T]) \overset{h}{\underset{K^B(X \text{ on } Z)}{\cup}} K^B(X[T^{-1}] \text{ on } Z[T^{k-1}]) \xrightarrow{b} K^B(X[T,T^{-1}] \text{ on } Z[T,T^{-1}]) \xrightarrow{d} \Sigma K^B(X \text{ on } Z).$$

(d) *For all integers $n \in \mathbb{Z}$, and all positive integers $k \geq 1$, the composition $\partial_k \cdot \ldots \cdot \partial_1(T_1 \cup \cdots \cup T_k \cup)$ is an isomorphism*

$$\begin{array}{c} K_n^B(X \text{ on } Z) \\ \big\downarrow T_1\cup\cdots\cup T_k\cup \\ K_{n+k}^B(X[T_1,T_1^{-1},\ldots,T_k,T_k^{-1}] \text{ on } Z[T_1,T_1^{-1},\ldots,T_k,T_k^{-1}]) \\ \big\downarrow \partial_{T_k}\cdot\partial_{T_{k-1}}\cdots\partial_{T_1} \\ K_n^B(X \text{ on } Z) \end{array}$$

In particular, this holds in the absolute case of $K^B(X) = K^B(X \text{ on } X)$.

(e) *For all positive integers $k \geq 1$, the composition $d_k \cdot \ldots \cdot d_1(T_1 \cup \cdots \cup T_k\cup)$ is a homotopy equivalence*

$$\begin{array}{c} \Sigma^k K^B(X \text{ on } Z) \\ \big\downarrow T_1\cup \cdots \cup T_k\cup \\ K^B(X[T_1,T_1^{-1},\ldots,T_k,T_k^{-1}] \text{ on } Z[T_1,T_1^{-1},\ldots,T_k,T_k^{-1}]) \\ \big\downarrow d_{T_k} \cdots d_{T_1} \\ \Sigma^k K^B(X \text{ on } Z) \end{array}$$

In particular, this holds in the absolute case of $K^B(X) = K^B(X \text{ on } X)$.

Proof. Parts (a), (b), (c) follow from 6.4, 6.3, 6.2, 6.1, and in fact just combine pieces of these.

Parts (d) and (e) follow by induction on k from 6.3(e), 6.1(b), and the definition 6.4.

6.7. By 6.6(d), for $n > 0$, $K_{-n}^B(X)$ is a natural retract of $K_0(X[T_1,T_1^{-1},\ldots,T_n,T_n^{-1}])$ and of $K_1(X[T_1,T_1^{-1},\ldots,T_{n+1},T_{n+1}^{-1}])$. Thus a statement that certain natural classes of maps (invariant under Laurent extensions to $X[T^{\pm}]$) induce isomorphisms or exact sequences on the K-groups K_n for $n > 0$ quickly extend to the K_n^B for all n. For a retract of an isomorphism is an isomorphism, and the retract of an exact sequence is exact. Section 7 will be devoted to results for K^B derived by this method. Similarly 6.6(e) allows one to deduce spectra level versions.

6.8 **Proposition.** *Let X be a regular noetherian scheme. Then*

(a) *$p^* : K(X) \xrightarrow{\sim} K(X[T])$ is a homotopy equivalence.*

(b) *$K(X) \xrightarrow{\sim} K^B(X)$ is a homotopy equivalence, so $K_n^B(X) = 0$ for $n < 0$.*

Proof. Statement (a) results from Poincaré duality 3.21 and the corresponding result for G-theory, [Q1] Section 7, 4.1. Similarly, Quillen's localization theorem for G-theory [Q1] Section 7, 3.2 applied as in (6.1.2) gives $G(X[T,T^{-1}]) \simeq G(X) \times \Sigma G(X)$ using $G(X[T]) \simeq G(X)$. By Poincaré duality, we have $K(X[T,T^{-1}]) \simeq K(X) \times \Sigma K(X)$. In particular, the map $K_0(X) \to K_0(X[T,T^{-1}])$ is onto, so $K^B_{-1}(X) = 0$ by 6.2(b). Now by descending induction on n, using 6.6(b), $K^B_n(X) = 0$ for $n < 0$.

6.9. *Exercise* (Optional). Show $K^B(X)$ is a homotopy ring spectrum. Let $K^B(X) > k <$ be the coPostnikov truncation killing off π_n for $n < k$. Thus $K^B(X) > 0 <\simeq K(X)$. Consider for k, $p \geq 0$ the map

$$\begin{array}{c}
\Sigma^k\left(K^B(X) > -k <\right) \wedge \Sigma^p\left(K^B(X) > -p <\right) \\
\downarrow T_1 \cup \ldots \cup T_k \wedge S_1 \cup \ldots \cup S_p \\
\left(K^B\left(X\left[T_1,T_1^{-1},\ldots,T_kT_k^{-1}\right]\right) > 0 <\right) \wedge \left(K^B\left(X\left[S_1,S_1^{-1},\ldots,S_p,S_p^{-1}\right]\right) > 0 <\right) \\
\wr| \\
K\left(X\left[T,T^{-1},\ldots,T_k,T_k^{-1}\right]\right) \wedge K\left(X\left[S_1,S_1^{-1},\ldots,S_p,S_p^{-1}\right]\right) \\
\downarrow \otimes^L_{\mathcal{O}_X} \\
K\left(X\left[T_1,T_1^{-1},\ldots,T_k,T_k^{-1},S_1.S_1^{-1},\ldots,S_p,S_p^{-1}\right]\right) \\
\wr| \\
K^B\left(X\left[T_1,T_1^{-1},\ldots,S_p.S_p^{-1}\right]\right) > 0 < \\
\downarrow \partial_{T_k} \quad \cdots \partial_{S_1} \\
\Sigma^{k+p}\left(K^B(X) > -p-k <\right)
\end{array}$$

This defines $(K^B(X) > -k <) \wedge (K^B(X) > -p <) \to K^B(X) > -p-k <$. Now take the colimit as $k \to \infty$ and $p \to \infty$.

7. Basic theorems for K^B, including the Localization Theorem

7.0. We recall that $K^B(X \text{ on } Z)$ is defined for X a quasi-compact and quasi-separated scheme X with a closed subspace Z such that $X - Z$ is quasi-compact. K^B has the same functorialities as K. There is a natural transformation $K(X \text{ on } Z) \to K^B(X \text{ on } Z)$ inducing an isomorphism on homotopy groups π_n for $n \geq 0$. However $K^B_n(X \text{ on } Z)$ could be non-zero for $n < 0$.

7.1. **Theorem** (Excision). *Let $f : X' \to X$ be a map of quasi-compact and quasi-separated schemes. Let Y be a closed subspace of X with $X - Y$ quasi-compact. Set $Y' = f^{-1}(Y)$.*

Suppose f is an isomorphism infinitely near Y, in the sense 2.6.2.2.

Then $f^* : K^B(X \text{ on } Y) \xrightarrow{\sim} K^B(X' \text{ on } Y')$ is a homotopy equivalence.

Proof. It suffices to show that $\pi_n f^*$ is an isomorphism on homotopy groups for all n. We use the trick of 6.7. As $\pi_{-k} f^*$ for X is a retract of $\pi_0 f^*$ for $X[T_1, T_1^{-1}, \ldots, T_k, T_k^{-1}]$ when $k > 0$ by 6.6(d), it suffices to show that $\pi_n f^*$ is an isomorphism for $n \geq 0$ on $K_n^B = K_n$. But this holds by excision 3.19.

7.2. **Theorem** (Continuity). *Let $X = \varprojlim X_\alpha$ be the limit of an inverse system of schemes X_α in which the bonding maps $f_{\alpha\beta} : X_\alpha \to X_\beta$ are affine. Suppose all the X_α are quasi-compact and quasi-separated. Let $Y_\alpha \subseteq X_\alpha$ be a system of closed subspaces with $f_{\alpha\beta}^{-1}(Y_\beta) = Y_\alpha$ and with $X_\alpha - Y_\alpha$ quasi-compact. Then the canonical maps are homotopy equivalences*

$$\varinjlim_\alpha K^B(X_\alpha) \xrightarrow{\sim} K^B(X)$$

$$\varinjlim_\alpha K^B(X_\alpha \text{ on } Y_\alpha) \xrightarrow{\sim} K^B(X \text{ on } Y).$$

Proof. We use the trick of 6.7. It suffices to show that the maps induce isomorphisms on homotopy groups π_n. By 6.6.(d), it suffices to do so for $n \geq 0$. Then the result follows from continuity 3.20 as $K_n^B = K_n$ for $n \geq 0$.

7.3. **Theorem** (Projective space bundle theorem). *Let X be a quasi-compact and quasi-separated scheme, and let $Y \subseteq X$ be a closed subspace with $X - Y$ quasi-compact. Let $\mathcal{E}$ be a vector bundle of rank r on X. Then the maps sending $(x_0, x_1, \ldots, x_{r-1})$ to $\Sigma \pi^*(x_i) \otimes [\mathcal{O}(-i)]$ (using the module structure of $K^B(\mathbb{P}\mathcal{E}_X \text{ on } \mathbb{P}\mathcal{E}_Y)$ over $K(\mathbb{P}\mathcal{E}_X)$) induce homotopy equivalences:*

$$\prod_1^r K^B(X) \xrightarrow{\sim} K^B(\mathbb{P}\mathcal{E}_X)$$

$$\prod_1^r K^B(X \text{ on } Y) \xrightarrow{\sim} K^B(\mathbb{P}\mathcal{E}_Y \text{ on } \mathbb{P}\mathcal{E}_Y).$$

Proof. Again, we use the trick of 6.7 to reduce this to 4.1.

7.4. **Theorem** (Localization Theorem). *Let X be a quasi-compact and quasi-separated scheme. Let $j : U \to X$ be an open immersion with U quasi-compact. Set $Y = X - U$. Let Z be a closed subspace of X with $X - Z$ quasi-compact. Then there are homotopy fibre sequences, induced by the obvious maps and nullhomotopies:*

$$\begin{aligned} &K^B(X \text{ on } Y) \to K^B(X) \to K^B(U) \\ &K^B(X \text{ on } Y \cap Z) \to K^B(X \text{ on } Z) \to K^B(U \text{ on } U \cap Z). \end{aligned} \tag{7.4.1}$$

There are resulting long exact sequences of homotopy groups:

(7.4.2)
$$\cdots \xrightarrow{\partial} K_n^B(X \text{ on } Y) \to K_n^B(X) \to K_n^B(U) \xrightarrow{\partial} K_{n-1}^B(X \text{ on } Y) \to \ldots$$

$$\begin{aligned} \cdots \xrightarrow{\partial} K_n^B(X \text{ on } Y \cap Z) \to K_n^B(X \text{ on } Z) \to K_n^B(U \text{ on } U \cap Z) \\ \to K_{n-1}^B(X \text{ on } Y \cap Z) \to \cdots . \end{aligned} \tag{7.4.3}$$

Proof. It suffices to prove that (7.4.1) are homotopy fibre sequences. For this, we must first specify a natural nullhomotopy of the composed map $K^B(X \text{ on } Y\cap Z) \to K^B(U \text{ on } U\cap Z)$. The map $K(X \text{ on } Y\cap Z) \to K(U \text{ on } U\cap Z)$ is canonically nullhomotopic, as any complex on X acyclic on $X-(Y\cap Z) \supseteq X-Y = U$ is naturally quasi-isomorphic to 0 on U. Thus 1.5.4 provides the nullhomotopy. This nullhomotopy is strictly natural in X, and in particular is natural for the maps $(X[T,T^{-1}]) \to X$. Thus the nullhomotopy is natural with respect to the map b in 6.3(b), and by inductive construction, as in 6.3, it induces a natural nullhomotopy of $F^k(X \text{ on } Y \cap Z) \to F^k(U \text{ on } U \cap Z)$ for $k = 0, -1, -2, \ldots$. By 6.4, on taking the colimit as k goes to $-\infty$, we get a natural nullhomotopy on K^B.

This specified natural nullhomotopy determines a natural map from $K^B(X \text{ on } Y\cap Z)$ to the canonical homotopy fibre of the map $K^B(X \text{ on } Z) \to K^B(U \text{ on } U \cap Z)$. It remains to show this map to the canonical homotopy fibre is a homotopy equivalence. It suffices to show it induces an isomorphism on homotopy groups π_n. By the trick of 6.7, it suffices to show that it induces an isomorphism on $\pi_n = K_n$ for $n \geq 0$. But this is true by the Proto-localization Theorem 5.1. This proves the result for $K^B(X \text{ on } Z)$, and so for $K^B(X) = K^B(X \text{ on } X)$.

7.5. **Theorem** (Bass Fundamental Theorem). *Let X be a quasi-compact and quasi-separated scheme, and let $Z \subseteq X$ be a closed subscheme with $X - Z$ quasi-compact. Then*

(a) *The natural map* $K(X \text{ on } Z) \to K^B(X \text{ on } Z)$ *induces isomorphisms on* π_n *for* $n \geq 0$: $K_n(X \text{ on } Z) \cong K_n^B(X \text{ on } Z)$, $K_n(X) \cong K_n^B(X)$.

(b) *For all integers* $n \in \mathbb{Z}$, *there are natural exact sequences:*

$$\begin{array}{c} 0 \\ K_n^B(X \text{ on } Z) \\ \downarrow \\ K_n^B(X[T] \text{ on } Z[T]) \oplus K_n^B(X[T^{-1}] \text{ on } Z[T^{-1}]) \\ \downarrow \\ K_n^B(X[T,T^{-1}] \text{ on } Z[T,T^{-1}]) \\ \downarrow \partial \\ K_{n-1}^B(X \text{ on } Z) \\ \downarrow \\ 0 \end{array}$$

$$0 \to K_n^B(X) \to K_n^B(X[T]) \oplus K_n^B(X[T^{-1}]) \to K_n^B(X[T,T^{-1}]) \xrightarrow{\partial} K_{n-1}^B(X) \to 0$$

(c) There is a homotopy fibre sequence:

$$\begin{array}{c} K^B(X[T] \text{ on } Z[T]) \underset{K^B(X \text{ on } Z)}{\overset{h}{\cup}} K^B(X[T^{-1}] \text{ on } Z[T^{-1}]) \\ \downarrow b \\ K^B(X[T,T^{-1}] \text{ on } Z[T,T^{-1}]) \\ \downarrow d \\ \Sigma K^B(X \text{ on } Z). \end{array}$$

Proof. This is part of 6.6.

7.6. **Theorem.** *If* X *has an ample family of line bundles, then* $K^{\text{naive}}(X) \to K(X) \to K^B(X)$ *induces a natural isomorphism for* $n \geq 0$: $K_n^{\text{naive}}(X) \cong K_n^B(X)$, *where* $K_n^{\text{naive}}(X)$ *is Quillen's* K_n *of* X *as in* [Q1].

Proof. Combine 7.5(a) with 3.8. and 3.10.

8. Mayer-Vietoris theorems

8.1. **Theorem.** *Let X be a quasi-separated scheme. Let U and V be quasi-compact open subschemes of X. Let $Z \subseteq U \cup V$ be a closed subspace, with $U \cup V - Z$ quasi-compact. Then the squares (8.1.1) are homotopy cartesian:*

(8.1.1)
$$\begin{array}{ccc} K^B(U \cup V) & \longrightarrow & K^B(U) \\ \downarrow & \square & \downarrow \\ K^B(V) & \longrightarrow & K^B(U \cap V) \end{array}$$

$$\begin{array}{ccc} K^B(U \cup V \text{ on } Z) & \longrightarrow & K^B(U \text{ on } U \cap Z) \\ \downarrow & \square & \downarrow \\ K^B(V \text{ on } V \cap Z) & \longrightarrow & K^B(U \cap V \text{ on } U \cap V \cap Z) \end{array}$$

Thus, there are long exact Mayer-Vietoris sequences:

(8.1.2)
$$\cdots \xrightarrow{\partial} K_n^B(U \cup V) \to K_n^B(U) \oplus K_n^B(V) \to K_n^B(U \cap V) \xrightarrow{\partial} K_{n-1}^B(U \cup V) \to \dots$$

There is a similar sequence for $K_n^B(\text{ on } Z)$.

Proof. Consider (8.1.3)

$$\begin{array}{ccc} K^B(U \cup V \text{ on } ((U \cup V) - V) \cap Z) & \xrightarrow{\simeq} & K^B(U \text{ on} (U - V) \cap Z) \\ \downarrow & & \downarrow \\ K^B(U \cup V \text{ on } Z) & \longrightarrow & K^B(U \text{ on } U \cap Z) \\ \downarrow & & \downarrow \\ K^B(V \text{ on } V \cap Z) & \longrightarrow & K^B(U \cap V \text{ on} U \cap V \cap Z) \end{array} \tag{8.1.3}$$

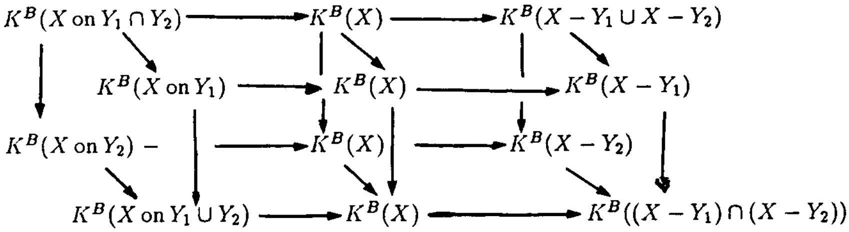
The columns of (8.1.3) are homotopy fibre sequences by the Localization Theorem 7.4. The induced map on the fibres is a homotopy equivalence by excision 7.1. Hence by Quetzalcoatl, the square on the bottom of (8.1.3) is homotopy cartesian, as required (or more naively, apply the 5-lemma to show the map of $K^B(U \cup V \text{ on } Z)$ into the homotopy pullback induces an isomorphism on homotopy groups).

8.1.4. **Corollary.** *Let X be a quasi-compact and quasi-separated scheme. Let Y_1 and Y_2 be two closed subspaces with both $X - Y_1$ and $X - Y_2$ quasi-compact. Then (8.1.5) is a homotopy cartesian square*

$$
(8.1.5) \qquad \begin{array}{ccc}
K^B(X \text{ on } Y_1 \cap Y_2) & \longrightarrow & K^B(X \text{ on } Y_1) \\
\downarrow & \square & \downarrow \\
K^B(X \text{ on } Y_2) & \longrightarrow & K^B(X \text{ on } Y_1 \cup Y_2)
\end{array}
$$

Proof. Consider diagram (8.1.6)

(8.1.6)



The rows are homotopy fibre sequences by Localization 7.4. The right vertical plane is homotopy cartesian by 8.1, as is the middle plane of $K^B(X)$'s. Hence by Quetzalcoatl, the left hand plane is homotopy cartesian, as required.

8.2. To formulate Mayer-Vietoris theorems for covers by more than two open sets, we recall the Čech hypercohomology spectrum of a cover, $\check{\mathbb{H}}^{\cdot}(\mathcal{U};\ )$, from [Th1] Section 1. (The reader unfamiliar with this may skip ahead to Section 9 and ignore the rest of Section 8).

8.2.1. Let $\mathcal{E}^{\sim}$ be a Grothendieck topos with terminal object X, and with a site $\mathcal{E}$ containing X and closed under pullbacks ([SGA 4]). For us, $\mathcal{E}$ will usually be the category of Zariski open subsets of a scheme X, and $\mathcal{E}^{\sim}$ the category of sheaves of sets on X.

Let $F : \mathcal{E}^{\mathrm{op}} \to$ Spectra be a presheaf of spectra defined on the site $\mathcal{E}$. (Upon replacing F by a homotopy equivalent presheaf, we may assume it satisfies the technical topological conditions to be a "presheaf of fibrant spectra," (see [Th1] 5.2.)).

Let $\mathcal{U} = \{U_\iota \to X \mid i \in I\}$ be a cover of X in the site $\mathcal{E}$.

8.2.2. Recall ([Th1] 1.9) that $\check{\mathbb{H}}^{\cdot}(\mathcal{U}; F)$ is defined as the homotopy limit of the Čech cosimplicial spectrum of F and $\mathcal{U}$

(8.2.3)
$$\begin{array}{c}\check{\mathbb{H}}^{\cdot}(\mathcal{U};F)\\ \wr\|\\ \operatorname*{holim}_{\Delta}\left(\prod_{i_0\in I} F(U_{i_0}) \overrightarrow{\rightrightarrows} \prod_{(i_0,i_1)\in I^2} F\left(U_{i_0}\times_X U_{i_1}\right) \overleftarrow{\overrightarrow{\rightrightarrows}} \cdots\right)\end{array}$$

There is a natural augmentation $F(X) \to \check{\mathbb{H}}^{\cdot}(\mathcal{U};F)$.

$\check{\mathbb{H}}^{\cdot}(\mathcal{U};F)$ is a covariant functor in F, and preserves homotopy fibre sequences and homotopy equivalences of presheaves F ([Th1] 1.15). $\check{\mathbb{H}}^{\cdot}(\mathcal{U};F)$ is a covariant functor with respect to maps of covers, where a map $\mathcal{U} \to \mathcal{V}$ of covers consists of a function $\varphi : J \to I$ from the indexing set of $\mathcal{V}$ to that of $\mathcal{U}$, and a family of maps over X, $V_j \to U_{\varphi(j)}$, one for each $j \in J$. Up to homotopy, the induced map $\check{\mathbb{H}}^{\cdot}(\mathcal{U};F) \to \check{\mathbb{H}}^{\cdot}(\mathcal{V};F)$ is independent of the choice of φ or the particular maps $V_j \to V_{\varphi(j)}$, and exists whenever $\mathcal{V}$ is a "refinement" of the cover $\mathcal{U}$ ([Th1] 1.20). Hence if $\mathcal{U}$ and $\mathcal{V}$ refine each other, their $\check{\mathbb{H}}^{\cdot}(\ ;F)$ are homotopy equivalent ([Th1] 1.21]).

Let $f : \mathcal{F} \to \mathcal{E}$ be a map of sites ([SGA 4] III 1, IV 4.9). Then the cover $\mathcal{U}$ on $\mathcal{E}$ induces a cover $f^{-1}(\mathcal{U}) = \{f^{-1}(U_i) | i \in I\}$ in $\mathcal{F}$. There is a canonical isomorphism $\check{\mathbb{H}}^{\cdot}(\mathcal{U}; f_{\#}G = G \cdot f^{-1}) \cong \check{\mathbb{H}}^{\cdot}(f^{-1}(\mathcal{U});G)$. This makes $\check{\mathbb{H}}^{\cdot}$ a contravariant functor with respect to the site $\mathcal{E}$.

There is a spectral sequence relating Čech hypercohomology $\check{\mathbb{H}}^{\cdot}(\mathcal{U};F)$ to the usual Čech cohomology of presheaves of abelian groups

(8.2.4)
$$E_2^{p,q} = \check{H}^p(\mathcal{U};\pi_q F) \Longrightarrow \pi_{q-p}\check{\mathbb{H}}^{\cdot}(\mathcal{U};F).$$

This spectral sequence converges strongly if either there exists an integer N so that $\pi_q F = 0$ for all $q \geq N$, or else, if there exists an integer M such that $\check{H}^p(\mathcal{U};\pi_q F) = 0$, for all $p \geq M$ and all q. See [Th1] 1.16, and note we follow [Th1] in using the Bousfield-Kan indexing of spectral sequences, so $d_r : E_r^{p,q} \to E_r^{p+r,q+r-1}$.

8.2.5. **Lemma**. *Assume* 8.2.1. *Let $\mathcal{U}$ and $\mathcal{V}$ be two covers of X, and suppose there is a map of covers $\mathcal{U} \to \mathcal{V}$, so $\mathcal{V}$ is a refinement of $\mathcal{U}$.*

Suppose that for every finite set I of $U_\iota \to X$ drawn from $\mathcal{U}$, and for the fibre product U_I over X of the elements of I and for the induced cover $\mathcal{V} \times_X U_I$ of

(8.2.6)
$$U_I = U_{\iota_0} \times_X U_{i_1} \times_X \cdots \times_X U_{\iota_n}$$

that the augmentation map (8.2.7) *is a homotopy equivalence*

(8.2.7) $$F(U_I) \xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}\left(\mathcal{V} \underset{X}{\times} U_I; F\right).$$

In particular, for $I = \phi$ we suppose that $F(X) \xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}(\mathcal{V}; F)$ is a homotopy equivalence.

Then the augmentation map (8.2.8) for $\mathcal{U}$ is also a homotopy equivalence

(8.2.8) $$F(X) \xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}(\mathcal{U}; F).$$

Proof. We consider the diagram of maps induced by the various augmentation maps ϵ

(8.2.9)
$$\begin{array}{ccc}
F(X) & \xrightarrow{\epsilon} & \check{\mathbb{H}}^{\cdot}(\mathcal{U}; F) \\
\epsilon \downarrow \sim & & \sim \downarrow \check{\mathbb{H}}^{\cdot}(\mathcal{U}; \epsilon) \\
\check{\mathbb{H}}^{\cdot}(\mathcal{V}; F) & \xrightarrow{\epsilon} & \check{\mathbb{H}}^{\cdot}(\mathcal{U}; \check{\mathbb{H}}^{\cdot}(\mathcal{V} \underset{X}{\times}\ ; F)) \\
\check{\mathbb{H}}^{\cdot}(\mathcal{V}; \epsilon) & \searrow & \wr\| \\
 & & \check{\mathbb{H}}^{\cdot}(\mathcal{V}; \check{\mathbb{H}}^{\cdot}(\mathcal{U} \underset{X}{\times}\ ; F))
\end{array}$$

By the hypothesis the left vertical map of (8.2.9) is a homotopy equivalence. By hypothesis (8.2.7) and the fact that holim preserves homotopy equivalences, inspection of formula (8.2.3) shows that the right vertical map $\check{\mathbb{H}}^{\cdot}(\mathcal{U}; \epsilon)$ is also a homotopy equivalence (cf. [Th1] 1.15).

The isomorphism at the bottom right of (8.2.9) is deduced from the fact that holims commute (e.g., [Th1] 5.7), so that there is an isomorphism:

(8.2.10)
$$\begin{aligned}
&\check{\mathbb{H}}^{\cdot}\left(\mathcal{U}; \check{\mathbb{H}}^{\cdot}\left(\mathcal{V} \underset{X}{\times}\ ; F\right)\right) \\
&= \operatorname*{holim}_{\Delta}\left(p \mapsto \prod_{(i_0, i_1, \ ,i_p)} \right. \\
&\qquad \left. \operatorname{holim}\left(q \mapsto \prod_{(j_0, j_1, \cdot ,j_q)} F\left(U_{i_0} \underset{X}{\times} \cdots \underset{X}{\times} U_{i_p} \underset{X}{\times} V_{j_0} \underset{X}{\times} \cdots \underset{X}{\times} V_{j_q}\right)\right)\right) \\
&\cong \operatorname*{holim}_{\Delta}\left(p \mapsto \left(\operatorname*{holim}_{\Delta}\left(q \mapsto \prod_{(i_0, i_1, \ ,i_p)} \prod_{(j_0, j_1, \ ,j_q)} F\left(U_{i_0} \underset{X}{\times} \cdots \underset{X}{\times} V_{j_q}\right)\right)\right)\right) \\
&\cong \operatorname*{holim}_{\Delta}\left(q \mapsto \operatorname*{holim}_{\Delta}\left(p \mapsto \prod_{(i_0, i_1, \ ,i_p)} \prod_{(j_0, j_1, \ ,j_q)} F\left(U_{i_0} \underset{X}{\times} \cdots \underset{X}{\times} V_{j_q}\right)\right)\right) \\
&\cong \check{\mathbb{H}}^{\cdot}\left(\mathcal{V}; \check{\mathbb{H}}^{\cdot}\left(\mathcal{U} \underset{X}{\times}\ ; F\right)\right).
\end{aligned}$$

It is easy to see that this isomorphism carries the augmentation ϵ to $\check{\mathbb{H}}^{\cdot}(\mathcal{V};\epsilon)$ as claimed in (8.2.9).

We claim that $\check{\mathbb{H}}^{\cdot}(\mathcal{V};\epsilon)$ is a homotopy equivalence since $\mathcal{V}$ is a refinement of $\mathcal{U}$. It will suffice to show for each $V_J = V_{j_0} \underset{X}{\times} \cdots \underset{X}{\times} V_{j_q}$ that $\epsilon : F(V_J) \to \check{\mathbb{H}}^{\cdot}(\mathcal{U} \underset{X}{\times} V_J;\ F)$ is a homotopy equivalence, for then we use the argument above that $\check{\mathbb{H}}^{\cdot}(\mathcal{V};\ )$ preserves homotopy equivalences. But as $\mathcal{V}$ is a refinement of $\mathcal{U}$, $V_{j_0} \to X$ factors as $V_{J_0} \to U_{\varphi(j_0)} \to X$ for some $U_{\varphi(j_0)} \to X$ in $\mathcal{U}$. But then the identity map $V_J \to V_J$ factors through an element of $\mathcal{U} \underset{X}{\times} V_J$:

(8.2.11)

$$\begin{array}{ccc}
V_J = V_{j_0} \underset{X}{\times} \cdots \underset{X}{\times} V_{j_q} & \xrightarrow{\Delta \times 1 \times \cdots \times 1} & V_{j_0} \underset{X}{\times} V_{j_0} \underset{X}{\times} V_{j_1} \underset{X}{\times} \cdots \underset{X}{\times} V_{j_q} \\
 & & \downarrow \\
\Big\downarrow 1 & & U_{\varphi(j_0)} \underset{X}{\times} V_J \\
 & & \downarrow \\
V_J & \xrightarrow{\quad\quad 1 \quad\quad} & V_J = X \underset{X}{\times} V_J
\end{array}$$

This gives a map of covers from $\mathcal{U} \underset{X}{\times} V_J$ to the trivial cover $\{V_J = V_J\}$ of V_J, so $\mathcal{U} \underset{X}{\times} V_J$ is refined by the trivial cover. There is a canonical map of covers from the trivial cover to any cover. So $\mathcal{U} \underset{X}{\times} V_J$ and the trivial cover of V_J refine each other. It follows that these maps of covers induce homotopy equivalences of Čech hypercohomologies for these covers ([Th1] 1.21). But the Čech hypercohomology for the trivial cover of V_J and with coefficients F is homotopy equivalent via the augmentation map to $F(V_J)$. (E.g., this is well-known for abelian group presheaves of coefficients, and the general case then follows by collapse of the spectral sequence (8.2.4); or else use (8.2.3) for the constant Čech cosimplicial spectrum coming from the trivial cover, and the dual of [Th1] 5.21.) It follows that $\epsilon : F(V_J) \xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}(\mathcal{U} \underset{X}{\times} V_J; F)$ is a homotopy equivalence, and hence that $\check{\mathbb{H}}^{\cdot}(\mathcal{V};\epsilon)$ is a homotopy equivalence, as claimed.

Now we have shown that three sides of the square in (8.2.9) are homotopy equivalences. It follows that the fourth side $\epsilon : F(X) \to \check{\mathbb{H}}^{\cdot}(\mathcal{U}; F)$ is also a homotopy equivalence, proving the lemma.

8.3. **Proposition** (cf. 8.4). *Let X be a quasi-compact and quasi-separated scheme, and let $Z \subseteq X$ be a closed subspace with $X - Z$ quasi-compact. Let $\mathcal{U} = \{U_1, \ldots, U_n\}$ be a cover of X by finitely many Zariski* *open subschemes, each of which is quasi-compact. Then the augmentation maps are homotopy equivalences*

$$\begin{aligned} K^B(X) &\xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}\left(\mathcal{U}; K^B\right) \\ K^B(X \text{ on } Z) &\xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}\left(\mathcal{U}; K^B((\ ) \text{ on } (\ )\cap Z)\right). \end{aligned} \tag{8.3.1}$$

There are strongly converging Mayer-Vietoris spectral sequences (with Bousfield-Kan indexing, so $d_r : E_r^{p,q} \to E_r^{p+r,q+r-1}$):

$$\begin{aligned} E_2^{p,q} &= \check{\mathbb{H}}^p\left(\mathcal{U}; K_q^B\right) \Longrightarrow K_{q-p}^B(X) \\ E_2^{p,q} &= \check{\mathbb{H}}^p\left(\mathcal{U}; K_q^B((\ ) \text{ on } (\ )\cap Z)\right) \Longrightarrow K_{p-q}^B(X \text{ on } Z). \end{aligned} \tag{8.3.2}$$

Proof. The last statement (8.3.2) follows from (8.3.1) by taking the canonical spectral sequence (8.2.4), and noting that $H^p(\mathcal{U};\ ) = 0$ for $p > n$ as $\mathcal{U}$ has at most n distinct elements, so the alternating Čech cochain complex for $\mathcal{U}$ vanishes in degree $p > n$ (cf. [EGA] 0_{III} 11.8 and [Th1] 1.49). Thus it suffices to prove (8.3.1). To simplify notation, we give the argument in the absolute case $K^B(X)$,which in fact implies the case with supports as a corollary, using 7.4, the fact $\check{\mathbb{H}}^{\cdot}(\mathcal{U};\ )$ preserves homotopy fibre sequences, and the 5-lemma.

We prove (8.3.1) by induction on n, the number of open sets in the cover $\mathcal{U}$. For $n = 1$, $\mathcal{U}$ is the trivial cover and (8.3.1) is trivially true (cf. proof 8.2.5).

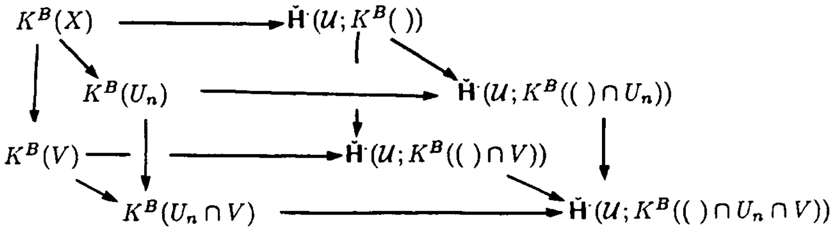
To do the induction step, we suppose the result is known for covers of schemes by $\leq n-1$ opens. We set $V = U_1 \cup \dots U_{n-1}$, and set $\mathcal{V}$ to be the cover $\{U_1, \dots, U_{n-1}\}$ of V.

By the Mayer-Vietoris Theorem 8.1, for any quasi-compact open W in X, there is a homotopy cartesian square, natural in W

$$\begin{array}{ccc} K^B(W) & \longrightarrow & K^B(W \cap U_n) \\ \downarrow & & \downarrow \\ K^B(W \cap V) & \longrightarrow & K^B(W \cap V \cap U_n) \end{array} \tag{8.3.3}$$

As W varies, this is a homotopy cartesian square of presheaves on X. Applying $\check{\mathbb{H}}^{\cdot}(\mathcal{U};\ )$ which preserves homotopy fibre sequences and hence preserves homotopy cartesian squares ([Th1] 1.15), we get a cube

(8.3.4)



The left-to-right arrows in the cube are all augmentation maps. The left and right sides of this cube are homotopy cartesian.

For W open in X, $\check{\mathbb{H}}^{\cdot}(\mathcal{U}; K^B((\ ) \cap W))$ is naturally isomorphic to $\check{\mathbb{H}}^{\cdot}(\mathcal{U} \cap W; K^B(\ ))$, as we see by inspection of (8.2.3) (cf. [Th1] 1.25).

In particular, $\check{\mathbb{H}}^{\cdot}(\mathcal{U}; K^B((\ ) \cap U_n))$ is $\check{\mathbb{H}}^{\cdot}(\mathcal{U} \cap U_n; K^B)$. As $\mathcal{U} \cap U_n$ contains $U_n \to U_n$, there is a map of covers from $\mathcal{U} \cap U_n$ to the trivial cover $\{U_n \to U_n\}$, in addition to the canonical map of covers going the other way. As before in the proof of 8.2.5, this shows that $\check{\mathbb{H}}^{\cdot}(\mathcal{U} \cap U_n; K^B)$ is homotopy equivalent to $K^B(U_n)$, and in fact that the augmentation map $K^B(U_n) \to \check{\mathbb{H}}^{\cdot}(\mathcal{U}; K^B((\ ) \cap U_n))$ is a homotopy equivalence.

Also, we get that $\check{\mathbb{H}}^{\cdot}(\mathcal{U}; K^B((\ ) \cap V))$ is $\check{\mathbb{H}}^{\cdot}(\mathcal{U} \cap V; K^B)$. As $\mathcal{V} = \{U_1, \ldots, U_{n-1}\}$ is a subset of $\{U_1, \ldots, U_{n-1}, U_n \cup V\} = \mathcal{U} \cap V$, there is a map of covers $\mathcal{U} \cap V \to \mathcal{V}$. By the induction hypothesis, for any $W \subseteq V$, $K^B(W) \to \check{\mathbb{H}}^{\cdot}(W \cap \mathcal{V}; K^B)$ is a homotopy equivalence. Then as $\mathcal{V}$ refines $\mathcal{U} \cap V$, Lemma 8.2.5 shows that $K^B(V) \to \check{\mathbb{H}}^{\cdot}(\mathcal{U} \cap V; K^B)$ is also a homotopy equivalence. Thus $K^B(V) \to \check{\mathbb{H}}^{\cdot}(\mathcal{U}; K^B((\ ) \cap V))$ is a homotopy equivalence. The argument of the preceding paragraph applies also to $V \cap U_n$, to show that $K^B(V \cap U_n) \to \check{\mathbb{H}}^{\cdot}(\mathcal{U} \cap V \cap U_n; K^B)$ is a homotopy equivalence.

Thus we have seen that three of the four left-to-right arrows in (8.3.4) are homotopy equivalences. As the left and right sides of (8.3.4) are homotopy cartesian, and as taking homotopy pullbacks preserves homotopy equivalences of diagrams, it follows that the fourth left-to-right arrow $K^B(X) \to \check{\mathbb{H}}^{\cdot}(\mathcal{U}; K^B)$ is also a homotopy equivalence. This completes the proof of the induction step, and hence of the theorem.

8.4. **Theorem.** *Let X be a quasi-compact and quasi-separated scheme. Let Z be a closed subspace of X with $X - Z$ quasi-compact. Let $\mathcal{U} = \{U_i | i \in I\}$ be any open cover of X by quasi-compact open U_i.*

Then the augmentation maps are homotopy equivalences:

$$K^B(X) \xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}\left(\mathcal{U}; K^B\right)$$
$$K^B(X \text{ on } Z) \to \check{\mathbb{H}}^{\cdot}\left(\mathcal{U}; K^B((\ ) \text{ on } (\ ) \cap Z)\right).$$

Proof. If $\mathcal{U}$ is a finite cover, the augmentation maps are homotopy equivalences by 8.3. If $\mathcal{U} = \{U_i | i \in I\}$ is not finite, it has a finite subcover $\mathcal{V} = \{U_{\varphi(k)} | k = 1, \ldots, n\}$ as X is quasi-compact. There is an obvious map of covers $\mathcal{U} \to \mathcal{V}$. It now follows from Lemma 8.2.5, where hypothesis (8.2.7) is met by 8.3 for the finite cover $\mathcal{V}$, that the augmentation map for $\mathcal{U}$ is a homotopy equivalence.

8.5. *Exercise* (Optional). (a) The homotopy cofibre of $K \to K^B$ has non-zero homotopy groups π_q only for $q < 0$. Conclude from the spectral sequence (8.2.4) that $\pi_n \check{\mathbb{H}}^{\cdot}(\mathcal{U}; \text{cofibre}(K \to K^B)) = 0$ for $n \geq 0$. Deduce that $K_n(X) \to \pi_n \check{\mathbb{H}}^{\cdot}(\mathcal{U}; K)$ is an isomorphism for $n \geq 0$.

(b) Let X be quasi-compact and quasi-separated. Take an open cover $\mathcal{U}$ of X by affines. Then the intersections $U_1 = U_{i_0} \cap \ldots \cap U_{i_n}$ of the U_i in $\mathcal{U}$ are quasi-affine, and hence the U_I have an ample family of line bundles. Thus $K^{\text{naive}}(U_I) \simeq K(U_I)$. Conclude that for $n \geq 0$, $K_n(X)$ is isomorphic to $\pi_n \check{\mathbb{H}}^{\cdot}(\mathcal{U}; K^{\text{naive}})$. Thus it is necessary to use K and not Quillen's K^{naive} to make the Mayer-Vietoris theorem work for any scheme X where these theories are not equivalent. (Recall that such a bad X cannot have an ample family of line bundles.)

8.6. *Exercise* (Optional). Let k be a field, and let n be an integer, $n \geq 2$. Let X be affine n-space with the origin doubled, the union of two copies of affine n-space $\mathbb{A}^n$ glued together on the open $\mathbb{A}^n - \{0\}$. This X is noetherian.

Using Poincaré duality 3.21, and Quillen's localization sequences for G-theory, $G(k) \to G(\mathbb{A}^n) \to G(\mathbb{A}^n - 0)$, $G(k) \to G(X) \to G(\mathbb{A}^n)$, and the homotopy equivalence $G(\mathbb{A}^n) \simeq G(k)$, show there are homotopy equivalences

$$K(X) \simeq G(X) \simeq G(\mathbb{A}^n) \times G(k) \simeq G(k) \times G(k) \simeq K(k) \times K(k).$$

On the other hand, show that the open immersion $\mathbb{A}^n \to X$ induces homotopy equivalences

$$K^{\text{naive}}(X) \xrightarrow{\sim} K^{\text{naive}}(\mathbb{A}^n) \simeq K(\mathbb{A}^n) \simeq K(k).$$

(Hint: $j : \mathbb{A}^n \to X$ induces an isomorphism of the categories of algebraic vector bundles. For j^* is fully faithful as $X - \mathbb{A}^n$ is a codimension ≥ 2 in X. Also any vector bundle on X consists of two vector bundles on the two copies of $\mathbb{A}^n$ together with a patching isomorphism on $\mathbb{A}^n - 0$. But as 0 has codimension ≥ 2 in $\mathbb{A}^n$, this isomorphism extends over $\mathbb{A}^n$ and the vector bundle on X is a pullback of a vector bundle on $\mathbb{A}^n$ via the canonical map identifying the two origins $X \to \mathbb{A}^n$. See [EGA] IV 5.10, 5.9).

Conclude that $K^{\text{naive}}(X) \neq K(X)$. Also note that $G(X) = K(X)$, but $G(X) \neq K^{\text{naive}}(X)$.

9. Reduction to the affine case, and the homotopy, closed Mayer-Vietoris, and invariance-under-infinitesimal -thickenings properties of *K*-theory with coefficients.

9.1. Suppose $F(X)$ and $F'(X)$ are homotopy limits of diagrams of $K^B(Z)$'s for a diagram of schemes Z over X natural with respect to base change. Suppose there is a natural map $F(X) \to F'(X)$ and one wishes to prove it is a homotopy equivalence. Then if this result is known for X affine, it will follow for X quasi-compact and quasi-separated. First one proves it for X quasi-compact and separated. Such an X has a finite open cover $\{U_i\}$ with all U_i, and hence all finite intersections $U_{i_0} \cap \cdots \cap U_{i_n}$ being affine schemes. Then $F(U_{i_0} \cap \cdots \cap U_{i_n}) \to F'(U_{i_0} \cap \cdots \cap U_{i_n})$ is a homotopy equivalence, and it follows that $\check{\mathbb{H}}^{\cdot}(\mathcal{U}; F) \xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}(\mathcal{U}; F')$ is a homotopy equivalence. But as homotopy limits commute, $\check{\mathbb{H}}^{\cdot}(\mathcal{U}; F)$ is a homotopy limit of the $\check{\mathbb{H}}^{\cdot}(\mathcal{U}; K^B((\ )\underset{X}{\times} Z))$ for $K^B(Z)$'s in the diagram for $F(X)$. Hence by the Mayer-Vietoris Theorem 8.3, it follows that $F(X) \xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}(\mathcal{U}; F)$ is a homotopy equivalence. Similarly $F'(X) \xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}(\mathcal{U}; F')$ is a homotopy equivalence, and it follows that $F(X) \to F'(X)$ is a homotopy equivalence for X is quasi-compact and and separated. In particular this is true for X quasi-affine. Now if X is quasi-compact and quasi-separated, it has a finite open cover $\{U_i\}$ by affines, and all the finite intersections $U_{i_0} \cap \cdots \cap U_{i_n}$ are quasi-affine. Now arguing as above, we conclude that $\check{\mathbb{H}}^{\cdot}(\mathcal{U}; F) \to \check{\mathbb{H}}^{\cdot}(\mathcal{U}; F')$ and $F(X) \to F'(X)$ are homotopy equivalences as required. We note we could also use 8.1 and induction on the cardinality of $\mathcal{U}$ in place of using 8.3 above.

9.2. To apply this method of reduction to the affine case, we need to find some K-theory results known in the affine case. There are not too many, since lack of a localization theorem like 7.4 for X affine but U quasi-affine has hindered the development of K-theory of commutative rings. For example, for X the Spec of an integral domain A, and $\mathcal{U} = \{U_1, \ldots, U_n\}$ a cover by affine opens of the form $U_i = \mathrm{Spec}(A[1/a_i])$, the Mayer-Vietoris Theorem 8.3 was hitherto known for $n = 2$, but not for $n \geq 3$, as the proof of the latter requires consideration of the typically non-affine scheme $U_1 \cup \ldots \cup U_{n-1}$.

However, a few results are known in the affine case. These are: the Bass Fundamental Theorem, which we have already proved in general; Swan's theorem on quadric hypersurfaces over an affine [Sw]; Gabber's rigidity theorem for henselian pairs of rings [Gab]; and the work of Stienstra, Vorst, van der Kallen, Goodwillie, Ogle, and especially Weibel on the failure of $K(A) \to K(A[T])$ to be a homotopy equivalence for a general commutative ring A. This last failure is closely related to the failure of $K(A) \to K(A/I)$ to be a homotopy equivalence when I is a nil ideal, and to the failure of K to send fibre squares of rings to homotopy fibre squares of K-spectra. These failures can be remedied by passing to K-theory with appropriate coefficients. In this section we will extend these results to schemes.

9.3. We recall K-theory with coefficients first considered by Karoubi and Browder.

For $n \geq 2$ an integer, let $K^B/n(X)$ be the mod n reduction of the spectrum $K^B(X)$, that is, its smash product with a mod n Moore spectrum Σ^∞/n. It fits in a homotopy fibre sequence

$$(9.3.1) \qquad K^B(X) \xrightarrow{n} K^B(X) \to K^B/n(X).$$

The long sequence of homotopy groups of (9.3.1) induces short exact universal coefficient sequences, (which are split if $n \geq 3$)

$$(9.3.2) \quad 0 \to K_k^B(X) \otimes \mathbb{Z}/n \to K_k^B(X) \to \mathrm{Tor}^1_{\mathbb{Z}}\left(K_{k-1}^B(X); \mathbb{Z}/n\right) \to 0.$$

$K^B/n(X)$ is a product of the $K^B/\ell^\nu(X)$ for the prime powers ℓ^ν dividing n, so usually we consider only the $K^B/\ell^\nu(X)$. As reduction mod ℓ^ν preserves homotopy equivalences and homotopy fibre sequences, and commutes with $\check{\mathbb{H}}^\cdot(\mathcal{U};\ )$ (all this is clear from the fibre sequence (9.3.1)), all results of Sections 7 and 8 immediately adapt to $K^B/\ell^\nu(X)$.

Similarly, for any multiplicative subset S of $\mathbb{Z}$, we form a spectrum $K^B(X) \otimes \mathbb{Z}_{(S)}$ by taking the colimit along the direct system of multiplication maps $n : K^B(X) \to K^B(X)$ for $n \in S$. Then clearly we have

$$(9.3.3) \qquad \pi_k\left(K^B(X) \otimes \mathbb{Z}_{(S)}\right) \cong K_k^B(X) \otimes \mathbb{Z}_{(S)}.$$

9.4. For X a quasi-compact and quasi-separated scheme, we define the group $NK_n^B(X)$ as the kernel of the map induced by the 0-section $X \to (X[T])$ embedding X as $(T = 0)$ in $(X[T])$

$$NK_n^B(X) = \ker\left(K_n^B(X[T]) \to K_n^B(X)\right).$$

As the 0-section splits the projection $p : (X[T]) \to X$, the map $K_*^B(X[T]) \to K_*^B(X)$ is a naturally split epimorphism, and $NK_n^B(X)$ is naturally isomorphic to the cokernel of p^*.

For $X = \operatorname{Spec}(A)$, Stienstra, following work of Almkvist and Grayson, showed that $NK_n^B(A)$ was a module over the ring of Witt vectors of A ([We6] or [We2] for $n \geq 0$, hence for $n < 0$ by 6.7). As Weibel noted, it follows that if $1/\ell \in A$, then $NK_n^B(A)$ is a $\mathbb{Z}[1/\ell]$-module, and if $\ell^m = 0$ in A, $NK_n^B(A)$ consists of ℓ-torsion elements (as it is a "continuous" module over the ring of Witt vectors). Then considering the universal coefficient sequence (9.3.2) and (9.3.3) leads to the affine case of the following results.

9.5. **Theorem** (cf. Weibel, [We2], [We3], [We6]). *Let X be a quasi-compact and quasi-separated scheme. Let ℓ be a prime integer, and ℓ^ν a prime power. Suppose $1/\ell \in \mathcal{O}_X$. Then*

(a) *The projection $p : (X[T]) \to X$ induces a homotopy equivalence*

$$K^B/\ell^\nu(X) \xrightarrow{\sim} K^B\ell^\nu(X[T]).$$

(b) *More generally, for $\mathcal{S}_. = \bigoplus_{n\geq 0} \mathcal{S}_n$ any sheaf of positively graded commutative quasi-coherent $\mathcal{O}_X$-algebras, with $\mathcal{S}_0 = \mathcal{O}_X$, the projection induces a homotopy equivalence*

$$K^B\ell^\nu(X) \xrightarrow{\sim} K^B/\ell^\nu\left(\operatorname{Spec}_X(\mathcal{S}_.)\right).$$

(c) *If $p : W \to X$ is a torsor under a vector bundle, then p induces a homotopy equivalence*

$$p^* : K^B/\ell^\nu(X) \xrightarrow{\sim} K^B/\ell^\nu(W).$$

Proof. The method of reduction to the affine case 9.1 shows that it suffices to prove the maps are homotopy equivalences when X is affine. For X affine, (a) is [We3] 1.1, generalized from K/ℓ^ν by the trick of 6.7. Similarly (b) follows from [We6]. We may also deduce it from (a). We show the zero section $X \to \operatorname{Spec}(\mathcal{S}_.)$ induced by $\mathcal{S}_. \twoheadrightarrow \mathcal{S}_0 = \mathcal{O}_X$ induces on K^B/ℓ^ν a homotopy inverse to the map of (b). Consider the map $\mathcal{S}_. \to \mathcal{S}_.[T]$ of algebras sending an element $s \in \mathcal{S}_n$ to sT^n, and the induced map $\operatorname{Spec}(\mathcal{S}_.[T]) \to \operatorname{Spec}(\mathcal{S}_.)$. When composed with the section at $T = 1$, $\operatorname{Spec}(\mathcal{S}_.) \to \operatorname{Spec}(\mathcal{S}_.[T])$, this map yields the identity map of $\operatorname{Spec}(\mathcal{S}_.)$. When composed with the section at $T = 0$, this yields the composite $\operatorname{Spec}(\mathcal{S}_.) \to X \to \operatorname{Spec}(\mathcal{S}_.)$. But both sections yield homotopic maps $K^B/\ell^\nu(\operatorname{Spec}(\mathcal{S}_.[T])) \to K^B/\ell^\nu(\operatorname{Spec}(\mathcal{S}_.))$, as they are both inverse to the homotopy equivalence of (a) for $\operatorname{Spec}(\mathcal{S}_.)$ in place of X. Thus the composite map $K^B/\ell^\nu(\operatorname{Spec}(\mathcal{S}_.)) \to K^B/\ell^\nu(X) \to K^B/\ell^\nu(\operatorname{Spec}(\mathcal{S}_.))$ is homotopic to the identity. As the composite $K^B/\ell^\nu(X) \to K^B/\ell^\nu(\operatorname{Spec}(\mathcal{S}_.)) \to K^B/\ell^\nu(X)$ is also the identity, this proves (b).

To prove (c), we reduce to the case X is affine. Torsors under a vector bundle space $\mathbf{V}(\mathcal{E})$ on X are classified by the cohomology groups $H^1(X;\mathcal{E}^\vee)$, which is 0 as X is affine. Thus the torsor is trivial on X, and W is isomorphic to $\mathbf{V}(\mathcal{E}) = \mathrm{Spec}(S^\cdot(\mathcal{E}))$. Thus the affine case of (c) reduces to (b). (See [Gir], [Jo], [We1] for torsors.)

9.6. **Theorem** (cf. Weibel, [We2], [We3], [We6]). *Let X be a quasi-compact and quasi-separated scheme. Let ℓ be a prime integer, and ℓ^ν a prime power. Suppose ℓ is nilpotent in $\mathcal{O}_X$. Then*

(a) *The projection $p : (X[T]) \to X$ induces a homotopy equivalence*

$$K^B(X) \otimes \mathbb{Z}[1/\ell] \xrightarrow{\sim} K^B(X[T]) \otimes \mathbb{Z}[1/\ell].$$

(b) *For $\mathcal{S}_.$ a sheaf of positively graded commutative quasi-coherent $\mathcal{O}_X$-algebras with $\mathcal{S}_0 = \mathcal{O}_X$, the projection induces a homotopy equivalence*

$$K^B(X) \otimes \mathbb{Z}[1/\ell] \xrightarrow{\sim} K^B(\mathrm{Spec}(\mathcal{S}_.)) \otimes \mathbb{Z}[1/\ell].$$

(c) *If $p : W \to X$ is a torsor under a vector bundle, then p induces a homotopy equivalence*

$$p^* : K^B(X) \otimes \mathbb{Z}[1/\ell] \xrightarrow{\sim} K^B(W) \otimes \mathbb{Z}[1/\ell].$$

Proof. First we observe that for a finite open cover $\mathcal{U}, \check{\mathbb{H}}^\cdot(\mathcal{U};\ )$ preserves direct colimits up to homotopy, as this is trivially true for $\check{\mathbb{H}}^p(\mathcal{U};\pi_p(\ ))$, and we have the strongly converging spectral sequence (8.2.4) with $\check{\mathbb{H}}^p(\mathcal{U};\ ) = 0$ for $p > N =$ number of open sets in the cover $\mathcal{U}$. In particular, $\check{\mathbb{H}}^\cdot(\mathcal{U};\ )$ commutes with formation of $\otimes\mathbb{Z}[1/\ell]$. Now the method of 9.1 goes through to reduce the problem to the case where X is affine. Then (a) follows from [We2] 5.2. As in 9.5 (a) implies (b) and (b) implies (c).

9.7. **Theorem** (cf. Weibel, [We2], [We3], [We1]). *Let X be a quasi-compact and quasi-separated scheme. Let $i : X' \to X$ be a closed immersion defined by a nil ideal of $\mathcal{O}_X$. Let ℓ be a prime integer, and ℓ^ν a prime power. Then*

(a) *If $1/\ell \in \mathcal{O}_X$, $i^* : K^B/\ell^\nu(X) \xrightarrow{\sim} K^B/\ell^\nu(X')$ is a homotopy equivalence.*

(b) *If ℓ is nilpotent in $\mathcal{O}_X$, $i^* : K^B(X) \otimes \mathbb{Z}[1/\ell] \to K^B(X') \otimes \mathbb{Z}[1/\ell]$ is a homotopy equivalence.*

Proof. The method of 9.1 reduces this to the case X, and hence X', is affine. The obvious direct colimit argument reduces us to the case where the nil ideal is finitely generated, and hence nilpotent. Then (a) follows by [We3] 1.4 and its proof. Similarly (b) follows by [We2] 5.4.

9.8. **Theorem** (Weibel, [We2], [We3], [We1]). *Let X be a quasi-compact and quasi-separated scheme. Let Y and Z be closed subschemes of X, such that $X = Y \cup Z$ as spaces. Let $Y \cap Z$ be the fibre product scheme of Y and Z over X. Let ℓ be a prime integer and ℓ^ν a prime power. Then*

(a) *If $1/\ell \in \mathcal{O}_X$, the square (9.8.1) is homotopy cartesian*

$$\begin{array}{ccc} K^B/\ell^\nu(X) & \longrightarrow & K^B/\ell^\nu(Y) \\ \downarrow & & \downarrow \\ K^B/\ell^\nu(Z) & \longrightarrow & K^B/\ell^\nu(Y \cap Z) \end{array} \tag{9.8.1}$$

(b) *If ℓ is nilpotent in $\mathcal{O}_X$, the square (9.8.2) is homotopy cartesian*

$$\begin{array}{ccc} K^B(X) \otimes \mathbb{Z}[1/\ell] & \longrightarrow & K^B(Y) \otimes \mathbb{Z}[1/\ell] \\ \downarrow & & \downarrow \\ K^B(Z) \otimes \mathbb{Z}[1/\ell] & \longrightarrow & K^B(Y \cap Z) \otimes \mathbb{Z}[1/\ell] \end{array} \tag{9.8.2}$$

Proof. The method of 9.1 reduces this to the case where $X = \operatorname{Spec}(A)$ is affine. Then Y and Z are affine, corresponding to ideals I and J of A. The hypothesis that $X = Y \cup Z$ says that any prime ideal of A either contains I or else contains J. Thus any prime of A contains $I \cap J$, so the ideal $I \cap J$ is nil. By 9.7, we may replace A by $A/I \cap J$ without changing the relevant K-theory spectrum, and so may assume that $I \cap J = 0$. Then $0 \to A \to A/I \oplus A/J \to A/I + J \to 0$ is exact, so A is the pullback of A/I and A/J over $A/I + J$. The result then follows by [We3] 1.3 and [We2] 5.5.

9.9. We remark that the integral K^B analogs of 9.5, 9.6, 9.7, 9.8 are false even for affine X unless one adds additional hypotheses. Indeed, $NK_*(A)$ may be non-zero (e.g., [We2] 4.4, 4.5), $K_1(\mathbb{Q}[x]/x^2) \to K_1(\mathbb{Q})$ is not an isomorphism as the unit $1 + x$ is in the kernel, and 9.8 fails as first shown by Swan's famous "counter-example to excision" (cf. [We3] 1.6).

9.10. *Exercise* (Optional). For X affine over $\mathbb{Q}$ and $i : X' \to X$ a closed immersion defined by a nil ideal, the fibre of $i^* : K^B(X') \to K^B(X)$ is homotopy equivalent to the fibre of i^* on cyclic cohomology HC^- by a theorem of Goodwillie [Goo]. As cyclic cohomology is calculable, this gives some control on the fibre of i^*.

Similarly, Ogle and Weibel have proved ([OW], after reducing to $I \cap J = 0$) that the double relative K^B measuring failure of 9.8 is homotopy equivalent to double relative HC^- for X affine over $\mathbb{Q}$.

Using the result of Brylinski, Loday, J. Bloch, *et. al.* that HC^- satisfies the Mayer-Vietoris property for open covers ([Lo] 3.4), generalize the results of Goodwillie, Ogle,and Weibel to quasi-compact and quasi-separated schemes over $\mathbb{Q}$. (This generalization is due to Weibel, who suggested this exercise.)

9.11. *Exercise* (Optional). For X a quasi-compact and quasi-separated scheme, define $K^B H(X)$ as the homotopy colimit of K^B applied to the product of X with the Gersten resolution of $\mathbb{Z}$ by polynomial rings

$$K^B H(X) = \operatorname*{hocolim}_{\Delta^{\mathrm{op}}} \left(p \mapsto K^B \left(X\left[T_0, T_1, \ldots, T_p\right]/T_0 + \cdots + T_p = 1\right)\right).$$

See [Wel] for details in the affine case.

(a) Show $K^B H(X) \xrightarrow{\sim} K^B H(X[T])$ is a homotopy equivalence, essentially by construction.

(b) Show for a finite cover $\mathcal{U}$ by open subsets, that $\check{\mathbb{H}}^{\cdot}(\mathcal{U};\ )$ commutes with homotopy colimits, not only with homotopy limits. For homotopy pullback squares of spectra are also homotopy pushout squares, and so commute with homotopy colimits. Now argue by induction on the number of open sets in the cover $\mathcal{U}$, as in the proof of 8.3.

(c) From (b), and 8.3, conclude that $K^B H(X) \xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}(\mathcal{U}; K^B H)$ is a homotopy equivalence for $\mathcal{U}$ a finite open cover of X.

(d) Use the method of 9.1 to generalize the results of [Wel] for $K^B H$ of affines to quasi-compact and quasi-separated schemes. In particular show that:

(e) If $i : X' \to X$ is a closed immersion defined by a nil ideal, then $K^B H(X) \to K^B H(X')$ is a homotopy equivalence.

(f) If $X = Y \cup Z$ as in 9.8, then the following square is homotopy cartesian

$$\begin{array}{ccc} K^B H(X) & \longrightarrow & K^B H(Y) \\ \downarrow & & \downarrow \\ K^B H(Z) & \longrightarrow & K^B H(Y \cap Z) \end{array}$$

(g) If $1/\ell \in \mathcal{O}_X$, then $K^B/\ell^\nu(X) \to K^B H/\ell^\nu(X)$ is a homotopy equivalence.

(h) If ℓ is nilpotent in $\mathcal{O}_X$, then $K^B(X) \otimes \mathbb{Z}[1/\ell] \to K^B H(X) \otimes \mathbb{Z}[1/\ell]$ is a homotopy equivalence.

9.12. *Exercise* (Optional). The Čech cohomological descent results of [Vo] 1.7 and [vdK] 1.3 for NK of commutative rings are cluttered with stupid hypotheses that the rings are reduced, or that every non-zero-divisor is contained in a minimal prime. These hypotheses are required to justify appeal to [Vo] 1.4, which needs the hypothesis that f is a non-zero-divisor (or more generally, that $\exists g$, $fg = 0$, $f+g$ a non-zero-divisor) to be able to appeal to Quillen's Localization Theorem for projective modules [Gr1]. Use the Excision Theorem 7.1 and Localization Theorem 7.4 to remove this hypothesis on f in [Vo] 1.4, and hence to remove the stupid hypotheses from [Vo] and [vdK].

(a) Note $NK_*(A) \cong \operatorname{coker}(K_*(A) \to K_*(A[T]))$.

(b) Argue as in [Vo] 1.4, but using the coker formulation of NK_* in place of Vorst's $\ker(K_*(A[T]) \to K_*(A))$ formulation to show that the critical Vorst isomorphism $NK_*(A_f) \simeq NK_*(A)_{[f]}$ results from showing the canonical map is an isomorphism (9.12.1):

(9.12.1)

$$\operatorname{coker}(K_n(A) \to K_n(A + XA_f[X])) \xrightarrow{\cong} \operatorname{coker}(K_n(A_f) \to K_n(A_f[X]))\,.$$

(c) Deduce that (9.12.1) is an isomorphism for any $f \in A$ by applying 7.1 and 7.4 to the diagram

(9.12.2)
$$\begin{array}{ccc} K^B(A \text{ on } (f=0)) & \xrightarrow{\sim} & K^B(A + XA_f[X] \text{ on } (f=0)) \\ \downarrow & & \downarrow \\ K^B(A) & \longrightarrow & K^B(A + XA_f[X]) \\ \downarrow & & \downarrow \\ K^B(A_f) & \longrightarrow & K^B(A_f[X]) \end{array}$$

9.13. *Exercise* (Optional). Let X be quasi-compact and quasi-separated.

(a) Let $\mathcal{E}$ be a vector bundle on X. The 0-section $i : X \to \mathbf{V}_X\mathcal{E}$ is a regular immersion. Using ideas of 2.7 as in 3.16.5, show there is a map $i_* : K^B(X) \to K^B(\mathbf{V}\mathcal{E} \text{ on } i(X))$.

(b) For $\mathcal{E} = \mathcal{O}_X$, $\mathbf{V}\mathcal{E} = X[T]$, consider the diagram (9.13.1)

(9.13.1)

$$\begin{array}{ccccc}
K^B\left(\mathbf{P}^1_X \text{ on } (T=0)\right) & \longrightarrow & K^B\left(\mathbf{P}^1_X\right) & \longrightarrow & K^B\left(X\left[T^{-1}\right]\right) \\
\downarrow \simeq & & & & \\
K^B(X[T] \text{ on } (T=0)) & & \uparrow \simeq & & \uparrow \pi^* \\
\uparrow i_* & & & & \\
K^B(X) & \longrightarrow & K^B(X) \times K^B(X) & \longrightarrow & K^B(X)
\end{array}$$

Note that the rows are homotopy fibre sequences, and conclude that there is a homotopy equivalence

$$\begin{aligned}
&\text{fibre}\left(\pi^* : K^B(X) \to K^B\left(X\left[T^{-1}\right]\right)\right) \simeq \\
&\qquad\qquad \text{cofibre}\left(i_* : K^B(X) \to K^B(X[T] \text{ on } (T=0))\right).
\end{aligned}$$

(c) Now suppose X has an ample family of line bundles. Use 5.7 and the ideas of [Gr1] to show $K(X[T] \text{ on } (T=0))$ is the K-theory of the exact category of vector bundles on X together with a nilpotent endomorphism. Thus the cofibre of i_* is a sort of Nil$K$, and in fact is the usual Nil$K$ in the case X is affine.

(d) Generalize the "Nil$K$" form of the Bass Fundamental Theorem from the affine case of [Gr1] to the case X has an ample family of line bundles.

9.14. It need not be true that $K^B/\ell^\nu_n(X) = 0$ for $n < 0$ if $1/\ell \in \mathcal{O}_X$. See [We4] 0.3 for $\ell = 2$, and X affine of finite type over $\mathbb{Q}$.

10. Brown-Gersten spectral sequences and descent

10.1. Let X be a scheme with underlying space a noetherian space of finite Krull dimension. For example, X should be a finite dimensional noetherian scheme.

10.2. A theorem of Grothendieck [Gro] 3.6.5 reveals that the Zariski cohomological dimension of such an X is at most its Krull dimension dim X, so $H^p_{\mathrm{Zar}}(X;\ ) = 0$ for $p > \dim X$.

10.3. **Theorem.** *Let X be an in 10.1, and $Y \subseteq X$ a closed subspace. Then the augmentation maps into Zariski hypercohomology spectra are homotopy equivalences* (10.3.1)*:*

$$(10.3.1)\qquad \begin{aligned} &K^B(X) \xrightarrow{\sim} \mathbb{H}^{\cdot}_{\mathrm{Zar}}\left(X; K^B(\ )\right) \\ &K^B(X \text{ on } Y) \xrightarrow{\sim} \mathbb{H}^{\cdot}_{\mathrm{Zar}}\left(X; K^B((\ ) \text{ on } (\ )\cap Y)\right). \end{aligned}$$

Thus there are stronlgy converging spectal sequences

$$(10.3.2)\qquad \begin{aligned} &E_2^{p,q} = H^p_{\mathrm{Zar}}\left(X; \tilde{K}^B_q\right) \Longrightarrow K^B_{q-p}(X) \\ &E_2^{p,q} = H^p_{\mathrm{Zar}}\left(X; \tilde{K}^B_q((\ ) \text{ on } (\ )\cap Y)\right) \Longrightarrow K^B_{q-p}(X \text{ on } Y). \end{aligned}$$

(In (10.3.2)*, $\tilde{K}^B_q$ is the sheafification of the presheaf $\pi_q K^B(\ )$, and the spectral sequences have the Bousfield-Kan indexing with differentials $d_r : E_r^{p,q} \to E_r^{p+r,q+r-1}$.)*

Proof. For Zariski hypercohomology $\mathbb{H}^{\cdot}_{\mathrm{Zar}}$, see [Th1] Section 1, and references there to earlier work of Brown.

The theorem follows from the Mayer-Vietoris property 8.1 by a result of Brown and Gersten ([BG] Thm 4). For the reader's convenience and to prepare for the generalization to the Nisnevich topology, we will give a complete proof following the sketch in [Th1] Exercise 2.5. For notational simplicity, we will give the proof for the absolute case $X = Y$. To prove the case with supports in Y, one just replaces all $K^B((\ ) \text{ on } (\ )\cap Z)$ below with $K^B((\ ) \text{ on } (\ )\cap Z \cap Y)$.

The spectral sequences (10.3.2) follow from (10.3.1) and 10.2 by [Th1] 1.36. It remains to prove (10.3.1).

Let $Z \subseteq X$ be a locally closed subspace, the intersection of a closed and open subspace of X. Then $Z = \bar{Z} \cap U$ for some open U and $\bar{Z}$ the closure of Z. We define $K^B(X \text{ on } Z)$ to be the direct colimit of $K^B(U \text{ on } Z)$ over the inverse system of such opens U with Z closed in U. Note by excision 7.1 that all $K^B(U \text{ on } Z)$ are homotopy equivalent, and hence all are homotopy equivalent to $K^B(X \text{ on } Z)$. (We need to take the colimit to avoid an arbitrary choice of U, so that $K^B(X \text{ on } Z)$ will be strictly functorial.)

If Z' and Z are locally closed in X with Z' closed in Z, the Localization Theorem 7.4 for $Z' \subseteq Z \subseteq U$, Z closed in U, shows that (10.3.3) is a homotopy fibre sequence

$$(10.3.3)\qquad \begin{array}{ccccc} K^B(X \text{ on } Z') & \longrightarrow & K^B(X \text{ on } Z) & \longrightarrow & K^B(X \text{ on } Z - Z') \\ \wr| & & \wr| & & \wr| \\ K^B(U \text{ on } Z') & \longrightarrow & K^B(U \text{ on } Z) & \longrightarrow & K^B(U - Z' \text{ on } Z - Z') \end{array}$$

This fibre sequence is natural for open immersions $V \to X$, $Z \cap V \to Z$, $Z' \cap V \to Z'$, and hence yields a homotopy fibre sequence of presheaves on X:

(10.3.4)
$$K^B((\ ) \text{ on } Z' \cap(\ )) \to K^B((\ ) \text{ on } Z \cap (\ )) \to K^B((\ ) \text{ on } (Z - Z')\cap(\ )).$$

Now for $p \geq 0$ a non-negative integer, we define

$$S^p K^B((\ ) \text{ on } Y \cap (\ )) = \varinjlim_{\bar{Z}'} K^B((\ ) \text{ on } Y \cap \bar{Z}' \cap (\ )) \tag{10.3.5}$$

as the direct colimit over all closed $\bar{Z}'$ of codimension $\geq p$ in X. Note $S^p K^B = 0$ if $p > \dim X$.

We claim that the obvious map $S^{p+1}K^B \to S^p K^B$ induced by inclusion of direct systems is part of a homotopy fibre sequence (10.3.6), where the last term is a wedge over the points x of codimension p in X of skyscraper sheaves $i_* F(x)$ supported at x:

$$S^{p+1}K^B(\ ) \to S^p K^B(\ ) \to \bigvee_{\substack{i \in X \\ \text{codim X} = \text{p}}} i_x F(x). \tag{10.3.6}$$

In fact, the $F(x)$ are given by

$$F(x) = K^B\,(\mathrm{Spec}\,(\mathcal{O}_{X,x}) \text{ on } x) \tag{10.3.7}$$

and $i_x F(x)(V) = F(x)$ if $x \in V$, and $= 0$ if $x \notin V$. (In the case with supports Y, $F(x) = K^B(\mathrm{Spec}\,(\mathcal{O}_{X,x}) \text{ on } x \cap Y)$, so $F(x) = 0$ if $x \notin Y$.)

To begin the proof of the claim, we fix a codimension p closed subspace Z in X. We look at the fibre sequence of (10.3.4) for all $Z' \subseteq Z$ closed and of codimension $\geq p+1$ in X.

$$\begin{aligned} \text{(10.3.8)} \quad & K^B(V \text{ on } Z' \cap V) \to K^B(V \text{ on } Z \cap V) \\ & \to K^B(V \text{ on } (Z - Z') \cap V) \simeq K^B(V - Z' \text{ on } (Z - Z') \cap V). \end{aligned}$$

As the space of X is noetherian, Z contains finitely many points $z_1, z_2, \ldots,$ z_k of codimension p in X. The union of the $\bar{z}_i \cap \bar{z}_j$ over all distinct pairs $z_i \neq z_j$ is a closed subspace of codimension $\geq p+1$ in X, and we consider only the cofinal system of Z' that contains this union. Then $Z - Z'$ is a disjoint union of k components $\bar{z}_i - Z'$. Thus $Z - Z'$ has an open *nbd* $V_1 \cup V_2 \cup \cdots \cup V_k$ so that $V_i \cap (Z - Z') = (\bar{z}_i - Z')$. Then $V_I \cap (Z - Z') = \phi$ for $V_I = V_{i_0} \cap \ldots V_{i_n}$ if $I = \{i_0, \ldots, i_n\}$ contains two distinct indices i.

Adding $X - Z$ to each V_i, we may assume that $\{V_i\} = \mathcal{V}$ is a cover of $X - Z'$. As $K^B(V_I \text{ on } V_I \cap (Z - Z')) = K^B(V_I \text{ on } \phi) \simeq 0$ if I contains two distinct indices,the Mayer-Vietoris spectral sequence (8.3.2) collapses (even at the E^1 term if this is calculated by alternating cochains) yielding homotopy equivalences

$$(10.3.9) \quad K^B(V - Z' \text{ on } Z - Z') \simeq \bigvee_{i=1}^{k} K^B(V \cap V_i \text{ on } \bar{z}_i \cap V_i - Z') \simeq \bigvee_{i=1}^{k} K^B(V - Z' \text{ on } \bar{z}_i - Z').$$

(One may also prove this from 8.1 by induction on k). Substituting (10.3.9) into (10.3.8) and taking the direct colimit over Z' contained in our fixed Z and of codimension $\geq p+1$ in X, we obtain a homotopy fibre sequence

$$(10.3.10) \quad \begin{array}{c} S^{p+1}K^B(V \text{ on } Z \cap V) \to K^B(V \text{ on } Z) \\ \downarrow \\ \bigvee_1^k \varinjlim_{Z'} K^B(V - Z', \bar{z}_i \cap V - Z'). \end{array}$$

By excision 7.1, the right hand term of (10.3.10) is not changed up to homotopy if $V - Z'$ is replaced by an open $W \subseteq V - Z'$ such that $\bar{z}_i \cap W = \bar{z}_i \cap V - Z'$. As Z' runs over the codimension $\geq p+1$ subspaces of Z, the inverse limit of $\bar{z}_i - Z'$ is the point z_i, and the inverse limit of the various W's for the various Z''s is the spectrum of the local ring $\operatorname{Spec}(\mathcal{O}_{X,z_i})$. The subsystem of such W's which are affine is cofinal in the full system. Applying continuity 7.2 to this subsystem, we get a homotopy equivalence

$$(10.3.11) \quad \varinjlim_{Z'} K^B(V - Z' \text{ on } \bar{z}_i \cap V - Z') \simeq K^B(V \cap \operatorname{Spec}(\mathcal{O}_{X,z_i}) \text{ on } z_i \cap V) = i_{z_i} F(z_i)(V).$$

Now substituting (10.3.11) into (10.3.10), and then taking the direct colimit over all Z closed and of codimension $\geq p$ in X yields a homotopy fibre sequence which is (10.3.6), proving our claim. We now prove the theorem.

As $\vee i_x F(x)$ is a skyscraper sheaf, it is Zariski cohomologically trivial, and the augmentation map (10.3.12) is a homotopy equivalence

$$\bigvee_x F(x) \xrightarrow{\sim} \mathbb{H}^{\cdot}_{\mathrm{Zar}}(X; \vee i_x F(x)). \tag{10.3.12}$$

(In more detail, $\pi_n(_x\vee i_x F(x)) = \oplus i_x \pi_n F(x)$ is for all n a skyscraper sheaf of abelian groups, hence flasque, hence Zariski cohomologically acyclic. Then the hypercohomology spectral sequence [Th1] 1.36 collapses, yielding (10.3.12).)

Now we prove by descending induction on p that the augmentation is a homotopy equivalence

$$S^p K^B(X) \xrightarrow{\sim} \mathbb{H}^{\cdot}_{\mathrm{Zar}}(X; S^p K^B(\ )). \tag{10.3.13}$$

For $p > \dim X$, this is trivial as $S^p K^B \simeq 0$. To do the induction step, suppose the augmentation is known to be a homotopy equivalence for $S^{p+1}K^B$. We consider the diagram

$$\begin{array}{ccc} S^{p+1}K^B(X) & \xrightarrow{\sim} & \mathbb{H}^{\cdot}_{\mathrm{Zar}}(X; S^{p+1}K^B(\ )) \\ \downarrow & & \downarrow \\ S^p K^B(X) & \longrightarrow & \mathbb{H}^{\cdot}_{\mathrm{Zar}}(X; S^p K^B(\ )) \\ \downarrow & & \downarrow \\ \bigvee_x F(X) & \xrightarrow{\sim} & \mathbb{H}^{\cdot}_{\mathrm{Zar}}\left(X; \bigvee_x i_x F(x)\right) \end{array} \tag{10.3.14}$$

The columns are homotopy fibre sequences by (10.3.6) and the fact that hypercohomology $\mathbb{H}^{\cdot}_{\mathrm{Zar}}(X;\ )$ preserves homotopy fibre sequences ([Th1] 1.35). The top horizontal arrow is a homotopy equivalence by induction hypothesis, and the bottom arrow is such by (10.3.12). Hence the middle arrow is a homotopy equivalence by the 5-lemma, completing the proof of the induction step. Hence (10.3.13) is a homotopy equivalence. For $p = 0$, $S^0 K^B = K^B$ so this yields the theorem.

10.4. *Remark.* The sheafification $\tilde{K}^B_q$ in the Zariski topology of the presheaf K^B_q has as stalk at a point $x \in X$ the K^B_q of the local ring $\mathcal{O}_{X,x}$ in X

$$\left(\tilde{K}^B_q\right)_x \cong K^B_q(\mathcal{O}_{X,x}). \tag{10.4.1}$$

This is immediate from continuity 7.2 (cf. [Th1] 1.44). Thus the spectral sequence (10.3.2) reduces problems in K-theory to the question of what happens at a local ring.

10.5. **Corollary.** *Let X have underlying space noetherian of finite Krull dimension, and let $Y \subseteq Z$ be a closed subspace. Then there is a natural homotopy equivalence*

$$K^B(X \text{ on } Y) \to \mathbf{H}^{\cdot}_Y\left(X; K^B(\ )\right)_{\mathrm{Zar}}. \tag{10.5.1}$$

Thus there is a strongly converging spectral sequence of cohomology with supports in Y

$$E_2^{p,q} = H^p_Y\left(X; \tilde{K}^B_q(\ )\right)_{\mathrm{Zar}} \Longrightarrow K^B_{q-p}(X \text{ on } Y). \tag{10.5.2}$$

Proof. (10.5.1) follows from the 5-lemma applied to diagram (10.5.3) where the indicated maps are homotopy equivalences by 10.3, and the rows are homotopy fibre sequences by localization 7.4, and the definition of $\mathbf{H}^{\cdot}_Y(X;\ )$ as the homotopy fibre of $\mathbf{H}^{\cdot}(X;\ ) \to \mathbf{H}^{\cdot}(X - Y;\ )$, (see Appendix D).

(10.5.3)

$$\begin{array}{ccccc} K^B(X \text{ on } Y) & \longrightarrow & K^B(X) & \longrightarrow & K^B(X-Y) \\ \downarrow & & \downarrow \simeq & & \downarrow \simeq \\ \mathbf{H}^{\cdot}_Y\left(X; K^B\right)_{\mathrm{Zar}} & \longrightarrow & \mathbf{H}^{\cdot}_{\mathrm{Zar}}\left(X; K^B\right) & \longrightarrow & \mathbf{H}^{\cdot}_{\mathrm{Zar}}\left(X - Y; K^B\right) \end{array}$$

(The canonical choice of nullhomotopy of $K^B(X \text{ on } Y) \to K^B(X - Y)$ and the strictly commutative right half of (10.5.3) determine a strictly natural map of fibres $K^B(X \text{ on } Y) \to \mathbf{H}^{\cdot}_Y(X; K^B)$.)

The spectral sequence (10.5.2) follows from (10.5.1) and D.4.

10.6. *Remark.* The analog of 10.3 for G-theory of noetherian schemes, and hence for K-theory of regular noetherian schemes is due to Brown and Gersten [BG] using Quillen's Localization Theorem for G-theory. Quillen constructed another version of the G-theory spectral sequence [Q1]. Much work has been done to perturb Quillen's ideas to produce 10.3 in special cases where X has dimension ≤ 1, or where X has isolated or otherwise very mild singularities. See [Co], [PW], [L1], [L2], [We4], [We5], [Gi3].

Note that the result 10.3 is new even for X affine of dimension ≥ 2. One was not able to prove this affine case using Quillen's projective module Localization Theorem [Gr1], as the proof involves considering all open sets $U \subseteq X$, which will not in general be affine.

10.7. By modifying the above argument, we can prove $K^B(X \text{ on } Y)$ has cohomological descent for the Nisnevich topology on X. This topology is close to the Zariski topology, but has as its local rings the henselian local rings. Thus the descent theorem for this topology allows us to reduce problems in K-theory to the case of henselian local rings, bringing them within range of Gabber's form of the Rigidity Theorem, 11.6. This step will be essential in the comparison of algebraic and topological K-theory in Section 11.

The Nisnevich topology also plays an essential role in the work of Kato and Saito on higher-dimensional global class field theory [KS], where it is called the henselian topology, and originated in Nisnevich's work on algebraic group schemes [N1], where it is called the completely decomposed topology.

The basic references for the Nisnevich topology are [N2], [N3] Section 1, and [KS] 1.1-1.2.5. We collect the basic facts in Appendix E, which the logical reader will turn to before resuming 10.8.

10.8. **Theorem.** *Let X be a noetherian scheme of finite Krull dimension, and let $Y \subseteq X$ be a closed subscheme. Then the augmentation maps into Nisnevich hypercohomology are homotopy equivalences*

$$(10.8.1)\qquad \begin{aligned} K^B(X) &\xrightarrow{\sim} \mathbf{H}^{\cdot}_{\mathrm{Nis}}\left(X; K^B(\ )\right) \\ K^B(X \text{ on } Y) &\xrightarrow{\sim} \mathbf{H}^{\cdot}_{\mathrm{Nis}}\left(X; K^B((\ ) \text{ on } (\ ) \cap Y)\right). \end{aligned}$$

Thus there are strongly converging spectral sequences

$$(10.8.2)\qquad \begin{aligned} E_2^{p,q} = H^p_{\mathrm{Nis}}\left(X; \tilde{K}^B_q\right) &\Longrightarrow K^B_{q-p}(X) \\ E_2^{p,q} = H^p_{\mathrm{Nis}}\left(X; \tilde{K}^B_q((\ ) \text{ on } (\ ) \cap Y)\right) &\Longrightarrow K^B_{q-p}(X \text{ on } Y). \end{aligned}$$

Proof. The spectral sequences result from (10.8.1) via the standard hypercohomology spectral sequence [Th1] 1.36. The strong convergence holds as X has finite Nisnevich cohomological dimension by E.6(c). So it suffices to prove (10.8.1).

As pullback along flat, *a fortiori* along étale, maps preserves local codimension at each point ([EGA] 0_{IV} 14.2.3, 14.2.4, IV 6.1.4, IV 2.4.6), the constructions $S^pK^B((\ ) \text{ on } Y \times_X (\ ))$ of (10.3.5) extend to presheaves on the Nisnevich site. The localization fibre sequence (10.3.6) is natural for flat, hence for étale maps, and so induces a fibre sequence of presheaves on the Nisnevich site, whose value at $U \to X$ is the sequence

$$(10.8.3)\qquad S^{p+1}K^B\left(U \text{ on } U \times_X Y\right) \to S^pK^B\left(U \text{ on } U \times_X Y\right) \to \bigvee_{\substack{z \in U \\ \operatorname{codim} z = p}} K^B\left(\operatorname{Spec}(\mathcal{O}_{U,z}) \text{ on } z \times_X Y\right).$$

The henselization $\mathrm{Spec}\,(\mathcal{O}^h_{U,z}) \to \mathrm{Spec}\,(\mathcal{O}_{U,z})$ is pro-étale, hence flat, and induces an isomorphism of residue fields $k(z) = k(z)$. Hence it is an isomorphism infinitely near z (2.6.2.2), and so by excision 7.1 it induces a homotopy equivalence

(10.8.4)
$$K^B\left(\mathrm{Spec}\,(\mathcal{O}_{U,z}) \text{ on } z \underset{X}{\times} Y\right) \xrightarrow{\sim} K^B\left(\mathrm{Spec}\,(\mathcal{O}^h_{U,z}) \text{ on } z \underset{X}{\times} Y\right).$$

For $x \in X$ of codimension p, let $F(x)$ be the presheaf of spectra on the Nisnevich site of the residue field $k(x)$, which associates to each étale $\mathrm{Spec}\,(k') \to \mathrm{Spec}\,(k(x))$, the spectrum $K^B(\mathrm{Spec}\,(\mathcal{O}^h_{U,z}) \text{ on } z \underset{X}{\times} Y)$ where $\mathcal{O}^h_{U,z}$ is the henselization of the local ring $\mathcal{O}_{X,x}$ at the residue extension $k(x) \subseteq k(z) = k'$, that is, $\mathcal{O}^h_{U,z}$ is the stalk of the structure sheaf $\mathcal{O}_X$ at the point of the Nisnevich topos corresponding to $\mathrm{Spec}(k(x)) \to \mathrm{Spec}\,(k(x)) \to X$ (see E.4, E.5). For the closed immersion of x into X, $i:\ \mathrm{Spec}\,(k(x)) \to X$, let $i_\# F(x)$ be the induced presheaf of spectra on X_{Nis}, so $(i_\# F)(V) = F(i^{-1}(V))$. Then using E.5, we get isomorphisms

$$(10.8.5)\qquad (i_\# F(x))\,(U) \cong F(x)\left(i^{-1}(U)\right) \cong \bigvee_{z \in U \underset{X}{\times} k(x)} F(x)(k(z))$$
$$\cong \bigvee_{z \in U \underset{X}{\times} k(x)} K^B\left(\mathcal{O}^h_{U,z} \text{ on } z \underset{X}{\times} Y\right).$$

Thus the fibration sequence (10.8.3) becomes a homotopy fibre sequence

$$(10.8.6)\qquad S^{p+1}K^B\left((\ ) \text{ on } (\ ) \underset{X}{\times} Y\right) \to S^p K^B\left((\ ) \text{ on } (\ ) \underset{X}{\times} Y\right)$$
$$\to \bigvee_{\substack{x \in X \\ \mathrm{codim}\ x = p}} i_\# F(x).$$

Now by descending induction on p, we show that the augmentation (10.8.7) is a homotopy equivalence for $S^p K^B$

$$(10.8.7)\qquad S^p K^B(X \text{ on } Y) \xrightarrow{\sim} \mathbb{H}^{\cdot}_{\mathrm{Nis}}\left(X; S^p K^B\left((\ ) \text{ on } (\ ) \underset{X}{\times} Y\right)\right).$$

For $p >$ Krull $\dim X$, both sides are 0, which starts the induction. To do the induction step, as in (10.3.14), by using the localization sequence (10.8.6) and the 5-lemma, we reduce to proving that the augmentation is a homotopy equivalence for $\vee i_\# F(x)$:

$$(10.8.8)\qquad \bigvee_{\substack{x\\ \operatorname{codim} x=p}} F(x)(k(x)) \xrightarrow{\sim} \mathbf{H}^{\cdot}_{\mathrm{Nis}}\left(X;\ \bigvee_{\substack{x\\ \operatorname{codim} x=p}} i_{\#}F(x)\right).$$

To show this, we consider the sheafification of the presheaf of homotopy groups $\tilde{\pi}_n(\vee i_{\#}F(x)) \cong \oplus\tilde{\pi}_n i_{\#}F(x) \cong \oplus i_*\tilde{\pi}_n F(x)$. In the Nisnevich topos of $k(x)$, the only covering sieve of a field k' is the trivial sieve of all objects over Spec(k'), by E.4. Thus every sheaf in the Nisnevich site of $k(x)$ is acyclic for the topos in the sense of [SGA 4] VI 4.1, because the Spec(k') have the cohomology of points, and every object in the site is a disjoint union of such Spec(k'). It follows that $i_*\tilde{\pi}_n F(x) = \tilde{\pi}_n i_{\#}F(x)$ for $i:$ Spec$(k(x)) \to X$ is acyclic for the Nisnevich site of X ([SGA 4] V 4.9). As $H^*_{\mathrm{Nis}}(X;\ )$ preserves direct colimits (E.6) and finite sums, it also preserves infinite sums. Hence we get $\oplus i_*\tilde{\pi}_n F(x)$ is acyclic, so

$$(10.8.9)\qquad \begin{aligned} H^0_{\mathrm{Nis}}\left(X;\tilde{\pi}_n\left(\bigvee_x i_{\#}F(x)\right)\right) &= \oplus\pi_n F(x)(k(x))\\ H^p_{\mathrm{Nis}}\left(X;\tilde{\pi}_n\left(\bigvee_x i_{\#}F(x)\right)\right) &= 0 \quad \text{for } p>0. \end{aligned}$$

Plugging this into the hypercohomology spectral sequence of [Th1] 1.36 for the right hand side of (10.8.8) yields that (10.8.8) is a homotopy equivalence. This completes the induction step to prove (10.8.7). For $p = 0$, (10.8.7) yields the theorem.

10.9. *Remark.* By continuity 7.2, and the description of stalks in the Nisnevich topology E.5, the stalks of the sheaves $\widetilde{K}^B_q((\ )$ on $(\ )\underset{X}{\times} Y)$ at the point corresponding to Spec$(k') \to$ Spec$(k(x)) \to X$ are the $K^B_q(\mathcal{O}^h_{U,z}$ on $z \underset{X}{\times} Y)$, where $\mathcal{O}^h_{U,z}$ is the henselization of the local ring $\mathcal{O}_{X,x}$ at an étale residue field extension $k(x) \to k' = k(z)$, or equivalently the usual henselization $\mathcal{O}^h_{U,z}$ of an appropriate U étale over X at a point z over x with $k(z) = k'$ over $k(x)$.

Thus the spectral sequence (10.8.2) reduces problems in the K-theory of X to the case of hensel local rings of schemes étale over X.

10.10. **Corollary.** *Under the hypotheses of 10.9, there is a natural homotopy equivalence*

$$(10.10.1)\qquad K^B(X \text{ on } Y) \xrightarrow{\sim} \mathbf{H}^{\cdot}_Y\left(X;K^B(\ )\right)_{\mathrm{Nis}}.$$

Thus, there is a strongly converging spectral sequence

(10.10.2) $$E_2^{p,q} = H_Y^p\left(X; \tilde{K}_q^B(\ )\right)_{\mathrm{Nis}} \Longrightarrow K_{q-p}^B(X \text{ on } Y).$$

Proof. This follows from 10.8 as 10.5 follows from 10.3.

10.11. *Remark.* The analog of 10.8 for G-theory of noetherian schemes, and hence for K-theory of regular neotherian schemes is due to Nisnevich, circa 1983, although the paper has just now appeared as a preprint [N3]. This paper also has a proof that under certain hypotheses on F, that $F(X) \simeq H^{\cdot}_{\mathrm{Nis}}(X; F)$, which hypotheses are met by K^B thanks to 7.1 and 7.4.

11. Étale cohomological descent and comparison with topological K-theory

11.1. *Hypotheses.*

11.1.0. Let X be a noetherian scheme of finite Krull dimension. Let ℓ be a prime integer, and ℓ^ν a prime power.

Suppose $1/\ell \in \mathcal{O}_X$. If $\ell = 2$, also suppose $\sqrt{-1} \in \mathcal{O}_X$.

11.1.1. Suppose there is a uniform bound on the étale cohomological dimension with respect to ℓ-torsion coefficient sheaves, of all the residue fields $k(x)$ of X.

11.1.2. Suppose further that the extension of each residue field $k(x)$ to its separable closure has a Tate-Tsen filtration by subextensions of cohomological dimension 1, [Th1] 2.112.

11.2. *Remark.* If k is a field with $1/\ell \in k$ (and $\sqrt{-1} \in k$ if $\ell = 2$), and k is of finite transcendence degree over a separably closed field, or over a global field (e.g., over $\mathbb{Q}$), or over a local field (e.g., over $\mathbb{F}_p$, $\mathbb{F}_q((t))$, $\mathbb{Q}_p^\wedge$); then k has finite étale cohomological dimension for ℓ-torsion sheaves (in fact, it is $\leq 2 +$ the transcendence degree of k), and has a Tate-Tsen filtration. This follows from [SGA 4] X, (cf. [Th1] 2.44).

11.3. *Remark.* If X is a scheme with $1/\ell \in \mathcal{O}_X$, (and with $\sqrt{-1} \in \mathcal{O}_X$ if $\ell = 2$), and if X is of finite type over any of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{F}_q$, $\mathbb{F}_q((t))$, $\mathbb{Q}_p^\wedge$, $\mathbb{Z}_p^\wedge$, or over a separably or algebraically closed field, then X satisfies all the hypotheses of 11.1. This results from [SGA 4] X and various obvious facts.

In particular, 11.1.1 and 11.1.2 follows from 11.1.0 in the cases of interest to number theory or classical algebraic geometry.

11.4. Let $K/\ell^\nu(X)[\beta^{-1}]$ be the localization of the ring spectrum $K/\ell^\nu\ (X)$ by inverting the Bott element as in [Th1] Appendix A. The discussion there extends immediately to our $K(X)$ from Quillen's $K^{\text{naive}}(X)$. (We note that for $\ell = 2$ or 3, there are technical complications in putting a ring spectrum structure on mod ℓ reductions, see [Th1] A for details of what to do in this case.)

As in [Th1] A, $K/\ell^\nu(X)[\beta^{-1}]$ is homotopy equivalent to the mod ℓ^ν reduction of the Bousfield K-localization of $K(X)$, $K/\ell^\nu(X)_K$.

We have by [Th1] A that for a suitable integer N depending on X and ℓ^ν ($N = 1$ for X over a separably closed field), there is a Bott element β^N in $K/\ell^\nu_{2N}(X)$, and so an isomorphism

(11.4.1)

$$K/\ell^\nu_n(X)[\beta^{-1}] \cong \varinjlim_k \left(\cdots \to K/\ell^\nu_{n+k(2N)}(X) \xrightarrow{\cup\beta^N} K/\ell^\nu_{n+(k+1)(2N)}(X) \to \cdots \right).$$

Similarly for $K^B/\ell^\nu(X)[\beta^{-1}]$, the localization of the module spectrum over $K/\ell^\nu(X)$. As $n + k(2N)$ becomes positive as k increases, and as $K/\ell^\nu_q(X) \cong K^B/\ell^\nu_q(X)$ for $q \geq 1$ (by 6.6(a) and the universal coefficient sequences like (9.3.2)), the canonical map $K(X) \to K^B(X)$ induces a homotopy equivalence

$$(11.4.2)\qquad \begin{array}{ccc} K/\ell^\nu(X)\,[\beta^{-1}] & \xrightarrow{\sim} & K^B/\ell^\nu(X)\,[\beta^{-1}] \\ \wr\| & & \wr\| \\ K/\ell^\nu(X)_K & & K^B/\ell^\nu(X)_K \end{array}$$

We also note that localization $(\ )[\beta^{-1}]$ and $(\ )_K = (\ )\wedge\Sigma_K$ of spectra preserve homotopy equivalences, direct colimits, and homotopy fibre sequences.

11.5. **Theorem.** *Under the hypotheses of* 11.1, *the augmentation is a homotopy equivalence into étale hypercohomology*

$$(11.5.1)\qquad K/\ell^\nu(X)\,[\beta^{-1}] \xrightarrow{\sim} \mathbb{H}^{\cdot}_{\text{et}}\left(X; K/\ell^\nu(\ )\,[\beta^{-1}]\right).$$

Moreover, the sheaf of homotopy groups in the étale topology is given by

$$(11.5.2)\qquad \tilde{\pi}_q K/\ell^\nu(\ )\,[\beta^{-1}] \cong \begin{cases} \mathbb{Z}/\ell^\nu(i) \cong \mu_{\ell^\nu}^{\otimes i} & q = 2i,\ i \in \mathbb{Z} \\ 0 & q \ \mathit{odd} \end{cases}$$

where $\mathbb{Z}/\ell^\nu(i)$ is generated locally by β^i. Hence there is a spectral sequence which converges strongly to $K/\ell^\nu(X)[\beta^{-1}]$ from étale cohomology

$$(11.5.3) \quad E_2^{p,q} = \left\{ \begin{matrix} H^p_{\text{et}}(X;\mathbb{Z}/\ell^\nu(i)) & q=2i \\ 0 & q \text{ odd} \end{matrix} \right\} \Longrightarrow K/\ell^\nu_{q-p}(X)\,[\beta^{-1}]\,.$$

(The spectral sequence has Bousfield-Kan indexing, so $d_r : E_r^{p,q} \to E_r^{p+r,q+r-1}$.)

Proof. As the hypotheses 11.1 imply that X has finite étale cohomological dimension for ℓ-torsion sheaves ([SGA 4] X), the strongly converging spectral sequence (11.5.3) results from (11.5.2) and (11.5.1) via the canonical hypercohomology spectral sequence [Th1] 1.36.

The proof of (11.5.1) and (11.5.2) will spread over 11.6 - 11.8. Both depend on Gabber's form of the Gabber-Gillet-Thomason Rigidity Theorem.

11.6. **Theorem** (Gabber [Gab]). *Let A be a commutative ring with $1/\ell \in A$, and let I be an ideal of A such that $(Spec(A),\ Spec(A/I))$ is a henselian pair* ([EGA] IV 18.5.5)*. Then the map $A \to A/I$ induces a homotopy equivalence*

$$(11.6.1) \qquad K/\ell^\nu(A) \xrightarrow{\sim} K/\ell^\nu(A/I).$$

In particular, if $\mathcal{O}^h_{U,z}$ is a hensel local ring with residue field $k(z)$, and if $1/\ell \in \mathcal{O}^h_{U,z}$, then $\mathcal{O}^h_{U,z} \to k(z)$ induces a homotopy equivalence

$$(11.6.2) \qquad K/\ell^\nu\left(\mathcal{O}^h_{U,z}\right) \xrightarrow{\sim} K/\ell^\nu(k(z)).$$

Proof. This is Theorem 1 of [Gab]. Gabber's proof extends ideas of Suslin who proved the special case of A an algebra over a field in [Su2] 2.1 from an even more special case due to Gabber and to Gillet and Thomason [GT], which in turn was inspired by a theorem of Suslin [Su1].

11.7. To prove (11.5.2), we note that the inclusion of the subgroup generated by β^i induces a map of sheaves $\mathbb{Z}/\ell^\nu(i) \to \widetilde{K}/\ell^\nu_{2i}(\ )[\beta^{-1}]$. It suffices to show that this map is an isomorphism and that $0 \to \widetilde{K}/\ell^\nu_q(\ )[\beta^{-1}]$ is an isomorphism for q odd. For this, it suffices to show the maps are isomorphisms on the stalks of sheaves for each point in the étale topology. The stalk of $\widetilde{K}/\ell^\nu_*(\ )[\beta^{-1}]$ at a point x is $K/\ell^\nu_*(\mathcal{O}^{\text{sh}}_{X,x})$ for $\mathcal{O}^{\text{sh}}_{X,x}$ the strict local henselization of X at x (e.g., [Th1] 1.29, 1.43). This $\mathcal{O}^{\text{sh}}_{X,z}$ is a hensel local ring whose residue field $\overline{k(x)}$ is separably closed. So by Gabber rigidity (11.6), $K/\ell^\nu(\mathcal{O}^{\text{sh}}_{X,z})[\beta^{-1}] \xrightarrow{\sim} K/\ell^\nu(\overline{k(x)})[\beta^{-1}]$ is a homotopy equivalence. Thus it suffices to show that (11.5.2) gives the values of $K/\ell_q^\nu(\overline{k(x)})[\beta^{-1}]$ for $\overline{k(x)}$ a separably closed field. But this is true by [Th1] 3.1 or by a trivial extension of the results of Suslin [Su1], [Su2] 3.13 from the algebraically closed to the separably closed case. This proves (11.5.2).

11.7.1. *Remark.* Suslin's result shows that without inverting β, $\widetilde{K}/\ell_q^\nu$ is 0 for $q \leq 0$ or for q odd, and is $\mathbf{Z}/\ell^\nu(i)$ for $q = 2i \geq 0$. Thus inverting β does not change $\pi_n \mathbf{H}_{\text{et}}^{\cdot}(X; K/\ell^\nu(\ ))$ for $n \geq 0$. The inversion of β is necessary to make this isomorphic to $K/\ell_n^\nu(X)[\beta^{-1}]$, and inverting β makes the minimum possible change to $K/\ell_n^\nu(X)$ to create this isomorphism. However, this minimum change is not zero in general, as one sees by the examples $K/\ell_0^\nu(X) \neq K/\ell_0^\nu(X)[\beta^{-1}]$ for X a $K3$ surface, or for $X =$ Spec$(R[1/\ell])$ for R any ring of integers which has at least two distinct primes over ℓ, ([Th1] 4.5).

11.8. It remains to prove (11.5.1) is a homotopy equivalence. The augmentation map (11.5.1) is strictly natural in X, and hence induces a map of presheaves on the Nisnevich site of X. Hence we have a commutative diagram

(11.8.1)

$$\begin{array}{ccc} K/\ell^\nu(X)\,[\beta^{-1}] & \longrightarrow & \mathbf{H}_{\text{et}}^{\cdot}\,(X; K/\ell^\nu(\ )\,[\beta^{-1}]) \\ \simeq\downarrow & & \downarrow\simeq \\ \mathbf{H}_{\text{Nis}}^{\cdot}\,(X; K/\ell^\nu(\ )\,[\beta^{-1}]) & \longrightarrow & \mathbf{H}_{\text{Nis}}^{\cdot}\,(X; \mathbf{H}_{\text{et}}^{\cdot}\,((\ ); K/\ell^\nu(\ )\,[\beta^{-1}])) \end{array}$$

The left vertical map of (11.8.1) is a homotopy equivalence by Nisnevich cohomological descent 10.8 (using also the facts that $\mathbf{H}_{\text{Nis}}^{\cdot}(X;\ )$ preserves homotopy fibre sequences as in reduction mod ℓ^ν, and direct colimits as in forming the localization $(\ )[\beta^{-1}]$, E.6(d)). The right hand vertical map is a homotopy equivalence because the étale topology is finer than the Nisnevich topology, and hence étale cohomology cohomologically descends for the Nisnevich topology. (In more detail: Apply the Cartan-Leray Theorem [Th1] 1.56 for the map of sites $f : X_{\text{et}} \to X_{\text{Nis}}$, considering [Th1] 1.55. The cohomological dimension hypotheses of [Th1] 1.56 hold by the proof of [Th1] 1.48.)

As the two vertical maps in (11.8.1) are homotopy equivalences, the top horizontal map will be a homotopy equivalence iff the bottom horizontal map is so. To show the latter is, it suffices by the strongly converging hypercohomology spectral sequence [Th1] 1.36 to show that $\widetilde{K}/\ell_q^\nu(\ )[\beta^{-1}] \to \tilde{\pi}_n \mathbf{H}_{\text{et}}^{\cdot}(\ ; K/\ell^\nu(\ )[\beta^{-1}])$ is an isomorphism of sheaves in the Nisnevich topos of X. For this, it suffices that it is an isomorphism on stalks. From the description of stalks in E.5, and the continuity of K-theory and étale cohomology (7.1, [Th1] 1.43, 1.45), this in turn reduces to showing that the augmentation map (11.5.1) is a homotopy equivalence whenever X is replaced by $\mathrm{Spec}(\mathcal{O}^h_{U,z})$, the henselization at a point z of a scheme U étale over X.

To prove this reduced statement, we consider diagram (11.8.2)

$$\begin{array}{ccc} K/\ell^\nu\,(\mathcal{O}^h_{U,z})\,[[\beta^{-1}] & \longrightarrow & \mathbf{H}^{\cdot}_{\mathrm{et}}\,(\mathcal{O}^h_{U,z}; K/\ell^\nu(\ )\,[\beta^{-1}]) \\ \simeq\downarrow & & \simeq\downarrow \\ K/\ell^\nu\,(k(z))\,[\beta^{-1}] & \longrightarrow & \mathbf{H}^{\cdot}_{\mathrm{et}}\,(k(z); K/\ell^\nu(\ )\,[\beta^{-1}]) \end{array} \tag{11.8.2}$$

The vertical maps are induced by the map of a hensel local ring $\mathcal{O}^h_{U,z}$ to its residue field $k(z)$. The left vertical map is a homotopy equivalence by Gabber 11.6. The right vertical map we see to be a homotopy equivalence by combining the formula (11.5.2) for the coefficients, the strongly converging hypercohomology spectral sequence [Th1] 1.36, and the isomorphism $H^*_{\mathrm{et}}(\mathcal{O}^h_{U,z}; \mathbb{Z}/\ell^\nu(i)) \cong H^*_{\mathrm{et}}(k(z); \mathbb{Z}/\ell^\nu(i))$ for the hensel local ring as provided by [SGA 4] VIII 8.6.

By diagram (11.8.2) we further reduce to proving that the augmentation is a homotopy equivalence in the special case of a field, $k(z)$. This is hard, but was done in [Th1] 2.43. This quote completes the proof of the theorem.

11.9. **Corollary**. *Under the hypotheses of* 11.1, *the Dwyer-Friedlander map induces a homotopy equivalence from $K/\ell^\nu(X)[\beta^{-1}]$ to the étale topological K-theory of X ([DF])*

$$\rho : K/\ell^\nu(X)\,[\beta^{-1}] \xrightarrow{\sim} K/\ell^{\nu\,\mathrm{top}}(X). \tag{11.9.1}$$

Proof. This follows from 11.5 by the method of [Th1] 4.11, 4.12. The naive idea is that the spectral sequence (11.5.3) has the same E_2 term as the Atiyah-Hirzebruch spectral sequence of [DF].

11.10. **Proposition**. *For X a noetherian scheme of finite Krull dimension, the augmentation map is a homotopy equivalence:*

$$K^B(X) \otimes \mathbb{Q} \xrightarrow{\sim} \mathbf{H}^{\cdot}_{\mathrm{et}}\,(X; K^B(\ ) \otimes \mathbb{Q})\,. \tag{11.10.1}$$

Proof. Note that $K^B(X) \otimes \mathbb{Q}$ is a direct colimit of $K^B(X)$ along a system of multiplication by intergers, so E.6(d) shows that $\mathbf{H}^{\cdot}_{\mathrm{Nis}}(X;\ )$ and $(\ ) \otimes \mathbb{Q}$ commute. We also note that all schemes étale over X have bounded étale cohomological dimension for $\mathbb{Q}$-sheaves, by the methods of [SGA 4] X (cf. [Th1], proof of 1.4.8).

Then by Nisnevich cohomological descent for the two sides of (11.10.1), as in (11.8.1), it suffices to prove that the map is a homotopy equivalence in the special case where $X = \mathrm{Spec}(R_x)$ is a hensel local ring with residue field $k(x)$.

Let k_α run over the finite Galois extensions of $k(x)$, and let G_α be the Galois group $\mathrm{Gal}(k_\alpha/k)$. As X is hensel local, to each k_α there is a corresponding finite étale covering of hensel local rings, $f_\alpha : X_\alpha \to X$, inducing $\mathrm{Spec}(k_\alpha) \to \mathrm{Spec}(k)$ over the closed point of X ([EGA] IV 18.5.15). As f_α is flat and finite, it induces a transfer map $f_* : K^B(X_\alpha) \to K^B(X)$, e.g., by 3.16.6 and 6.5. By 3.17 and 6.5, the composite $f_*f^* : K^B(X) \to K^B(X)$ is multiplication by $[f_*f^*\mathcal{O}_X] = [f_*\mathcal{O}_{X_\alpha}]$ in $K_0(X)$. As X is local, $f_*\mathcal{O}_{X_\alpha}$ is free of rank equal to the degree $[k_\alpha : k]$ of the extension. So f_*f^* is multiplication by the integer $[k_\alpha : k]$ and $f_*f^* \otimes \mathbb{Q}$ is a homotopy equivalence. By 3.18 and 6.5, we see that the other composite $f^*f_* : K^B(X_\alpha) \to K^B(X_\alpha)$ is induced by the functor

$$\mathcal{O}_{X_\alpha} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha} \otimes_{\mathcal{O}_X} (\ ).$$

Galois theory gives an isomorphism

$$\kappa : \mathcal{O}_{X_\alpha} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha} \xrightarrow{\cong} \prod_{G_\alpha} \mathcal{O}_{X_\alpha} \tag{11.10.2}$$

where $\kappa(x \otimes y)$ has component $x \cdot gy$ in the factor indexed by $g \in G_\alpha$. Indeed, κ corresponds to the action map $G_\alpha \times X_\alpha \to X_\alpha \underset{X}{\times} X_\alpha$ sending $(g, x) \mapsto (x, gx)$, which is an isomorphism as $X_\alpha \to X$ is a Galois covering, hence a torsor under $G_\alpha = \mathrm{Gal}(X_\alpha/X) = \mathrm{Gal}(k_\alpha/k)$ ([EGA] IV 18.5.15, [SGA 1] V). This isomorphism κ shows that $f^*f_* = \Sigma g^*$ equals the sum of the g^* for $g : X_\alpha \to X_\alpha$ in the Galois group (cf. [Th1] 1.50, 2.12 - 2.13). It follows by a standard transfer argument from $f^*f_* = \Sigma g^*$ and $f_*f^* = [k_\alpha : k] =$ order of G_α that the augmentation induces an isomorphism (cf. [Th1] 2.14)

(11.10.3)
$$K_n^B(X) \otimes \mathbb{Q} \xrightarrow{\cong} H^0(G_\alpha;\ K_n^B(X_\alpha) \otimes \mathbb{Q}) \cong \varinjlim_\alpha H^0(G_\alpha; K_n^B(X_\alpha) \otimes \mathbb{Q}).$$

As G_α is a finite group, its rational cohomology is trivial by a standard transfer argument, so

$$0 \cong \varinjlim_\alpha H^p(G_\alpha;\ K_*^B(X_\alpha) \otimes \mathbb{Q}) \quad \text{for } p \geq 1. \tag{11.10.4}$$

But by ([SGA 4] VIII 8.6, 2.3), for any sheaf of abelian groups $\widetilde{K}_*(\ )$ on the étale site of the hensel local ring X, there is a canonical isomorphism

$$\varinjlim_{\alpha} H^p\left(G_\alpha;\ \widetilde{K}_*(X_\alpha)\right) \cong H^p_{\text{et}}\left(X;\ \widetilde{K}_*\right). \tag{11.10.5}$$

Combining (11.10.5), (11.10.4), and (11.10.3), we see that the hypercohomology spectral sequence [Th1] 1.36 collapses to yield an isomorphism for X hensel local and all integers n

$$K^B_n(X) \otimes \mathbb{Q} \cong \pi_n \mathbb{H}^{\cdot}_{\text{et}}\left(X;\ K^B(\ ) \otimes \mathbb{Q}\right). \tag{11.10.6}$$

This proves that (11.10.1) is a homotopy equivalence for X hensel local. But by our previous reductions, this proves the theorem.

11.11. **Theorem.** *Let X be a noetherian scheme of finite Krull dimension. Let S be a set of prime integers such that the hypotheses of 11.1 hold for every $\ell \in S$. Let $K^B(\ )_K$ denote the Bousfield K-localization of K^B,* [Th1] A*), and let $K^B(\ )_K \otimes \mathbb{Z}_{(S)}$ be the further localization by inverting all primes not in S.*

Then the augmentation is a homotopy equivalence

$$K^B(X)_K \otimes \mathbb{Z}_{(S)} \xrightarrow{\sim} \mathbb{H}^{\cdot}_{\text{et}}\left(X;\ K^B(\ )_K \otimes \mathbb{Z}_{(S)}\right). \tag{11.11.1}$$

Proof. This follows from 11.10 and 11.5 for the various ℓ in S. Indeed the homotopy fibre of $K^B(X)_K \to \mathbb{H}^{\cdot}_{\text{et}}(X;\ K^B(\ )_K)$ becomes trivial upon forming $\otimes\mathbb{Q}$ by 11.10, and so its homotopy groups are torsion. By 11.5, the mod ℓ^ν reductions of the fibre are trivial for ℓ in S, and hence the homotopy groups are uniquely ℓ-divisible torsion groups for ℓ in S (9.3.2). Thus the torsion groups have no ℓ-torsion for ℓ in S, and $\otimes\mathbb{Z}_{(S)}$ of them are zero. Thus $\otimes\mathbb{Z}_{(S)}$ of the homotopy fibre is homotopy trivial, so (11.11.1) is a homotopy equivalence.

11.12. The analogs of 11.5, 11.10, 11.11 for G-theory appeared in [Th1].

Appendix A

Exact categories and the Gabriel-Quillen embedding

A1. We recall Quillen's definition of an exact category [Q1] Section 2. An exact category $\mathcal{E}$ is an additive category together with a choice of a class of sequences $\{E_1 \rightarrowtail E_2 \twoheadrightarrow E_3\}$ said to be exact. This determines two classes of morphisms: the admissible epimorphisms $E_2 \twoheadrightarrow E_3$ and the admissible monommorphisms $E_1 \rightarrowtail E_2$. The exact category is to satisfy the following axioms: The class of admissible monomorphisms is closed under composition and is closed under cobase change by pushout along an arbitrary map $E_1 \to E_1'$ (cf. 1.2.1.3). Dually, the class of admissible epimorphisms is closed under composition and under base change by pullback along an arbitrary map $E_3' \to E_3$. Any sequence isomorphic to an exact sequence is exact, and any "split" sequence

$$E \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} E \oplus F \xrightarrow{[0,1]} F$$

is to be exact. In any exact sequence $E_1 \rightarrowtail E_2 \twoheadrightarrow E_3$, the map $E_1 \rightarrowtail E_2$ is a kernel for $E_2 \twoheadrightarrow E_3$, and $E_2 \twoheadrightarrow E_3$ is a cokernel for $E_1 \rightarrowtail E_2$. Finally, there is the obscure axiom:

A.1.1. Let $i : E \to F$ be a map in $\mathcal{E}$ which has a cokernel in $\mathcal{E}$. If there exists a map $k : F \to G$ such that $ki : E \rightarrowtail G$ is an admissible monomorphism, then $i : E \rightarrowtail F$ is itself an admissible monomorphism.

Dually if $i : F \to E$ has a kernel in $\mathcal{E}$, and if there exists a $k : G \to F$ such that $ik : G \twoheadrightarrow$E is an admissible epimorphism, then $i : F \twoheadrightarrow E$ is an admissible epimorphism.

A.2. The concept of exact category is self-dual, so $\mathcal{E}$ is exact iff the opposite category $\mathcal{E}^{\text{op}}$ is exact, where $E_1 \to E_2 \to E_3$ is exact in $\mathcal{E}$ iff $E_1 \leftarrow E_2 \leftarrow E_3$ is exact in $\mathcal{E}^{\text{op}}$.

A.3. An exact functor $f : \mathcal{E}_1 \to \mathcal{E}_2$ is one that sends exact sequences in $\mathcal{E}_1$ to exact sequence in $\mathcal{E}_2$. An exact functor *reflects* exactness if whenever f of a sequence in $\mathcal{E}_1$ is exact in $\mathcal{E}_2$, the original sequence is exact in $\mathcal{E}_1$.

If $\boldsymbol{C}$ is a full subcategory of an exact category $\mathcal{E}$, we say $\boldsymbol{C}$ is closed under extensions in $\mathcal{E}$ if whenever $A \rightarrowtail B \twoheadrightarrow C$ is exact in $\mathcal{E}$ with A and C in $\boldsymbol{C}$, then B is isomorphic to an object of $\boldsymbol{C}$.

A.4. Let $\mathcal{E}$ be an additive full subcategory of an abelian category $\mathcal{A}$. Suppose $\mathcal{E}$ is closed under extensions in $\mathcal{A}$. Declare a sequence $E_1 \to$

$E_2 \to E_3$ in $\mathcal{E}$ to be exact iff it is short exact in $\mathcal{A}$; i.e., iff $E_1 \to E_2$ is the kernel of $E_2 \to E_3$, and $E_2 \to E_3$ is the cokernel of $E_1 \to E_2$ in $\mathcal{A}$. Then $\mathcal{E}$ is an exact category.

A.5. Many exact categories satisfy a stronger version of A.1.1, namely:

A.5.1. Axiom: If $f : E \to F$ is a map in $\mathcal{E}$, and there is a map $s : F \to E$ which splits f so $f \cdot s = 1_F$, then f is an admissible epimorphism $E \twoheadrightarrow F$.

A.5.2. Assuming that A.5.1 holds for $\mathcal{E}$, suppose $g : F \to E$ has a $t : E \to F$ with $t \cdot g = 1_F$. Then by A.5.1, t is an admissible epimorphism and so has a kernel in $\mathcal{E}$. Then there is an isomorphism $E \cong F \oplus \ker t$, under which g corresponds to the canonical inclusion of the summand F. Thus g is an admissible monomorphism. Hence A.5.1 implies its dual in the presence of the axioms A.1.1.

A.6.1. *Definition.* An additive category $\mathcal{E}$ is Karoubian (in Karoubi's terminology [K] 1.2.1, "pseudo-abelienne") if whenever $p : E \to E$ is an idempotent endomorphism in $\mathcal{E}$ (i.e., $p^2 = p$), then there is an isomorphism in $\mathcal{E}$, $E \cong E' \oplus E''$ under which p corresponds to the endomorphism $1 \oplus 0$. Note then E' is an image for p, and E'' is a kernel for p.

A.6.2. **Lemma.** *If an exact category $\mathcal{E}$ is Karoubian, it satisfies the extra axiom* A.5.1.

Proof. Given f, s as in A.5.1, $sf : E \to E$ is idempotent as $sfsf = s(1)f = sf$. Hence $E \cong \operatorname{im}(sf) \oplus \ker(sf)$. Clearly $s : F \to E$ induces an isomorphism of F onto $\operatorname{im}(sf)$, and $f : E \to F$ corresponds to the projection $F \oplus \ker(sf) \to F$. Thus f is an admissible epi, as required by A.5.1.

A.7.1. **Theorem** (Gabriel-Quillen Embedding Theorem) (cf. [Ga] II Section 2, [Q1] Section 2). *Let $\mathcal{E}$ be a small exact category. Then there is an abelian category $\mathcal{A}$, and a fully faithful exact functor $i : \mathcal{E} \to \mathcal{A}$ that reflects exactness. Moreover $\mathcal{E}$ is closed under extensions in $\mathcal{A}$.*

$\mathcal{A}$ may be canonically chosen to be the category of left exact functors $\mathcal{E}^{\mathrm{op}} \to \mathbb{Z}$-modules, and $i : \mathcal{E} \to \mathcal{A}$ to be the Yoneda embedding $i(E) = \mathrm{H}om_{\mathcal{E}}(\ ,E)$.

A.7.2. The proof of A.7.1 and its elaborations will occupy all of Section A.7, and is derived from the Grothendieck-Verdier theory of sheafification in [SGA 4] II.

Let $\mathcal{B}$ be the abelian category of additive functors $F : \mathcal{E}^{\mathrm{op}} \to \mathbb{Z}$-mod where $\mathbb{Z}$-mod is the category of abelian groups. Limits and colimits exist in $\mathcal{B}$, and are formed pointwise, so $(\varprojlim F_\alpha)(E) = \varprojlim(F_\alpha(E))$, etc. Then it is clear that direct colimits in $\mathcal{B}$ are exact, i.e., Grothendieck's axiom AB5 holds. Also, $\mathcal{B}$ has a set of generators consisting of the functors $yE = \mathrm{Hom}(\ ,E)$ for E in $\mathcal{E}$. The Yoneda embedding $y : \mathcal{E} \to \mathcal{B}$ is fully faithful by the Yoneda lemma. Thus $\mathcal{B}$ is a Grothendieck abelian category, as is well-known.

A.7.3. *Definition.* Let $G : \mathcal{E}^{\mathrm{op}} \to \mathbb{Z}\text{-mod}$ be an object of $\mathcal{B}$. One says G is *separated* if for all admissible epimorphisms $E \twoheadrightarrow F$ in $\mathcal{E}$, the induced map $G(F) \to G(E)$ is a monomorphism. One says that G is "*left exact*" if for all admissible epimorphisms $E \twoheadrightarrow F$ in $\mathcal{E}$, then (A.7.4) is a difference kernel, where the maps d are induced by the two projections $p : E \underset{F}{\times} E \to E$:

$$G(F) \to G(E) \underset{d^1=G(p_1)}{\overset{d^0=G(p_0)}{\rightrightarrows}} G\left(E \underset{F}{\times} E\right). \tag{A.7.4}$$

Thus $G(F)$ is the kernel of $d^0 - d^1 : G(E) \to G(E \underset{F}{\times} E)$.

A.7.5. Let $\mathcal{A}$ be the full subcategory of $\mathcal{B}$ consisting of the "left exact" functors $\mathcal{E}^{\mathrm{op}} \to \mathbb{Z}\text{-mod}$. Let $j_* : \mathcal{A} \to \mathcal{B}$ be the inclusion. Later, we will show that j_* has a left adjoint j^* so that $j^* j_* = 1_{\mathcal{A}}$. Then $\mathcal{A}$ will be a Grothendieck abelian category such that j^* is an exact functor, and j_* is left exact (in the covariant abelian sense that j_* preserves kernels).

A.7.6. The Yoneda embedding $y : \mathcal{E} \to \mathcal{B}$ factors through $\mathcal{A}$, so $y = j_* \cdot i$ for a functor $i : \mathcal{E} \to \mathcal{A}$. To show this, it suffices to show that $yG = \mathrm{Hom}(\ ,G)$ is "left exact" for all G in $\mathcal{E}$. But for any admissible epimorphism $E \twoheadrightarrow F$, $E \underset{F}{\times} E \rightarrowtail E \oplus E \twoheadrightarrow F$ is exact in $\mathcal{E}$; so $E \oplus E \twoheadrightarrow F$ is the cokernel of $E \underset{F}{\times} E \to E \oplus E$. Thus

(A.7.7)
$$0 \to \mathrm{Hom}(F,G) \to \mathrm{Hom}(E,G) \oplus \mathrm{Hom}(E,G) \to \mathrm{Hom}\left(E \underset{F}{\times} E, G\right)$$

is exact, so $\mathrm{Hom}(F,G)$ is the kernel of $d^0 - d^1 : \mathrm{Hom}(E,G) \to \mathrm{Hom}(E \underset{F}{\times} E, G)$. Thus $\mathrm{Hom}(\ ,G)$ is "left exact" as required.

A.7.8. (cf. [SGA 4] II 3.0.5, 2.4). For E in $\mathcal{E}$, let $\mathcal{C}_E$ be the following directed category. The objects of $\mathcal{C}_E$ are admissible epimorphisms $e' : E' \twoheadrightarrow E$. There is at most one map between any two objects of $\mathcal{C}_E$. There exists a map $(e' : E' \twoheadrightarrow E) \to (e'' : E'' \twoheadrightarrow E)$ in $\mathcal{C}_E$ iff there is a map (backwards!) $a : E'' \to E'$ in $\mathcal{E}$ such that $e'a = e''$. It is easy to check that $\mathcal{C}_E$ is directed.

Note that any two choices of $a_1,\ a_2 : E'' \to E'$ with $e' \cdot a_i = e''$ induce the same map on the kernels of $d^0 - d^1$ in (A.7.9)

$$\begin{array}{ccc} \ker\left(G(E')\right. & \xrightarrow{d^0-d^1} & G(E' \underset{E}{\times} E')) \\ \Big\downarrow G(a_i) & & \Big\downarrow G(a_i \times a_i) \\ \ker\left(G(E'')\right. & \xrightarrow{d^0-d^1} & G(E'' \underset{E}{\times} E'')) \end{array} \tag{A.7.9}$$

For $a_1 \perp a_2 : E'' \to E' \underset{E}{\times} E'$ induces a "homotopy" $h : G(E' \underset{F}{\times} E') \to G(E'')$ such that $h \cdot (d^0 - d^1) = G(a_1) - G(a_2)$ on $G(E')$, showing that $G(a_1) - G(a_2) = 0$ on $\ker(d^0 - d^1)$.

Thus sending E' to $\ker(G(E') \to G(E' \underset{F}{\times} E'))$ is a functor from the directed category $\mathcal{C}_E$ to the category of abelian groups. Define $LG(E)$ to be the direct colimit

$$LG(E) = \varinjlim_{(E' \twoheadrightarrow E) \text{ in } \mathcal{C}_E} \ker\left(G(E') \overset{d^0-d^1}{\longrightarrow} G\left(E' \underset{E}{\times} E'\right)\right). \tag{A.7.10}$$

We note that LG is a covariant functor in E, and that LG is an additive functor as G is additive and as $\mathcal{C}_{E_1 \oplus E_2} = \mathcal{C}_{E_1} \times \mathcal{C}_{E_2}$, since $E' \twoheadrightarrow E_1 \oplus E_2$ canonically decomposes as $E' = E' \underset{E}{\times} E_1 \oplus E' \underset{E}{\times} E_2$ for $E' \underset{E}{\times} E_i \twoheadrightarrow E_i$ in $\mathcal{C}_{E_i}$. There is a natural transformation $\eta : G \to LG$ induced by the obvious map $G(E) \to \ker(G(E') \to G(E' \underset{E}{\times} E'))$. As kernels and directed colimits commute with finite limits in $\mathbb{Z}$-mod, and as limits in $\mathcal{B}$ are formed by pointwise taking limits in $\mathbb{Z}$-mod, the functor $L : \mathcal{B} \to \mathcal{B}$ preserves finite limits.

A.7.11. **Lemma.** (a) *For any $x \in LG(E)$, there exists an admissible epimorphism $e : E' \twoheadrightarrow E$ in $\mathcal{E}$, and a $y \in G(E')$, such that $\eta(y) = LG(e)(x)$ in $LG(E')$.*

(b) *For any $x \in G(E)$, then $\eta(x) = 0$ in $LG(E)$ iff there exists an admissible epimorphism $e : E' \twoheadrightarrow E$ such that $G(e)(x) = 0$ in $G(E')$.*

(c) *$LG = 0$ iff for all E in $\mathcal{E}$ and all $x \in G(E)$, there exists an admissible epimorphism $e : E' \twoheadrightarrow E$ such that $G(e)(x) = 0$ in $G(E')$.*

(d) *If G is separated, then for all E in $\mathcal{E}$ the map $\eta(E) : G(E) \to LG(E)$ is a monomorphism.*

(e) *If G is left exact, then for all E in $\mathcal{E}$, $\eta(E) : G(E) \to LG(E)$ is an isomorphism.*

Proof. Statements (a), (b), (d), and (e) are clear from the definitions. Statement (c) follows immediately from (a) and (b).

A.7.12. **Lemma** (cf. [SGA 4] II 3.2).

(a) *For all G in $\mathcal{B}$, LG is separated.*

(b) *For all separated G in $\mathcal{B}$, LG is "left-exact."*

Proof. (a) Suppose $x \in LG(A)$, and that $b : B \twoheadrightarrow A$ is an admissible epimorphism for which $LG(b)(x) = 0$. We need to show then that $x = 0$. By construction of $LG(A)$, x is represented by $y \in \ker(G(C) \rightrightarrows G(C \underset{A}{\times} C))$ for some $C \twoheadrightarrow A$ in $\mathcal{C}_A$. As x goes to 0 in $LG(B)$, the image of y in $\ker(G(C \underset{A}{\times} B) \rightrightarrows G((C \underset{A}{\times} B) \underset{B}{\times} (C \underset{A}{\times} B)))$ is equivalent to 0 in the direct colimit over $\mathcal{C}_B$ that defines $LG(B)$. Hence there is a map $D \to C \underset{A}{\times} B$ in $\mathcal{E}$ such that the composite with the projection to B is an admissible epimorphism $D \twoheadrightarrow B$, and such that y goes to 0 in $G(D)$. But then $D \twoheadrightarrow B \twoheadrightarrow A$ is in $\mathcal{C}_A$, and y is equivalent to 0 in the direct colimit over $\mathcal{C}_A$ that defines $LG(A)$. Hence $x = 0$ in $LG(A)$, as required.

(b) Suppose G is separated, we must show for any admissible epi $B \twoheadrightarrow A$ in $\mathcal{E}$, that $LG(A) \to LG(B) \rightrightarrows LG(B \underset{A}{\times} B)$ is a difference kernel. As G is separated, $LG(A) \to LG(B)$ is a monomorphism. It remains to show that if $x \in LG(B)$ has $d^0 x = d^1 x$, i.e., $LG(p_1)(x) = LG(p_2)(x)$ in $LG(B \underset{A}{\times} B)$, then x is in the image of $LG(A)$. But by A.7.11, there is a $c : C \twoheadrightarrow B$ and a $y \in G(C)$ such that $\eta(y) = LG(c)(x)$. Then $\eta G(p_1)(y) = \eta G(p_2)(y)$ in $LG(C \underset{A}{\times} C) \supseteq LG(B \underset{A}{\times} B)$. As G is separated, $\eta : G \to LG$ is a monomorphism, so $G(p_1)(y) = G(p_2)(y)$ in $G(C \underset{A}{\times} C)$. Hence $y \in \ker(G(C) \rightrightarrows G(C \underset{A}{\times} C))$ is a class in $LG(A)$ which represents x. This shows x is in $LG(A)$, as required.

A.7.13. **Proposition.** *Let $j^* : \mathcal{B} \to \mathcal{A}$ be $j^* = L \cdot L$. Then j^* is left adjoint to $j_* : \mathcal{A} \to \mathcal{B}$, and the adjunction map $j^* j_* \to 1_{\mathcal{A}}$ is an isomorphism. Hence $\mathcal{A}$ is a reflexive subcategory of $\mathcal{B}$, and j_* is fully faithful.*

$\mathcal{A}$ is an abelian category, and j^* is an exact functor. The functor j_* is left exact, i.e., *it preserves kernels.*

$\mathcal{A}$ has all limits and colimits, and is a Grothendieck abelian category.

Proof. The adjointness of j^* to j_* and the isomorphism $j^* j_* \cong 1_{\mathcal{A}}$ follow immediately from the fact $\eta : G \to LG$ is an isomorphism for G in the subcategory $\mathcal{A}$ of "left exact" functors.

The cokernel of a map in $\mathcal{A}$ is simply j^* of the cokernel taken in $\mathcal{B}$. As $j^* = L \cdot L$ preserves finite limits, it preserves kernels. It is then clear that $\mathcal{A}$ is abelian since $\mathcal{B}$ is, and that $j^* : \mathcal{B} \to \mathcal{A}$ is exact. Then the right adjoint j_* must be left exact.

$\mathcal{A}$ has all limits and colimits, and is a Grothendieck abelian category, since it is a retract of $\mathcal{B}$ which has these properties.

A.7.14. **Proposition.** *The Yoneda functor $i : \mathcal{E} \to \mathcal{A}$ of* A.7.6 *is fully faithful and exact.*

Proof. As $y : \mathcal{E} \to \mathcal{B}$ is fully faithful by the Yoneda lemma, and as $j_* : \mathcal{A} \to \mathcal{B}$ is fully faithful, and as $y = j_* \cdot i$, it follows that $i : \mathcal{E} \to \mathcal{A}$ is fully faithful.

As $y : \mathcal{E} \to \mathcal{B}$ is clearly left exact, and $i = 1 \cdot i = j^* \cdot j_* \cdot i = j^* \cdot y$, the functor $i : \mathcal{E} \to \mathcal{A}$ is left exact. It remains to show that the functor i is right exact.

Let $A \rightarrowtail B \twoheadrightarrow C$ be an exact sequence in $\mathcal{E}$. We have already shown that $0 \to iA \to iB \to iC$, i.e., $0 \to \mathrm{Hom}(\ ,A) \to \mathrm{Hom}(\ ,B) \to \mathrm{Hom}(\ ,C)$ is left exact in $\mathcal{A}$. Let H be the cokernel of $\mathrm{Hom}(\ ,B) \to \mathrm{Hom}(\ ,C)$ in $\mathcal{B}$. For $0 \to iA \to iB \to iC \to 0$ to be exact in $\mathcal{A}$, it suffices that $j^*H = 0$. For this, it suffices to show that $LH = 0$. We show $LH = 0$ by applying criterion A.7.11(c). Take any $\bar{x} \in H(E)$. We must show $\bar{x}$ goes to 0 in $H(E')$ for some $E' \twoheadrightarrow E$. As $\bar{x} \in H(E) = \mathrm{Hom}(E,C)/\mathrm{Hom}(E,B)$, $\bar{x}$ is represented by a map $x : E \to C$. We consider the pullback along x of $B \twoheadrightarrow C$, $B \underset{C}{\times} E \twoheadrightarrow E$. Then $\bar{x}$ goes to 0 in $H(B \underset{C}{\times} E) = \mathrm{Hom}(B \underset{C}{\times} E, C)/\mathrm{Hom}(B \underset{C}{\times} E, B)$ as the map $x' : B \underset{C}{\times} E \twoheadrightarrow E \to C$ factors through B as $B \underset{C}{\times} E \to B \twoheadrightarrow C$. Thus $B \underset{C}{\times} E$ is the required E'.

A.7.15. **Lemma.** *Let $e : E \to F$ be a map in $\mathcal{E}$. Then $i(e)$ is an epimorphism in $\mathcal{A}$ iff there is a $k : E' \to E$ in $\mathcal{E}$ such that $ek : E' \twoheadrightarrow F$ is an admissible epimorphism.*

More generally, for any A in $\mathcal{A}$ and F in $\mathcal{E}$, a map $e : A \to i(F)$ in $\mathcal{A}$ is an epimorphism in $\mathcal{A}$ iff there is a $k : i(E') \to A$ (i.e., a $k \in A(E')$) such that $ek : E' \twoheadrightarrow F$ is an admissible epimorphism in $\mathcal{E}$.

Proof. If $ek : E' \twoheadrightarrow F$ is an admissible epimorphism in $\mathcal{E}$, the exact $i : \mathcal{E} \to \mathcal{A}$ preserves the exact sequence $\ker(ek) \rightarrowtail E' \twoheadrightarrow F$, so $ek : i(E') \to i(F)$ is an epimorphism in $\mathcal{A}$. Hence $e : A \to i(F)$ is an epimorphism in $\mathcal{A}$.

Conversely suppose that $e : A \to i(F)$ is epi in $\mathcal{A}$. We let H be the cokernel of e in $\mathcal{B}$. Then $j^*H = \mathrm{coker}\ e = 0$ in $\mathcal{A}$. As $0 = j^*H = LLH$, and LH is separated, it follows that $LH = 0$. Consider $\bar{x} \in H(F) = \mathrm{Hom}(F,F)/\mathrm{Hom}(F,A)$ corresponding to 1 in $\mathrm{Hom}(F,F)$. As $\eta(\bar{x}) = 0$ in $LH(F)$, by A.7.11(b) there is an admissible epimorphism $E' \twoheadrightarrow F$ in $\mathcal{E}$ such that $\bar{x}$ goes to 0 in $H(E') = \mathrm{Hom}(E',F)/\mathrm{Hom}(E',A)$. That is to say, that $iE' \twoheadrightarrow iF$ factors as the composite ke of a $k : iE' \to A$ and $e : A \to iF$, as required.

A.7.16. **Proposition.** (a) *The embedding $i : \mathcal{E} \to \mathcal{A}$ reflects exactness.*

(b) *If $\mathcal{E}$ satisfies the extra axiom* A.5.1, *and if e is a map in $\mathcal{E}$ such that $i(e)$ is an epimorphism in $\mathcal{A}$, then e is an admissible epimorphism in $\mathcal{E}$.*

Proof. Let $A \to B \to C$ be a sequence in $\mathcal{E}$ such that $0 \to iA \to iB \to iC \to 0$ is a short exact in A. Then $iA \to iB$ is the kernel of $iB \to iC$ in $\mathcal{A}$. As $i : \mathcal{E} \to \mathcal{A}$ is fully faithful, $A \to B$ is the kernel of $B \to C$ in $\mathcal{E}$.

By A.7.15, as $iB \to iC$ is epi in $\mathcal{A}$, there is a $B' \to B$ such that the composite $B' \to B \to C$ is an admissible epimorphism $B' \twoheadrightarrow C$ in $\mathcal{E}$. As $B \to C$ has a kernel in $\mathcal{E}$, this implies that $B \twoheadrightarrow C$ is an admissible epimorphism in $\mathcal{E}$, by the hitherto obscure axiom A.1.1. Then the kernel $A \to B$ is an admissible monomorphism and $A \rightarrowtail B \twoheadrightarrow C$ is exact in $\mathcal{E}$. This proves (a).

Suppose now that $\mathcal{E}$ satisfies A.5.1, and that $e : B \to C$ is a map in $\mathcal{E}$ such that $i(e)$ is an epimorphism in $\mathcal{A}$. By A.7.15, there is a $B' \to B$ in $\mathcal{E}$ such that the composite $B' \to B \to C$ is an admissible epimorphism in $\mathcal{E}$. If we knew that $B \to C$ had a kernel in $\mathcal{E}$, we would conclude that $e : B \twoheadrightarrow C$ is an admissible epimorphism by A.1.1. Hence it suffices to show that $e : B \to C$ has a kernel. We consider the pullback square in $\mathcal{E}$:

$$\begin{array}{ccc} B \underset{C}{\times} B' & \longrightarrow & B' \\ \downarrow\!\!\downarrow & \square & \downarrow\!\!\downarrow \\ B & \longrightarrow & C \end{array} \tag{A.7.17}$$

The map $1 : B' \to B'$ and $B' \to B$ induce a map $B' \to B \underset{C}{\times} B'$ that splits the map $B \underset{C}{\times} B' \to B'$. By axiom A.5.1, we conclude that $B \underset{C}{\times} B' \to B'$ is an admissible epimorphism in $\mathcal{E}$, and so has a kernel in $\mathcal{E}$. But as (A.7.17) is a pullback, the kernel of $B \underset{C}{\times} B' \to B'$ is also a kernel of $B \to C$ in $\mathcal{E}$, as sought.

A.7.18. **Lemma.** *$\mathcal{E}$ is closed under extension in $\mathcal{A}$.*

Proof. Let $0 \to A \to G \to B \to 0$ be a short exact sequence in $\mathcal{A}$ with A and B in $\mathcal{E}$. By A.7.15, the condition that $G \to B$ is epi in $\mathcal{A}$ implies that there is an admissible epimorphism $C \twoheadrightarrow B$ in $\mathcal{E}$ that factors as $C \to G \to B$. We consider the pullback diagram in $\mathcal{A}$:

(A.7.19)
$$\begin{array}{ccccc} & & A & = & A \\ & & \downarrow & & \downarrow \\ & & C \underset{B}{\times} G & \longrightarrow & G \\ & & \downarrow & \square & \downarrow \\ K & \rightarrowtail & C & \twoheadrightarrow & B \\ & & \downarrow & & \downarrow \\ & & 0 & & 0 \end{array}$$

The maps $1 : C \to C$ and $C \to G$ induce a map $C \to C \underset{B}{\times} G$ that splits $C \underset{B}{\times} G \to C$. Hence in $\mathcal{A}$, $C \underset{B}{\times} G$ is isomorphic to $A \oplus C$, an object of $\mathcal{E}$. Let $i : K \rightarrowtail C$ be the kernel of the admissible epi $C \twoheadrightarrow B$. As (A.7.19) is a pullback, $K \to C \underset{B}{\times} G$ is the kernel of the epimorphism in $\mathcal{A}$, $C \underset{B}{\times} G \to G$. We write $K \to C \underset{B}{\times} G \cong A \oplus C$ as $\begin{bmatrix} a \\ i \end{bmatrix}$, $a : K \to A$, $i : K \rightarrowtail C$. Then the exact sequence $K \to A \oplus C \to G \to 0$ in $\mathcal{A}$ shows that G is isomorphic to the pushout

(A.7.20)
$$\begin{array}{ccc} K & \xrightarrow{a} & A \\ {\scriptstyle i}\downarrow & & \downarrow \\ C & \longrightarrow & G \end{array}$$

But as K, A, C are in $\mathcal{E}$, and as i is an admissible monomorphism in $\mathcal{E}$, this square also has a pushout G' in $\mathcal{E}$. Then by Lemma A.8.1 below applied to the exact functor $i : \mathcal{E} \to \mathcal{A}$, we have an isomorphism $G \cong G'$ of G to an object of $\mathcal{E}$, as required.

A.7.21. Modulo A.8.1, this completes the proof of A.7.1.

A.8.1. **Lemma.** *Let $f : \mathcal{E} \to \mathcal{E}'$ be an exact functor between exact categories. The f preserves pushouts along an admissible mono, and f preserves pullbacks along an admissible epi.*

Proof. Consider $A \rightarrowtail B$ and $A \to C$ in $\mathcal{E}$, and the pushout $C \underset{A}{\cup} B$. We have an exact sequence $C \rightarrowtail C \underset{A}{\cup} B \twoheadrightarrow B/A$. Taking f of this sequence yields an exact sequence in $\mathcal{E}'$. Also, $fA \rightarrowtail fB$ is an admissible mono in $\mathcal{E}'$, so $\mathcal{E}'$ has a pushout $fC \underset{fA}{\cup} fB$ and an exact sequence $fC \rightarrowtail fC \underset{fA}{\cup} fB \twoheadrightarrow fB/fA$. Consider the diagram in $\mathcal{E}'$:

$$\begin{array}{ccccc} fC & \rightarrowtail & fC \underset{fA}{\cup} fB & \twoheadrightarrow & fB/fA \\ \| & & \downarrow & & \downarrow \cong \\ fC & \rightarrowtail & f(C \underset{A}{\cup} B) & \twoheadrightarrow & f(B/A) \end{array}$$

Note $f(B/A) \cong fB/fA$ by exactness. We consider the fully faithful exact embedding $\mathcal{E}' \to \mathcal{A}'$ of A.7.14. By the 5-lemma in the abelian category $\mathcal{A}'$ applied to the diagram, we see that $fC \underset{fA}{\cup} fB \to f(C \underset{A}{\cup} B)$ is an isomorphism in $\mathcal{A}'$, hence in $\mathcal{E}'$, as required. Dually, f preserves pullbacks along admissible epis.

A.8.2. **Proposition.** *Let $f : \mathcal{E} \to \mathcal{E}'$ be an exact functor between exact categories. Let $i : \mathcal{E} \to \mathcal{A}$ and $i' : \mathcal{E}' \to \mathcal{A}'$ be the Gabriel-Quillen embeddings into the categories of "left exact" functors.*

Then there is a right exact additive functor $f^* : \mathcal{A} \to \mathcal{A}'$ extending f in that $f^* \cdot i \cong i' \cdot f$. This f^* has an additive left exact right adjoint functor $f_* : \mathcal{A}' \to \mathcal{A}$.

Proof. We follow the analogy with [SGA 4] III. Consider $f_\# : \mathcal{B}' \to \mathcal{B}$ given by sending the additive functor $G : \mathcal{E}'^{\mathrm{op}} \to \mathbb{Z}$-mod to $f_\# G = G \cdot f$ with $(f_\# G)(E) = G(f(E))$ for E in $\mathcal{E}$. We claim that if G is 'left exact," so is $f_\# G$, so that $f_\#$ restricts to a functor $f_* : \mathcal{A}' \to \mathcal{A}$. For let G be "left exact" on $\mathcal{E}'$, and let $E \twoheadrightarrow F$ be an admissible epi in $\mathcal{E}$. Then $fE \twoheadrightarrow fF$ is an admissible epi in $\mathcal{E}'$, and $f(E \underset{F}{\times} E) \cong fE \underset{fF}{\times} fE$ by A.8.1. Then (A.8.3) is a difference kernel, as required (A.7.3)

$$G(fF) \to G(fE) \rightrightarrows G\left(f\left(E \underset{F}{\times} E\right)\right). \tag{A.8.3}$$

Clearly $f_\# : \mathcal{B}' \to \mathcal{B}$ preserves all limits. As the inclusions $\mathcal{A}' \to \mathcal{B}'$, $\mathcal{A} \to \mathcal{B}$ preserve and reflect all limits, it follows that the induced $f_* : \mathcal{A}' \to \mathcal{A}$ preserves all limits. In particular, it preserves finite products and kernels, so is additive and left exact.

As $\mathcal{A}'$ has limits and has a set of generators, the special adjoint functor theorem shows that f_* has a left adjoint $f^* : \mathcal{A} \to \mathcal{A}'$. This f^* must preserve all colimits, in particular direct sums and cokernels. So f^* is additive and right exact. As $\mathrm{Hom}(fE, G) \cong G(fE) \cong (f_*G)(E) \cong \mathrm{Hom}(E, f_*G)$ for E in $\mathcal{E}$ and G in $\mathcal{A}'$, it is clear that this adjoint f^* is isomorphic to f when restricted to $\mathcal{E}$.

A.8.4. In general $f^* : \mathcal{A} \to \mathcal{A}'$ need not be an exact functor of abelian categories. If R is a ring, and $\mathcal{E}$ is the exact category of finitely generated projective R-modules, $\mathcal{A}$ is the abelian category of all R-modules. For $R \to S$ a ring map, $f = S\underset{R}{\otimes}$ extends to $f^* = S\underset{R}{\otimes} : R\text{-mod} \to S\text{-mod}$, which need not be exact. Compare [SGA 4] IV 4.9.1.

However if $\mathcal{E}$ and $\mathcal{E}'$ have all pushouts and if $f : \mathcal{E} \to \mathcal{E}'$ preserves all pushouts, that is if $\mathcal{E}$ and $\mathcal{E}'$ are abelian categories with some exactness structure (possibly not the canonical one) and if $f : \mathcal{E} \to \mathcal{E}'$ is exact with respect to both the chosen and the canonical exactness structures, then $f^* : \mathcal{A} \to \mathcal{A}'$ is exact. We will not need this, but the interested reader may prove it as an exercise, guided by [SGA 4] IV 4.9.2, III 1.3.5, ...

A.8.5. Although $f^* : \mathcal{A} \to \mathcal{A}'$ may not be exact, it does preserve the exact sequences in $\mathcal{E}$, as $\mathcal{E} \to \mathcal{E}' \to \mathcal{A}'$ does so.

A.9.1. **Theorem.** *Let $\mathcal{E}$ be an exact category. Then*

(a) *There is a Karoubian* (A.6.1) *additive category $\mathcal{E}'$ and a fully faithful additive functor $f : \mathcal{E} \to \mathcal{E}'$ such that any additive functor from $\mathcal{E}$ to a Karoubian additive category factors uniquely-up-to-natural-isomorphism through $\mathcal{E} \to \mathcal{E}'$.*

(b) *Every object in $\mathcal{E}'$ is a direct summand in $\mathcal{E}'$ of an object in $\mathcal{E}$. We say a sequence in $\mathcal{E}'$ is exact iff it is a direct summand of an exact sequence in $\mathcal{E}$. This makes $\mathcal{E}'$ an exact category. The inclusion functor $f : \mathcal{E} \to \mathcal{E}'$ is exact and reflects exactness, and $\mathcal{E}$ is closed under extensions in $\mathcal{E}'$.*

(c) *$K(\mathcal{E})$ is a covering spectrum of $K(\mathcal{E}')$, in fact f induces an isomorphism of Quillen K-groups $K_n(\mathcal{E}) \xrightarrow{\cong} K_n(\mathcal{E}')$ for $n \geq 1$, and a monomorphism $K_0(\mathcal{E}) \subseteq K_0(\mathcal{E}')$.*

Proof. (Compare Karoubi [K] 1.2.2.) Let $\mathcal{E}'$ be the category whose objects are pairs (E,p), with E an object of $\mathcal{E}$ and $p = p^2$ an idempotent endomorphism of E. A map $e : (E,p) \to (E',p')$ in $\mathcal{E}'$ is a map $e : E \to E'$ such that $p'e = ep$. (The identity map of (E,p) is p.)

The functor $f : \mathcal{E} \to \mathcal{E}'$ sending E to $(E,1)$ is fully faithful.

The category $\mathcal{E}'$ is additive with $(E,p)\oplus(E',q) = (E\oplus E', p\oplus q)$. (E,p) is a summand of E, as there are obvious isomorphisms $(E,p)\oplus(E,1-p) \cong (E \oplus E, p \oplus 1-p) \cong (E \oplus E, 1 \oplus 0) \cong (E,1) = E$. It is easy to check that $\mathcal{E}'$ is Karoubian, and has the universal mapping property claimed for $\mathcal{E} \to \mathcal{E}'$.

To show that $\mathcal{E}'$ is an exact category, we consider the Gabriel-Quillen embedding $\mathcal{E} \to \mathcal{A}$. This induces a fully faithful functor between Karoubianizations, $\mathcal{E}' \to \mathcal{A}'$. By definition of exact sequence in the Karoubianization, and the fact $\mathcal{E} \to \mathcal{A}$ preserves and reflects exactness, the induced functor $\mathcal{E}' \to \mathcal{A}'$ preserves and reflects exact sequences. But as $\mathcal{A}$ already has images of idempotents, $\mathcal{A}'$ is equivalent to the abelian category $\mathcal{A}$. We claim that $\mathcal{E}'$ is closed under extensions in $\mathcal{A}$. For let

$0 \to A \to B \to C \to 0$ be an exact sequence in $\mathcal{A}$ with A and C in $\mathcal{E}'$. Then there are A', C' in $\mathcal{E}'$ so that $A \oplus A'$ and $C \oplus C'$ are isomorphic to objects of $\mathcal{E}$. The sequence $0 \to A \oplus A' \to C' \oplus B \oplus A' \to C' \oplus C \to 0$ is exact in $\mathcal{A}$, and shows that $C' \oplus B \oplus A'$ is isomorphic to an object of $\mathcal{E}$, since $\mathcal{E}$ is closed under extensions in $\mathcal{A}$. Thus B is a summand of an object of $\mathcal{E}$, hence is isomorphic to the image of an idempotent in $\mathcal{E}$, and hence is isomorphic to an object of $\mathcal{E}'$. This proves the claim. Now $\mathcal{E}'$ is an exact category by A.4. As the functors $\mathcal{E} \to \mathcal{A}$ and $\mathcal{E}' \to \mathcal{A}$ preserve and reflect exactness, so does the functor $\mathcal{E} \to \mathcal{E}'$.

It remains to prove part (c). But this follows from (b) and the classical cofinality theorems in Quillen K-theory, e.g., [Gr2] 6.1 or [Sta] 2.1.

A.9.2. The point is, that because of A.9.1(c), it is no real loss of generality in K-theory to consider only Karoubian exact categories $\mathcal{E}$. For these exact categories, A.6.2 and A.7.16(b) show that the Gabriel-Quillen embedding $\mathcal{E} \to \mathcal{A}$ satisfies hypothesis 1.11.3.1. So it is harmless to make this hypothesis in K-theory.

Appendix B

Modules vs. Quasi-coherent modules

B.0. This appendix reviews the relations between the categories of quasi-coherent $\mathcal{O}_X$-modules and of all $\mathcal{O}_X$-modules in the Zariski topos of a scheme X. Most of these facts are well-known in outline, although many people exhibit some confusion and fuzziness on the details when pressed. The theory of the "coherator" is more esoteric, but essential for this paper.

The results in this appendix are all due to Grothendieck and his school, and are collected from scattered parts of [Gro], [EGA], [SGA 4], and [SGA 6], with some slight sharpening due to the new concept of "semi-separated."

B.1. For X a scheme, let $\mathcal{O}_X$-Mod be the abelian category of all sheaves of $\mathcal{O}_X$-modules (for the Zariski topology on X), and let $D(X) = D(\mathcal{O}_X\text{-Mod})$ be the derived category of $\mathcal{O}_X$-Mod.

The category $\mathcal{O}_X$-Mod has all limits and colimits, and has a set of generators. Direct colimits are exact. Hence $\mathcal{O}_X$-Mod is a Grothendieck abelian category and has enough injectives. Also it has an internal hom sheaf, Hom(,) and a tensor product $\otimes_{\mathcal{O}_X}$ ([Gro], [SGA 4] IV).

B.2. Let Qcoh(X) be the full subcategory of $\mathcal{O}_X$-Mod consisting of the quasi-coherent $\mathcal{O}_X$-modules, i.e., those which locally on X have a presentation by free $\mathcal{O}_X$-modules. This category Qcoh(X) includes all $\mathcal{O}_X$-modules of finite presentation.

Let φ : Qcoh$(X) \to \mathcal{O}_X$-Mod be the inclusion. Then Qcoh(X) is an abelian category, closed under extensions and tensor products in $\mathcal{O}_X$-Mod. The functor φ is exact and reflects exactness. In particular, φ preserves all finite limits and colimits. It also preserves and reflects infinite direct sums, and hence all colimits. Thus Qcoh(X) has all colimits, and satisfies Grothendieck's axiom AB5 that direct colimits are exact. For $\mathcal{F}$ a finitely presented $\mathcal{O}_X$-module, and $\mathcal{G}$ a quasi-coherent $\mathcal{O}_X$-module, Hom$(\mathcal{F}, \mathcal{G})$ is quasi-coherent. ([EGA] I 2.2).

It seems to be unknown whether, for general schemes X, Qcoh(X) has a set of generators, enough injectives, or even all limits.

When X is affine, say $X = \text{Spec}(A)$, the category Qcoh(X) is equivalent to the category of A-modules. Of course, in this case Qcoh(X) has all limits, a set of generators, and enough injectives.

In general, let $D(\text{Qcoh}(X))$ be the derived category of Qcoh(X).

B.3. If X is a quasi-compact and quasi-separated scheme, every sheaf in Qcoh(X) is a direct colimit of its sub-$\mathcal{O}_X$-modules of finite type. Also, every sheaf in Qcoh(X) is a filtering colimit of finitely presented $\mathcal{O}_X$-modules. ([EGA] I 6.9.9, 6.9.12.) In this case, the set of finitely presented $\mathcal{O}_X$-modules forms a set of generators for Qcoh(X), which is then a Grothendieck abelian category and has enough injectives (cf. B.12.).

B.4. DANGER: For a general quasi-compact and quasi-separated X, φ : Qcoh (X) $\to \mathcal{O}_X$-Mod does *not* preserve injectives, nor need it send injectives in Qcoh(X) to flasque sheaves in $\mathcal{O}_X$-Mod. For Verdier's counterexample in which X is even affine with noetherian underlying space, see [SGA 6] II App. I 0.1.

On the other hand, if X is a noetherian scheme, then φ does preserve injectives. For let $\mathcal{F}$ be injective in Qcoh(X). By [H] II 7.18, there is a $\mathcal{G}$ in Qcoh(X) with $\varphi(\mathcal{G})$ injective in $\mathcal{O}_X$-Mod, and a monomorphism $\mathcal{F} \rightarrowtail \mathcal{G}$. In Qcoh($X$) this splits as $\mathcal{F}$ is injective. Then $\varphi(\mathcal{F})$ is a direct summand of the injective $\varphi(\mathcal{G})$, and so is injective in $\mathcal{O}_X$-Mod, as required.

The fact that φ does preserve injectives in the noetherian case can lure one to a false sense of security. In general, when one computes by injective resolutions various derived functors evaluated on a quasi-coherent sheaf $\mathcal{F}$, one must distinguish between the possibly different derived functors taken in $\mathcal{O}_X$-Mod and those taken in Qcoh(X) (see [SGA 6] II App. I 0.2). We may add "$\mathcal{O}_X$-Mod" or "Qcoh(X)" to the name of the derived functor to indicate the distinction, so that we have $R^n(\mathcal{O}_X\text{-Mod})f_*(\mathcal{F})$ vs. $R^n(\text{Qcoh}(X))f_*(\mathcal{F})$ for a quasi-coherent $\mathcal{F}$. When these are shown to be equivalent in some cases, the notation reverts back to $R^n f_*(\mathcal{F})$ (e.g., B.8).

B.5. If $j : U \to X$ is an open immersion of schemes, and $\mathcal{F}$ is an injective in $\mathcal{O}_X$-Mod, then $j^*\mathcal{F}$ is an injective in $\mathcal{O}_X$-Mod. For j^* has an exact left adjoint functor $j!$, extension by 0 off U ([SGA 4] V 4.11, IV 11.3.1). As $j!$ does not preserve quasi-coherence, this argument does not apply to injectives in Qcoh(X), and in fact j^* need not send them to injectives in Qcoh(U) ([SGA 6] II App. I).

However, if X is noetherian and $\mathcal{F}$ is injective in Qcoh(X) then $j^*\mathcal{F}$ is injective in Qcoh(U). For by B.4, $\varphi\mathcal{F}$ is injective in $\mathcal{O}_X$-Mod, so $j^*\varphi\mathcal{F} = \varphi j^*\mathcal{F}$ is injective in $\mathcal{O}_U$-Mod. But as φ is exact and fully faithful, this implies that $j^*\mathcal{F}$ is injective in Qcoh(U). (For another proof that $j^*\mathcal{F}$ is injective in the noetherian case, use the pro-existing left adjoint denoted $j!$ in Deligne's letter in [H] p.411.)

B.6. **Lemma.** *For X a quasi-compact and quasi-separated scheme, the cohomology functors $H^k(X;\ ) : \mathcal{O}_X$- Mod $\to \mathbb{Z}$-Mod preserve direct colimits.*

For $f : X \to Y$ a quasi-compact and quasi-separated map of schemes,

$R^k f_* = R^k(\mathcal{O}_X\text{-}Mod)f_* : \mathcal{O}_X\text{-}Mod \to \mathcal{O}_Y\text{-}Mod$ preserves direct colimits. Also, if $\mathcal{F}$ is a quasi-coherent sheaf, $f_*\mathcal{F}$ and indeed the $R^k(\mathcal{O}_X\text{-}Mod)f_*(\mathcal{F})$ are quasi-coherent for such an f.

Proof. To prove the first statement, we note by [SGA 4] VI 1.22 that the Zariski topos is coherent in the sense of [SGA 4] VI, and then we appeal to [SGA 4] VI 5.2. Similarly, $R^k f_*$ preserves direct colimits by appeal to [SGA 4] VI 5.1, or by the obvious reduction to the first statement.

The last statement is [EGA] III 1.4.10, IV 1.7.21, and is based on using Čech complexes of finite affine hypercovers ([SGA 4] V Section 7) to compute cohomology.

B.7. We say a scheme is *semi-separated* if there is a set $\mathcal{A} = \{U_\alpha\}$ of open subschemes of X which is a basis for the topology of X such that each U_α in $\mathcal{A}$ is affine, and which is such that the intersection $U_\alpha \cap U_\beta$ of any two members of $\mathcal{A}$ is also in $\mathcal{A}$. This $\mathcal{A}$ is then said to be a *semi-separating affine basis.* Note that any open or closed subscheme of a semi-separated scheme is itself semi-separated.

We say an open cover $\mathcal{B} = \{V_\alpha\}$ of a scheme X is a *semi-separating cover* if all the V_α, and also all the pairwise intersections $V_\alpha \cap V_\beta$ are affine schemes. Then the open immersions $V_\alpha \to X$ are affine maps, and it follows all finite intersections of V_α are affine. If X has a semi-separating cover $\mathcal{B}$, then X is semi-separated, for we take a semi-separating affine basis $\mathcal{A}$ to consist of all open affine subschemes U of X for which there is some V_β in $\mathcal{B}$ with $U \subseteq V_\beta$. Similarly, X is semi-separated if it has an open cover $\{V_\alpha\}$ with each V_α semi-separated and each open immersion $V_\alpha \cap V_\beta \to V_\beta$ an affine morphism.

We say a map $f : X \to Y$ of schemes is semi-separated if for every affine scheme Z and map $Z \to Y$, then the fibre product $Z \times_Y X$ is a semi-separated scheme. The class of semi-separated maps is closed under composition and base-change. If $f : X \to Y$ is a semi-separated map and Y is a semi-separated scheme, then X is a semi-separated scheme. (Consider $\mathcal{B} = \{f^{-1}(U_\alpha)\}$ on X for $\{U_\alpha\}$ a semi-separating basis for Y. Note each $f^{-1}(U_\alpha)$ is semi-separated, and that each $f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) \to f^{-1}(U_\beta)$ is an affine map as it is the base-change of the affine map $U_\alpha \cap U_\beta \to U_\beta$.)

Given a morphism $f : X \to Y$, suppose Y has an open cover by affines $\{V_\alpha\}$ with all $V_\alpha \cap V_\beta \to V_\alpha$ being affine morphisms and all $f^{-1}(V_\alpha)$ being semi-separated schemes. Then $f : X \to Y$ is a semi-separated morphism. In particular, any map between semi-separated schemes is a semi-separated morphism.

A semi-separated scheme or morphism is quasi-separated ([EGA] I

6.1.12). A separated scheme is semi-separated, with semi-separating basis $\mathcal{A}$ consisting of all affine open subschemes of X. A separated map is semi-separated.

A scheme with an ample family of line bundles (2.1.1, or [SGA 6] II 2.2.4) is semi-separated. For let $\mathcal{A}$ be the set of all affine opens of the form $X_f = \{x | f(x) \neq 0\}$ as f runs over the set of all those global sections of tensor powers of line bundles in the family for which X_f is indeed affine. Then $\mathcal{A}$ is a basis for the topology by ampleness. Also $X_f \cap X_g = X_{fg}$, and this is affine if either X_f or X_g is by [EGA] II 5.5.8.

B.8. **Proposition.** *Let X be either noetherian, or else quasi-compact and semi-separated* (B.7). *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then the canonical map is an isomorphism for all integers $k \geq 0$:*

$$\text{(B.8.1)} \qquad R^k(Qcoh)\Gamma(X;\mathcal{F}) \xrightarrow{\cong} R^k\,(\mathcal{O}_X\text{-}Mod)\,\Gamma(X;\mathcal{F}) = H^k(X;\mathcal{F}).$$

Moreover, if $\mathcal{V} = \{V_\alpha\}$ is a semi-separating open cover of X (so the V_α and all their finite intersections are affine), then there is a canonical isomorphism to the Čech cohomology of $\mathcal{V}$

$$\text{(B.8.2)} \qquad H^k(X;\mathcal{F}) \cong \check{H}^k(\mathcal{V};\mathcal{F}).$$

Proof. B.8.2 follows from the collapse of the Cartan-Leray spectral sequence in the usual way, [God] II 5.4.1, just as in [EGA] III 1.4.1. The key point is that since the intersections $V_{\alpha_1} \cap \ldots \cap V_{\alpha_n}$ are affine, $H^q(V_{\alpha_1} \cap \ldots V_{\alpha_n}; \varphi\mathcal{F}) = 0$ for $q > 0$ by Serre's Theorem [EGA] III 1.3.1.

For X noetherian, B.8.1 holds as $\varphi :$ Qcoh$(X) \to \mathcal{O}_X$-Mod preserves injective resolutions by B.4.

To prove B.8.1 for X quasi-compact and semi-separated, we take a semi-separating cover $\mathcal{V} = \{V_\alpha\}$. We may assume $\mathcal{V}$ is a finite cover, on passing to a subcover on the quasi-compact X. For all finite sequences of indices $I = (\alpha_1, \ldots, \alpha_n)$, let $V_I = V_{\alpha_1} \cap \ldots \cap V_{\alpha_n}$, and let $j_I = V_I \to X$ be the open immersion. As $\mathcal{V}$ is semi- separating, each V_I is an affine scheme and each j_I is an affine map. In particular j_{I*} preserves quasi-coherence.

We consider the Čech complex of quasi-coherent sheaves

$$\text{(B.8.3)} \qquad 0 \to \mathcal{F} \to \bigoplus_\alpha j_{\alpha *} j_\alpha^* \mathcal{F} \to \bigoplus_{I=(\alpha_0,\alpha_1)} j_{I*} j_I^* \mathcal{F} \to \bigoplus_{I=(\alpha_0,\alpha_1,\alpha_2)} j_{I*} j_I^* \mathcal{F} \to \ldots .$$

This is an exact sequence of sheaves, and in fact the complex has a canonical chain contraction when restricted to any V_α.

As V_I is affine, $\Gamma(V_I;\ )$ is exact on Qcoh(V_I), so $R^k($Qcoh$)\Gamma(V_I;\mathcal{F}) = 0$ for $k > 0$. Similarly, as j_I is an affine map, $R^k($Qcoh$)j_{I*} = 0$ for

$k > 0$. So in the derived category $R^{\cdot}(\text{Qcoh})\Gamma(V_I;\ ) = \Gamma(V_I;\ )$, and $R^{\cdot}(\text{Qcoh})j_{I*} = j_{I*}$.

We consider the Grothendieck spectral sequence (B.8.4) for the derived functors of the composite $\Gamma(V_I;\ ) = \Gamma(X;\ )\cdot j_{I*}$

(B.8.4)
$$E_2^{p,q} = R^p(\text{Qcoh})\Gamma(X; R^q(\text{Qcoh})j_{I*}(\ )) \Longrightarrow R^{p+q}(\text{Qcoh})\Gamma(V_I;\ ).$$

By the above, it collapses to yield isomorphisms for $k > 0$, $R^k(\text{Qcoh})\Gamma(X; j_{I*}(\ )) \cong R^k(\text{Qcoh})\Gamma(V_I;\ ) \cong 0$. So the sheaves $j_{I*}j_I^*\mathcal{F}$ are acyclic for $R^{\cdot}(\text{Qcoh})\Gamma(X;\ )$. Now the usual hypercohomology spectral sequence that results from applying $R^{\cdot}(\text{Qcoh})\Gamma(X;\ )$ to the Čech resolution B.8.3 of $\mathcal{F}$ collapses, yielding isomorphisms for $k \geq 0$

(B.8.5) $R^k(\text{Qcoh})\Gamma(X;\mathcal{F})$
$$\cong H^k\left(\Gamma\left(X; \bigoplus_\alpha j_{\alpha*}j_\alpha^*\mathcal{F}\right) \to \ldots \Gamma\left(X; \bigoplus_I j_{I*}j_I^*\mathcal{F}\right) \to \ldots\right) \cong \check{H}^k(\mathcal{V};\mathcal{F}).$$

Comparing this with (B.8.2) yields (B.8.1).

B.9. **Corollary.** *Let $f : X \to Y$ be either a quasi-compact and semi-separated map of schemes, or else a quasi-compact and quasi-separated map of schemes with X locally noetherian. Then for an quasi-coherent sheaf $\mathcal{F}$ on X, the canonical map is an isomorphism for all integers k*

$$\varphi : R^k(\mathit{Qcoh})f_*\mathcal{F} \xrightarrow{\cong} R^k(\mathcal{O}_X\text{-}\mathit{Mod})f_*\mathcal{F}.$$

Proof. This follows from B.8, as R^kf_* is the sheafification of $V \mapsto R^k\Gamma(f^{-1}(V);\ )$. We apply B.8 to $f^{-1}(V)$ for V affine open in Y.

(Note Qcoh(X) might not have enough injectives under our hypothesis, but that the Qcoh($f^{-1}(V)$) will for V affine in Y, and this suffices to define $R^*(\text{Qcoh})f_*$.)

B.10. The conclusion of B.9 is that there is a natural isomorphism

$$R^{\cdot}(\mathcal{O}_X\text{-Mod})f_*\cdot\varphi \cong \varphi\cdot R(\text{Qcoh})f_* : D^+(\text{Qcoh}(X)) \to D^+(Y).$$

B.11. **Proposition.** *Let $f : X \to Y$ be a quasi-compact and quasi-separated map with Y quasi-compact. Then there exists an integer N such that for all $k \geq N$ and all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$, one has $R^kf_*(\mathcal{F}) = 0$ (i.e., $R^k(\mathcal{O}_X\text{-Mod})f_*(\mathcal{F}) = 0$ for $k \geq N$).*

Moreover N can be chosen to be universal in that same N works for any pullback $f' : X' \to Y'$ of f by any map $Y' \to Y$.

Proof. ([EGA] III 1.4.12, IV 1.7.21). The question is local on Y for Y quasi-compact, so we reduce to the case where Y is affine, and hence where $X = f^{-1}(Y)$ is quasi-compact and quasi-separated. We first consider the case where X is also semi-separated. Then let $\mathcal{W}$ be a finite semi-separating cover, and let N be the number of opens in $\mathcal{W}$. By B.8, $H^k(X, \mathcal{F}) \cong \check{H}^k(\mathcal{W}; \mathcal{F})$. But computing $\check{H}^k(\mathcal{W}; \mathcal{F})$ with the Čech complex of alternating cochains shows that it is 0 for $k > N$. Moreover, given any $Y' \to Y$, let V' be any affine open in Y'. Then $V' \underset{Y}{\times} \mathcal{W}$ is a semi-separating cover of $V \underset{Y}{\times} X$, so $H^k(V' \underset{Y}{\times} X; \mathcal{F}') = 0$ for $k > N$. Sheafifying this yields $R^k f'_*(\mathcal{F}') = 0$ for $k > N$ and $\mathcal{F}'$ quasi-coherent on $X' = Y' \underset{Y}{\times} X$.

Now we do the general case without assuming X is semi-separated. Let $\mathcal{W}$ be a finite affine cover of X. As X is quasi-separated, the $W_I = W_{i_0} \cap \ldots \cap W_{I_n}$ are quasi-compact open in the affine W_{i_0}, so all the W_I are quasi-affine, hence semi-separated. (For V' affine in Y', $V' \to Y$ is affine as Y is affine, so $V' \underset{Y}{\times} \mathcal{W}$ is an affine cover of $V' \underset{Y}{\times} X$.) By the semi-separated case, there is an integer N_1 such that for all the finitely many W_I, $H^k(W_I; \mathcal{F}) = 0$ for $k > N_1$. (Moreover, N_1 is universal in that $H^k(V' \underset{Y}{\times} W_I; \mathcal{F}') = 0$ for $k > N_1$ for the $V' \underset{Y}{\times} W_I$ which are affine over W_I, and any quasi-coherent sheaf $\mathcal{F}'$.) We now consider the Cartan-Leray spectral sequence

$$\text{(B.11.1)} \qquad E_2^{p,q} = \check{H}^p\left(\mathcal{W}; H^q\left(W_J; \mathcal{F}\right)\right) \Longrightarrow H^{p+q}(X; \mathcal{F}).$$

If $\mathcal{W}$ has N_2 open sets, $\check{H}^p(\mathcal{W};\ ) = 0$ for $p > N_2$, and it follows that $H^k(X; \mathcal{F}) = 0$ for $k > N_1 + N_2$ and $\mathcal{F}$ quasi-coherent. (This holds also for $V' \underset{Y}{\times} X$.)

B.12. **Lemma** ([SGA 6] II 3.2). *Let X be a quasi-compact and quasi-separated scheme. Then the exact inclusion functor $\varphi : Qcoh(X) \to \mathcal{O}_X$-Mod has a right adjoint, the coherator $Q : \mathcal{O}_X$-Mod $\to Qcoh(X)$.*

The adjunction map $1 \to Q \cdot \varphi$ is an isomorphism, so $Qcoh(X)$ is a reflexive subcategory of $\mathcal{O}_X$-Mod. In particular, $Qcoh(X)$ has all limits.

Proof. By B.3 and B.2, Qcoh(X) has a set of generators and all colimits. As φ preserves colimits, the special adjoint functor theorem insures that φ has a right adjoint Q. As φ is fully faithful, the adjunction map $1 \to Q\varphi$ is an isomorphism. (For, under the adjunction isomorphism $\text{Mor}(\varphi\mathcal{F}, (\ )) \cong \text{Mor}(\mathcal{F}, Q(\ ))$, the map induced by $1 \to$

$Q\varphi$, $\text{Mor}(\mathcal{F},\mathcal{G}) \to \text{Mor}(\mathcal{F},Q\varphi\mathcal{G})$ is the isomorphism $\varphi : \text{Mor}(\mathcal{F},\mathcal{G}) \cong \text{Mor}(\varphi\mathcal{F},\varphi\mathcal{G})$.)

As the category $\mathcal{O}_X$-Mod has all limits, so does its reflexive subcategory Qcoh(X), as Q sends limits taken in $\mathcal{O}_X$-Mod to limits in Qcoh(X).

B.13. The coherator $Q : \mathcal{O}_X\text{-Mod} \to \text{Qcoh}(X)$ preserves all limits as it is the right adjoint of the functor φ. As φ is exact, hence preserves monomorphisms, Q sends injectives in $\mathcal{O}_X$-Mod to injectives in Qcoh(X).

Let $f : X \to Y$ be a quasi-compact and quasi-separated map of schemes. As f^* preserves quasi-coherence, $\varphi_X \cdot f^* = f^* \cdot \varphi_Y$. As Q is right adjoint to φ and f_* is right adjoint to f^*, it follows that $Q_Y \cdot f_* = f_* \cdot Q_X$. (We note that indeed f_* restricts to a functor $f_* : \text{Qcoh}(X) \to \text{Qcoh}(Y)$ by B.6.)

B.14. For $X = \text{Spec}(A)$ an affine scheme, Q_X is clearly the functor sending an $\mathcal{O}_X$-module $\mathcal{F}$ to the quasi-coherent sheaf associated to the A-module $\Gamma(X;\mathcal{F})$. For this functor is the adjoint to φ.

To deduce a formula for Q_X on a general quasi-compact and quasi-separated scheme X, we let $\{U_i\}$ be a finite cover of X by open affines. Each $U_i \cap U_j$ is quasi-compact, so we can choose a finite cover $\{U_{ijk}\}$ of $U_i \cap U_j$ by open affines. We denote all the various open immersions $U_{ijk} \to X$ as $j : U_{ijk} \to X$.

For any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, the sheaf axioms show that $\mathcal{F}$ is the difference kernel of the start of a Čech hypercover complex, where the two right maps are induced by $U_{ijk} \to U_i \cap U_j \to U_i$ and $U_{ijk} \to U_i \cap U_j \to U_j$

$$0 \to \mathcal{F} \to \bigoplus_i j_* (\mathcal{F}|U_i) \rightrightarrows \bigoplus_{(i,j,k)} j_* (\mathcal{F}|U_{ijk}) . \tag{B.14.1}$$

Hence by B.13, we have

$$Q_X\mathcal{F} = \ker\left(\bigoplus_i j_* (Q_{U_i} (\mathcal{F}|U_i)) \rightrightarrows \bigoplus_{(ijk)} j_* \left(Q_{U_{ijk}} (\mathcal{F}|U_{ijk})\right)\right) . \tag{B.14.2}$$

Here, Q_{U_i} and $Q_{U_{ijk}}$ are given by the first paragraph, as the U_i and U_{ijk} are affine.

B.15. **Lemma.** *For X a quasi-compact and quasi-separated scheme, $Q : \mathcal{O}_X$-Mod $\to$ Qcoh(X) preserves direct colimits.*

Proof. The global section functor $\Gamma(X;\ )$ preserves direct colimits by B.6. For A a commutative ring, the "associated sheaf" equivalence A-Mod $\to$ Qcoh(Spec(A)) preserves direct colimits. This proves the lemma for $X = \text{Spec}(A)$ affine, by B.14. The case for a general X now follows from (B.14.2), the fact that difference kernels commute with the exact direct colimits of a Grothendieck abelian category, the result for Q_U in the case U is affine, and B.6 for the maps j_*.

B.16. **Proposition** (cf. [SGA 6] II 3.5). *Let X be either quasi-compact and semi-separated, or else noetherian. Then for any positive integer $q > 0$ and any quasi-coherent sheaf $\mathcal{F}$ on X, one has $(R^q Q)(\varphi\mathcal{F}) = 0$. We recall that $R^0 Q\varphi\mathcal{F} = Q\varphi\mathcal{F} = \mathcal{F}$.*

For any complex $E^\cdot$ in $D^+(Qcoh(X))$, the canonical map $E^\cdot \to (RQ)(\varphi E^\cdot)$ is a quasi-isomorphism.

For any complex $F^\cdot$ in $D^+(\mathcal{O}_X\text{-Mod})$ with quasi-coherent cohomology, the canonical map $\varphi(RQ(F^\cdot)) \to F^\cdot$ is a quasi-isomorphism.

Proof. The last two statements result from the first by the collapse of the usual hypercohomology spectral sequence

$$R^p Q\left(H^q\left(G^\cdot\right)\right) \Longrightarrow H^{p+q}\left(R^\cdot Q(G^\cdot)\right). \tag{B.16.1}$$

To prove the first statement, we first consider the case where X is affine. Then by B.14, $R^q Q(\varphi\mathcal{F}) = H^q(X;\varphi\mathcal{F})$. But this is 0 for $q > 0$ and X affine, by Serre's Theorem [EGA] III 1.3.1.

For the general case of X quasi-compact and quasi-separated, we consider the exact Čech complex of sheaves (B.8.3) of a finite semi-separating cover $\mathcal{V}$. Applying $RQ \cdot \varphi$ and considering the resulting bicomplex, we get a canonical spectral sequence

$$H^p\left(\ldots \to \bigoplus_I R^q\left(Qj_{I*}\right)\left(j_I^*\varphi\mathcal{F}\right) \to \ldots\right) \Longrightarrow \left(R^{p+q}Q\right)(\varphi\mathcal{F}). \tag{B.16.2}$$

As $j_I : V_I \to X$ is an affine map, $R^k j_{I*}(\mathcal{G}) = 0$ for $k > 0$ and $\mathcal{G}$ quasi-coherent. As V_I is affine $R^k Q(\mathcal{G}) = 0$ for $k > 0$ and $\mathcal{G}$ an $\mathcal{O}_{V_I}$-module by the affine case done above. As $Q_X \cdot j_* = j_* \cdot Q_{V_I}$, the resulting collapse of the Grothendieck spectral sequence $R^a j_*(R^b Q_{V_I}(\mathcal{G})) \Longrightarrow R^{a+b}(Q_X j_{I*})(\mathcal{G})$ yields that $R^q(Qj_{I*}) = 0$ for $q > 0$. Then the spectral sequence (B.16.2) also collapses. This yields that $R^k Q(\varphi\mathcal{F})$ is just H^k of the complex formed by applying Q to the complex (B.8.3) for $\varphi\mathcal{F}$. But commuting φ past the j_* and j^*, and using $Q\varphi \cong 1$, this is just the complex (B.8.3) for $\mathcal{F}$, which is exact. Thus $R^k Q(\varphi\mathcal{F}) = 0$ for $k > 0$.

To prove the first statement in the noetherian case, we take an injective resolution of $\mathcal{F}$ in Qcoh(X). By B.4, for X noetherian, φ of this resolution is an injective resolution of $\varphi\mathcal{F}$ in $\mathcal{O}_X$-Mod. Taking Q of this yields the original resolution as $Q\varphi \cong 1$, so using this resolution to compute $R^* Q$ yields that $R^k Q(\varphi\mathcal{F}) = 0$ for $k > 0$.

B.17. The right exact functor Q induces a derived functor $RQ : D^+(\mathcal{O}_X\text{-Mod}) \to D^+(\text{Qcoh}(X))$. But this derived functor does not extend to unbounded complexes without further assumptions. Suppose however that RQ has finite cohomological dimension, i.e., that there is an integer N such that for all $q > N$ and all $\mathcal{O}_X$-modules $\mathcal{G}$, $R^q Q(\mathcal{G}) = 0$. Then RQ extends to derived functors $RQ : D(\mathcal{O}_X\text{-Mod}) \to D(\text{Qcoh}(X))$ and $RQ : D^-(\mathcal{O}_X\text{-Mod}) \to D^-(\text{Qcoh}(X))$, by [H] I Section 7, or [V] II Section 2 no. 2 Corollary 2-2 to Theorem 2.2 (one learns to appreciate the elegant and complete Wittgenstein-Grothendieck multi-decimals).

If X is quasi-compact and semi-separated, and if there is a bound on the $\mathcal{O}_U$-Mod cohomological dimension of the functor $H^*(U;\ )$ for all finite intersections $U = V_I = V_{i_0} \cap \ldots \cap V_{i_n}$ of the opens in some finite semi-separating cover $\mathcal{V} = \{V_i\}$ of X, then RQ has finite cohomological dimension, as we see by examining the proof of B.16. Indeed, the argument of [SGA 6] II 3.7 to prove this in the separated case immediately generalizes to the semi-separated case.

If X is neotherian, and if there is a uniform bound on the $\mathcal{O}_U$-Mod cohomological dimension of $H^*(U;\ )$ for all open U in X, then RQ has finite cohomological dimension by [SGA 6] II 3.7.

We recall that if X has a noetherian underlying space of finite Krull dimension, then for any open U in X, the $\mathcal{O}_U$-Mod cohomological dimension of $H^*(U;\ )$ is at most the Krull dimension of X by [Gro] 3.6.5.

In any case where RQ has finite cohomological dimension and extends to a derived functor on $D(\mathcal{O}_X\text{-Mod})$, the canonical maps $E^\cdot \xrightarrow{\sim} RQ(\varphi E^\cdot)$ and $\varphi RQ(F^\cdot) \xrightarrow{\sim} F^\cdot$ will be quasi-isomorphisms for $E^\cdot$ in $D(\text{Qcoh}(X))$ and for $F^\cdot$ in $D(\mathcal{O}_X\text{-Mod})$ with quasi-coherent cohomology. This follows as in B.16 by collapse of the spectral sequence B.16.1, which converges strongly even for unbounded complexes thanks to the finite cohomological dimension of RQ.

In particular, B.16 remains true if we delete the hypotheses that $E^\cdot$ and $F^\cdot$ are cohomologically bounded below, and at the same time add the hypothesis that either X is noetherian of finite Krull dimension, or else is semi-separated and has underlying space a noetherian space of finite Krull dimension.

Appendix C:
Absolute noetherian approximation

In this appendix we review part of Grothendieck's theory of inverse limits of schemes from [EGA] IV Section 8. We then extend his theory of noetherian approximation to the case of general quasi-compact and quasi-separated schemes which are not necessarily finitely presented over an affine. Presumably, this would have been in [EGA] V or VI.

C.1. ([EGA] IV 8.2) Consider an inverse system of quasi-compact and quasi-separated schemes X_α, where the maps of the system (the "bonding maps") $f_{\alpha\beta} : X_\alpha \to X_\beta$ are all affine morphisms. Then an inverse limit scheme $X = \varprojlim X_\alpha$ exists, and the canonical maps $f_\alpha : X \to X_\alpha$ are all affine morphisms. Indeed, over an affine open $\mathrm{Spec}(A_\beta) \subseteq X_\beta$, $f^{-1}(\mathrm{Spec}(A_\beta))$ is $\mathrm{Spec}(A)$ for $A = \varinjlim A_\alpha$, where A_α runs over the direct system of rings A_α such that $\mathrm{Spec}(A_\alpha) = f_{\alpha\beta}^{-1}(\mathrm{Spec}(A_\beta)) \subseteq X_\alpha$ for $\alpha \geq \beta$.

C.2. ([EGA] IV 8.3.11, 8.6.3). For a system as in C.1, given any quasi-compact open $U \subseteq X = \varprojlim X_\alpha$, there is an α and a quasi-compact open $U_\alpha \subseteq X_\alpha$ such that $U = f_\alpha^{-1}(U_\alpha)$. If we set $U_\beta = X_\beta \times_{X_\alpha} U_\alpha$, then $U = \varprojlim U_\beta$, with the limit taken over the cofinal system $\beta \geq \alpha$. Also the closed subspace $X - U$ will be $f_\beta^{-1}(X_\beta - U_\beta)$ for any $\beta \geq \alpha$.

C.3. ([EGA] IV 8.8, 8.10.5). Given a system X_α as in C.1, let $g : Y \to X$ be a scheme finitely presented over X. Then there is an α and a finitely presented $g_\alpha : Y_\alpha \to X_\alpha$ such that

$$g = g_\alpha \times_{X_\alpha} X : Y = Y_\alpha \times_{X_\alpha} X \to X.$$

Then $Y = \varprojlim Y_\beta$ for $Y_\beta = Y_\alpha \times_{X_\alpha} X_\beta$ over the cofinal system of $\beta \geq \alpha$.

If $h : Z \to X$ is also finitely presented, and $k : Z \to Y = \varprojlim Y_\beta$ is any map over X, it follows that $k : Z \to Y$ is finitely presented. Hence k is $Y \times_{Y\alpha} k_\alpha$ for some α and some $k_\alpha : Z_\alpha \to Y_\alpha$. The finitely presented map $k : Z \to Y$ is respectively an immersion, a closed immersion, and open immersion, separated, surjective, affine, quasi-affine, finite, quasi-finite, proper, projective, or quasi-projective, iff k_β has the same property for all $\beta \geq \alpha$ for some α, iff k_α has the same property for some α.

C.4. ([EGA] IV 8.5). Suppose $X = \varprojlim X_\alpha$ for a system as in C.1. Then for any finitely presented, hence quasi-coherent, $\mathcal{O}_X$-module $\mathcal{F}$, there exists an α, a finitely presented $\mathcal{O}_X$-module $\mathcal{F}_\alpha$ on X_α, and an isomorphism $\mathcal{F} \cong f_\alpha^* \mathcal{F}_\alpha$. Let $\mathcal{F}_\beta = f_{\beta\alpha}^* \mathcal{F}_\alpha$ on X_β for all $\beta \geq \alpha$. Then $\mathcal{F} \cong \varinjlim \mathcal{F}_\beta$ as a module over $\mathcal{O}_X = \varinjlim \mathcal{O}_{X_\beta}$ as a sheaf on X_α.

The sheaf $\mathcal{F}$ is a vector bundle on X iff there is an α such that $\mathcal{F}_\alpha$ is a vector bundle on X_α. When the later condition is satisfied, $\mathcal{F}_\beta$ will be a vector bundle on X_β for all $\beta \geq \alpha$.

For any map $k : \mathcal{F} \to \mathcal{G}$ between finitely presented $\mathcal{O}_X$-modules, there will be an α and a map $k_\alpha : \mathcal{F}_\alpha \to \mathcal{G}_\alpha$ such that k corresponds to $f_\alpha^*(k_\alpha)$ under the isomorphisms $\mathcal{F} \cong \mathcal{F}_\alpha$, $\mathcal{G} \cong f_\alpha^* \mathcal{G}_\alpha$. The map k is an isomorphism on X iff there is an α such that k_α is an isomorphism on X_α, and hence such that k_β is an isomorphism on X_β for all $\beta \geq \alpha$. A sequence $\mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ of finitely presented modules is right exact on X iff there is an α such that $\mathcal{F}_\alpha \to \mathcal{G}_\alpha \to \mathcal{H}_\alpha \to 0$ is right exact on X_α, and hence such that $\mathcal{F}_\beta \to \mathcal{G}_\beta \to \mathcal{H}_\beta \to 0$ is right exact on X_β for all $\beta \geq \alpha$.

A sequence of finitely presented modules on X is locally split short exact iff there is an α such that the corresponding sequence on X_α is locally split short exact. In particular, a sequence of algebraic vector bundles on X is short exact iff there is an α such that the corresponding sequence on X_α is short exact, iff there is an α such that the corresponding sequence on X_β is short exact for all $\beta \geq \alpha$.

C.5. ([EGA] IV 8.14.1). Let $f : T \to S$ be a map of schemes. Then f is locally finitely presented iff for all inverse systems of schemes over S, $\{X_\alpha\}$, satisfying the conditions of C.1, the canonical map (C.5.1) is an isomorphism

$$\varinjlim_\alpha \mathrm{Mor}_S (X_\alpha, T) \xrightarrow{\cong} \mathrm{Mor}_S \left(\varprojlim X_\alpha, T\right). \qquad \text{(C.5.1)}$$

C.6. **Proposition.** *Let Λ be a commutative ring. Let $\{X_\alpha\}$ be an inverse system of schemes as in* C.1, *and with all X_α finitely presented over* Spec(Λ) *and all bonding maps $f_{\beta\alpha} : X_\beta \to X_\alpha$ being maps over* Spec(Λ). *Then if $X = \varprojlim X_\alpha$ is an affine scheme, there exists an α such that X_β is affine for all $\beta \geq \alpha$.*

Proof. First note that we cannot quote C.3 or [EGA] IV 8.10.5, since we do not assume that X is finitely presented over Spec(Λ).

As the $f_{\beta\alpha}$ are all affine maps, it suffices to show some X_α is affine.

Let $A = \Gamma(X, \mathcal{O}_X)$, so $X = \mathrm{Spec}(A)$. We write $A = \varinjlim A_\gamma$ as a direct colimit of algebras finitely presented over Λ. Applying C.5 to $\varprojlim \mathrm{Spec}(A_\gamma) = X \to X_\beta$, we see that there is a γ so that $X \to X_\beta$ factors as $X = \mathrm{Spec}(A) \to \mathrm{Spec}(A_\gamma) \to X_\beta$. We now apply C.5 to $\varprojlim X_\alpha = X \to \mathrm{Spec}(A_\gamma)$ to see that there is an α such that $X \to \mathrm{Spec}(A_\gamma)$ factors as

$X \to X_\alpha \to \mathrm{Spec}(A_\gamma)$. By C.5, we may choose α sufficiently large so that the composite $X_\alpha \to \mathrm{Spec}(A_\gamma) \to X_\beta$ is $f_{\alpha\beta}$. As $\mathrm{Spec}(A_\gamma)$ is affine, hence a separated scheme, the map $\mathrm{Spec}(A_\gamma) \to X_\beta$ is a separated map. Hence the graph of the map $X_\alpha \to \mathrm{Spec}(A_\gamma)$ gives a closed immersion of X_α into the fibre product of X_α and $\mathrm{Spec}(A_\gamma)$ over X_β, as in (C.6.1)

$$\text{(C.6.1)} \qquad \begin{array}{ccccc} X_\alpha & \longrightarrow & X_\alpha \underset{X_\beta}{\times} \mathrm{Spec}\,(A_\gamma) & \longrightarrow & \mathrm{Spec}\,(A_\gamma) \\ & & \downarrow & & \downarrow \\ & & X_\alpha & \xrightarrow[f_{\alpha\beta}]{} & X_\beta \end{array}$$

As $f_{\alpha\beta}$ is an affine map, the right top arrow of (C.6.1) is affine. As closed immersions are affine maps, the composite map $X_\alpha \to \mathrm{Spec}(A_\gamma)$ is then an affine map, and X_α is an affine scheme as required (cf. [EGA] I 9.1, esp. 9.1.16(v) and 9.1.11.).

C.7. **Proposition.** *Let Λ be a commutative ring, and let $\{X_\alpha\}$ be an inverse system over* $\mathrm{Spec}(\Lambda)$ *of schemes finitely presented over* $\mathrm{Spec}(\Lambda)$, *satisfying the conditions of* C.1. *Then if $X = \varprojlim X_\alpha$ is a separated scheme, there is an α such that for all $\beta \geq \alpha$, X_β is separated.*

Proof. As the maps $X_\beta \to X_\alpha$ are affine, hence separated, it suffices to show some X_α is separated.

Let $\{U_i\}$ for $i = 1, \ldots, n$ be a finite open cover of X by affines. As X is separated, the maps $U_i \cap U_j \to U_i \times U_j$ are closed immersions, so the $U_i \cap U_j$ are affine and the maps of rings $\Gamma(U_i, \mathcal{O}) \otimes_{\mathbb{Z}} \Gamma(U_j, \mathcal{O}) \to \Gamma(U_i \cap U_j, \mathcal{O})$ are onto. Passing to a cofinal system of α by C.2, we may assume that $U_i = \varprojlim U_{i\alpha}$, $U_i \cap U_j = \varprojlim U_{i\alpha} \cap U_{j\alpha}$. By C.6, on passing to a cofinal subsystem, we may assume that all $U_{i\alpha}$ and all $U_{i\alpha} \cap U_{j\alpha}$ are affine. We denote $\Gamma(U_{i\alpha}, \mathcal{O}) = A_{i\alpha}$, $\Gamma(U_{i\alpha} \cap U_{j\alpha}, \mathcal{O}) = A_{ij\alpha}$. Then if for all (i,j) the map $A_{i\alpha} \otimes A_{j\alpha} \to A_{ij\alpha}$ is onto, $U_{i\alpha} \cap U_{j\alpha} \to U_{i\alpha} \times U_{j\alpha}$ will be a closed immersion for all (i,j), and X_α will be separated ([EGA] I 5.3.6).

As $U_{i\beta} \cap U_{j\beta}$ is the pullback of $U_{i\alpha} \cap U_{j\alpha}$ under $X_\beta \to X_\alpha$ or $U_{i\beta} \to U_{i\alpha}$, there is an isomorphism of coordinate rings

$$\text{(C.7.1)} \qquad A_{ij\beta} \cong A_{ij\alpha} \otimes_{A_{i\alpha}} A_{i\beta}.$$

As X is separated, $A_i \otimes A_j = \varinjlim(A_{i\alpha} \otimes A_{j\alpha}) \to \varinjlim(A_{ij\alpha}) = A_{ij}$ is onto. Fix a γ in the direct system of rings. As $A_{ij\gamma}$ is finitely generated as an algebra over Λ, there is an $\alpha \geq \gamma$ in the direct system of rings such that for all $\beta \geq \alpha$ and all the finitely many pairs (i,j), the image of $A_{i\alpha} \otimes A_{j\alpha}$ in $A_{ij\beta}$ contains all the generators of, and hence the image of, $A_{ij\gamma}$. Thus the image of $A_{i\alpha} \otimes A_{j\alpha}$ in $A_{ij\alpha}$ contains the image of $A_{i\alpha} \otimes A_{ij\gamma}$, and hence is all of $A_{ij\alpha}$ by (C.7.1). Thus $A_{i\alpha} \otimes A_{j\alpha} \to A_{ij\alpha}$ is onto for all (i,j), and X_α is separated, as required.

C.8. **Proposition.** *Let* Λ *be a commutative ring. Suppose* $X = \varprojlim X_\alpha$ *is an inverse limit of an inverse system over* $\mathrm{Spec}(\Lambda)$, *satisfying* C.1, *and with the schemes* X_α *finitely presented over* $\mathrm{Spec}(\Lambda)$. *Then if* X *has an ample family of line bundles* (2.1.1, or [SGA6] II 2.2), *there is an* α *such that for all* $\beta \geq \alpha$, X_β *has an ample family of line bundles.*

Proof. As all bonding maps $X_\beta \to X_\alpha$ are affine, it suffices to show some X_α has an ample family (2.1.2(g)).

If X has an ample family, there is a finite set of line bundles, $\mathcal{L}_i^{k_i}$, $i = 1, \ldots, n$, which are tensor powers of line bundles in the family, and sections $s_i \in \Gamma(X, \mathcal{L}_i^{k_i})$ such that each X_{s_i} is affine, and $X = \bigcup_{i=1}^{n} X_{s_i}$. By C.4, by taking α sufficiently large we may assume that the $\mathcal{L}_i$ are $f_\alpha^* \mathcal{L}_{i\alpha}$ for line bundles $\mathcal{L}_{i\alpha}$ on X_α, and that the sections s_i on X are induced by sections $s_{i\alpha} \in \Gamma(X_\alpha, \mathcal{L}_{i\alpha}^{\otimes k_i})$ on X_α. Then $X_{s_i} = f^{-1}(X_{\alpha s_{i\alpha}})$. As each X_{s_i} is affine, by C.6, on taking α sufficiently large, we may assume the $X_{\alpha s_{i\alpha}}$ are affine. As the X_{s_i} cover X, taking α larger still, we may assume that the $X_{\alpha s_{i\alpha}}$ cover X_α (apply C.3 to the finitely presented surjection $\coprod X_{s_i} \to X$). But then $\{\mathcal{L}_{i\alpha}\}$ is an ample family of line bundles for X_α by criterion 2.1.1(c).

C.8.1. In the situation of C.8, if X has a single line bundle $\mathcal{L}$ which is ample, there is an α such that for all $\beta \geq \alpha$, X_β has an ample line bundle $\mathcal{L}_\beta$ with $f_\beta^* \mathcal{L}_\beta = \mathcal{L}$, as the proof of C.8 shows.

C.9. **Theorem.** *Let* Λ *be a commutative noetherian ring (e.g.,* $\mathbb{Z}$*). Let* X *be a quasi-compact and quasi-separated scheme over* $\mathrm{Spec}(\Lambda)$. *Then* X *is the limit* $\varprojlim X_\alpha$ *of an inverse system over* $\mathrm{Spec}(\Lambda)$ *of schemes* X_α *finitely presented over Spec*(Λ). *The bonding maps of the system* $X_\alpha \to X_\beta$ *are all affine maps, and are schematically dominant, so* $\mathcal{O}_{X_\beta} \to f_{\alpha\beta}^* \mathcal{O}_{X_\alpha}$ *is a monomorphism ([EGA] I 5.4).*

All the X_α *are noetherian. If* Λ *has finite Krull dimension, all the* X_α *will have finite Krull dimension.*

If X *has an ample family of line bundles, we may arrange that all the* X_α *do. If* X *is semi-separated, we may arrange that all the* X_α *are semi-separated. If* X *is separated, we may arrange that all the* X_α *are separated.*

Proof. As X is quasi-compact, it has a finite cover $\{U_1, \ldots, U_n\}$ by affine open subschemes. The construction of the inverse system proceeds by induction on the number n in such a cover.

If $n = 1$, X is affine. Say $X = \mathrm{Spec}(A)$. We write $A = \varinjlim A_\alpha$ as the direct colimit of those subrings A_α which are finitely generated over Λ. As Λ is noetherian, each A_α is then finitely presented over Λ, and $X = \varprojlim X_\alpha$ for $X_\alpha = \mathrm{Spec}(A_\alpha)$.

To do the induction step, we suppose the result is known for such X as are covered by $n-1$ affines, and in particular for $V = U_2 \cup U_3 \cup \ldots \cup U_n$. So we write $V = \varprojlim V_\alpha$. Set $U = U_1$. As X is quasi-separated, $U \cap V$ is quasi-compact, and so the open immersion $W = U \cap V \to V$ is finitely presented. By C.3, on passing to a cofinal system of α, we may assume that $W = \varprojlim W_\alpha$ for a system of finitely presented open immersions $W_\alpha \to V_\alpha$. Then each W_α is finitely presented over Spec(Λ).

As $W \subseteq U$ is quasi-affine, $\mathcal{O}_W$ is an ample line bundle for W. By C.8.1 and C.4, on passing to a cofinal system of α, we may assume that $\mathcal{O}_{W_\alpha}$ is an ample line bundle for the quasi-compact W_α. Then each W_α is quasi-affine. In fact, let $A = \Gamma(U, \mathcal{O}_U)$, so $U = \mathrm{Spec}(A)$. As W is quasi-affine in U, there are elements $g_i \in A$ for $i = 1, \ldots, n$ such that the $W_{gi} = \mathrm{Spec}(A[1/g_i])$ are affine and cover W. For α sufficiently large, the $g_i \in \Gamma(W, \mathcal{O}_W) \supseteq \Gamma(U, \mathcal{O}_U)$ are in $\Gamma(W_\alpha, \mathcal{O}_{W_\alpha})$. By C.6, on taking α large, we may assume the $W_{\alpha g_i}$ are affine. Then $W_{\alpha g_i} = \mathrm{Spec}(\Gamma(W_\alpha, \mathcal{O}_{W_\alpha})[1/g_i])$, and it is of finite type over Λ since W_α is.

Consider now the pullback diagram of rings, where the indicated maps are monomorphisms by the schematic dominance of the maps $V_\alpha \to V_\beta$, and of the cover $\cup W_{\alpha g_i} \to W_\alpha$

(C.9.1)

$$\begin{array}{ccccccc} B_\alpha & & \rightarrowtail & & & & A \\ \downarrow & & & \square & & & \downarrow \\ \Gamma(W_\alpha, \mathcal{O}_{W_\alpha}) & \rightarrowtail & \prod_1^n \Gamma(W_{\alpha g_i}, \mathcal{O}_{W_\alpha}) & \rightarrowtail & \prod_{i=1}^n \Gamma(W_{g_i}, \mathcal{O}_X) & = & \prod_1^n A[1/g_i] \end{array}$$

As localization and direct colimits commute with pullbacks and finite products, we see that $B_\alpha[1/g_i] = \Gamma(W_{\alpha g_i}, \mathcal{O}_{W_\alpha})$, and that $A = \varinjlim B_\alpha$.

We consider the direct system whose objects (α, A') consist of an α in the system of V_α, and a subring $A' \subseteq B_\alpha$, such that A' is of finite type over Λ, contains the g_i for $i = 1, \ldots, n$, and satisfies $A'[1/g_i] = \Gamma(W_{\alpha g_i}, \mathcal{O}_{W_\alpha})$ for all i. As $W_{\alpha g_i}$ is affine and of finite type of Λ, and as $B_\alpha[1/g_i] = \Gamma(W_{\alpha g_i}, \mathcal{O}_{W_\alpha})$, such A' exist for each α. A morphism $(\alpha_1, A_1') \to (\alpha_2, A_2')$ is an $\alpha_1 \leq \alpha_2$ in the system of V_α (corresponding to a map $V_{\alpha_2} \to V_{\alpha_1}$), and an inclusion of rings $A_1' \subseteq A_2'$ induced by the monomorphism $B_{\alpha_1} \subseteq B_{\alpha_2}$.

For each $\gamma = (\alpha, A')$ in this system, the map $W_\alpha \to \mathrm{Spec}(A')$ induced by $A' \to B_\alpha \to \Gamma(W_\alpha, \mathcal{O}_{W_\alpha})$ is an open immersion. Let X_γ be the scheme obtained by patching V_α and Spec(A') along the open W_α. Then the X_γ form an inverse system of schemes, with affine and schematically dominant bonding maps. The inverse system $X_\gamma \cap V_\alpha$ is clearly cofinal with the original system of V_α, so $\varprojlim X_\gamma \cap V_\alpha \cong \varprojlim V_\alpha = V \subseteq X$. For each fixed

α, $\varinjlim A' = B_\alpha$. As $\varinjlim B_\alpha = A$, taking the colimit of A' for all $\gamma = (\alpha, A')$ yields $\varinjlim A' = A$. Hence $\varprojlim X_\gamma \cap \operatorname{Spec}(A') = \operatorname{Spec}(A) = U \subseteq X$. Thus $\varprojlim X_\gamma = X$, as required. This completes the induction step, proving the first paragraph of the statement of the theorem.

We note the X_γ are of finite type over the noetherian Λ, hence are noetherian, and have finite Krull dimension if Λ does.

If X has an ample family of line bundles, C.8 shows that all the X_α will after passing to a cofinal subsystem. If X is semi-separated, applying C.6 to a semi-separating cover of X by affines shows that all the X_α in a cofinal subsystem will be semi-separated (B.7). By C.7, if X is separated, a cofinal subsystem of X_α will be separated.

Appendix D

Hypercohomology with supports

D.1. Let X be a topos with enough points. Let $Y \subseteq X$ be a closed subtopos, and $X - Y$ its open complement. Recall that $X - Y$ and Y have enough points ([SGA 4] IV 6, 9).

Let G be a presheaf of spectra on a site for X. (As in [Th1], one should assume G is a presheaf of "fibrant spectra." We can always attain this by replacing G by a homotopy equivalent presheaf.)

D.2. *Definition.* Let $\mathbf{H}^{\cdot}_Y(X;G)$ be the canonical homotopy fibre of the restriction map on the hypercohomology spectra of [Th1], $\mathbf{H}^{\cdot}(X;G) \to \mathbf{H}^{\cdot}(X-Y;G)$

$$\mathbf{H}^{\cdot}_Y(X;G) \to \mathbf{H}^{\cdot}(X;G) \to \mathbf{H}^{\cdot}(X-Y;G). \tag{D.2.1}$$

D.3. **Lemma.** *$\mathbf{H}^{\cdot}_Y(X;\ )$ preserves homotopy equivalences and homotopy fibre sequences of coefficient spectra. If both $\mathbf{H}^{\cdot}(X;\ )$ and $\mathbf{H}^{\cdot}(X-Y;\ )$ preserve direct limits up to homotopy, so does $\mathbf{H}^{\cdot}_Y(X;\ )$.*

Proof. This is clear from the homotopy fibre sequence (D.2.1) using the 5-lemma, the Quetzalcoatl lemma, the fact direct colimits preserve homotopy fibre sequences, and the corresponding properties of $\mathbf{H}^{\cdot}(X;\ )$ and $\mathbf{H}^{\cdot}(X-Y;\ )$, [Th1] 1.35, 1.39.

D.4. **Theorem.** *There is a hypercohomology spectral sequence (with Bousfield-Kan indexing) of homotopy groups*

$$E_2^{p,q} = H^p_Y(X;\tilde{\pi}_q G) \Longrightarrow \pi_{q-p} H^{\cdot}_Y(X;G). \tag{D.4.1}$$

Here the E_2 term is cohomology with supports ([SGA 4] V Section 6) *with the coefficients in the sheafification $\tilde{\pi}_q G$ of the presheaf $\pi_q G$ on X.*

The spectral sequence converges strongly if either there exists an integer M such that $\tilde{\pi}_q G = 0$ for all $q \geq M$, or else if $H^*_Y(X;\ )$ has finite cohomological dimension for the $\tilde{\pi}_* G$ so there exists an integer N such that $H^p_Y(X,\tilde{\pi}_q G) = 0$ for all q and for all $p \geq N$. (We note that $H^*_Y(X;\ )$ has finite cohomological dimension for $\tilde{\pi}_* G$ if both $H^*(X;\ )$ and $H^*(X-Y;\ )$ do, thanks to the long exact sequence of [SGA 4] V (6.5.4).)

Proof. Let $\{G\langle n\rangle\}$ for $n \in \mathbb{Z}$ be the Postnikov tower of G as in [Th1] 5.51, so $\pi_q G\langle n\rangle = \pi_q G$ for $q \leq n$, and $\pi_q G\langle n\rangle = 0$ for $q > n$. The map to the homotopy inverse limit of $H^{\cdot}_Y(X;\ )$ of this tower is a homotopy equivalence

$$\text{(D.4.2)} \qquad \mathbf{H}^{\cdot}_Y(X;G) \xrightarrow{\sim} \underset{\underset{n}{\longleftarrow}}{\text{holim}}\, \mathbf{H}^{\cdot}_Y(X;G\langle n\rangle).$$

This homotopy equivalence follows from the 5-lemma and (D.2.1), and the corresponding equivalences for $\mathbf{H}^{\cdot}(X;\ )$ and $\mathbf{H}^{\cdot}(X-Y;\ )$ given by [Th1] 1.37. The spectral sequence (D.4.1) will be the canonical spectral sequence of this $\underset{\longleftarrow}{\text{holim}}$, or of the $\underset{\longleftarrow}{\lim}$ of a homotopy equivalent tower of fibrations, as in [Th1] 5.54, 5.43. Aside from the identification of the E_2 term, all results follow from [Th1] 5.43. The E_2 term of the canonical spectral sequence is given as $E_2^{p,q} = \pi_{q-p}\mathbf{H}^{\cdot}_Y(X;K(\pi_q G,q))$, where $K(\pi_q G,q)$ is the homotopy fibre of $G\langle q\rangle \to G\langle q-1\rangle$, and thus is equivalent to the presheaf of Eilenberg-Maclane spectra associated to the presheaf of abelian groups $\pi_q G$ shifted q degrees [Th1] 5.52. It remains to identify this E_2 term with the cohomology with supports as in (D.4.1).

Shifting degrees q times by looping, we reduce to showing that if A is a presheaf of abelian groups, with sheafification $\tilde{A}$, and if $K(A,0)$ is a presheaf of spectra with $\pi_q K(A,0) = 0$ for $q \neq 0$, $\pi_0 K(A,0) = A$, then there is a natural isomorphism for all p

$$\text{(D.4.3)} \qquad \pi_{-p}\mathbf{H}^{\cdot}_Y(X;K(A,0)) \cong H^p_Y(X;\tilde{A}).$$

We know from [Th1] 1.36 that we do have natural isomorphisms

$$\text{(D.4.4)} \qquad \begin{aligned} &\pi_{-p}\mathbf{H}^{\cdot}(X;K(A,0)) \cong H^p(X;\tilde{A}) \\ &\pi_{-p}\mathbf{H}^{\cdot}(X-Y;K(A,0)) \cong H^p(X-Y;\tilde{A}). \end{aligned}$$

As $K(A,0) \to K(\tilde{A},0)$ induces isomorphisms on $\pi_*\mathbf{H}^{\cdot}(X;\ )$ and $\pi_*\mathbf{H}^{\cdot}(X-Y;\ )$ by D.4.4, the 5-lemma and the long exact sequence of homotopy groups resulting from the defining fibration sequence D.2.1 shows that the map $\pi_*\mathbf{H}^{\cdot}_Y(X;K(A,0)) \to \pi_*\mathbf{H}^{\cdot}_Y(X;K(\tilde{A},0))$ is also an isomorphism. Thus we may assume that $A = \tilde{A}$ is a sheaf.

The isomorphisms (D.4.4) and the long exact sequence homotopy groups of (D.2.1), together with the obvious fact $H^p(X;\ ) = H^p(X-Y;\ ) = H^p_Y(X;\ ) = 0$ for $p < 0$, show that (D.4.3) trivially holds for $p < 0$, as both sides are 0. This argument also shows that $\pi_0\mathbf{H}^{\cdot}_Y(X;K(\tilde{A},0))$ is the kernel of $H^0(X;\tilde{A}) \to H^0(X-Y;\tilde{A})$, which is $H^0_Y(X;\tilde{A})$ by definition ([SGA 4] V 6). This proves (D.4.3) for $p = 0$.

If $\tilde{A}$ is an injective sheaf on X, it is also injective on U, so $H^p(X;\tilde{A}) = 0 = H^p(X-Y;\tilde{A})$ for $p > 0$. Also for $\tilde{A}$ injective, the long exact sequence [SGA 4] V 6.5.4 collapses into the short exact sequence

(D.4.5) $$0 \to H^0_Y(X;\tilde{A}) \to H^0(X;\tilde{A}) \to H^0(X-Y;\tilde{A}) \to 0.$$

Comparing this with the long exact sequence of homotopy groups induced by (D.2.1), using the isomorphisms already established, we see that $\pi_{-p}\mathbb{H}^{\cdot}_Y(X;K(\tilde{A},0)) = 0$ for $p > 0$ when $\tilde{A}$ is injective, so (D.4.3) is an isomorphism for all p when $\tilde{A}$ is injective.

Now suppose $0 \to \tilde{A} \to \tilde{B} \to \tilde{C} \to 0$ is some short exact sequence of sheaves. Then $\tilde{A} \to \tilde{B}$ is still a monomorphism in the category of presheaves. Let C be the cokernel presheaf, so $0 \to \tilde{A} \to \tilde{B} \to C \to 0$ is exact in the category of presheaves. (Note that $\tilde{C}$ is indeed the sheafification of C.) From the exact sequence, it follows that $K(\tilde{A},0) \to K(\tilde{B},0) \to K(C,0)$ is a homotopy fibre sequence of presheaves of spectra. Then by D.3, we have a homotopy fibre sequence of spectra

(D.4.6) $$\begin{array}{ccccc} \mathbb{H}^{\cdot}_Y(X;K(\tilde{A},0)) & \to & \mathbb{H}^{\cdot}_Y(X;K(\tilde{B},0)) & \to & \mathbb{H}^{\cdot}_Y(X;K(C,0)) \\ & & & & \wr\| \\ & & & & \mathbb{H}^{\cdot}_Y(X;K(\tilde{C},0)). \end{array}$$

This induces a long exact sequence of homotopy groups $\pi_*\mathbb{H}^{\cdot}_Y(X;K(\ ,0))$ for $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$. Thus $\pi_*\mathbb{H}^{\cdot}_Y(X;K(\ ,0))$ is a cohomological ∂-functor on the category of sheaves. We recall $\pi_0\mathbb{H}^{\cdot}_Y(X;K(\ ,0)) \cong H^0_Y(X;\ )$ and that $\pi_{-p}\mathbb{H}^{\cdot}_Y(X;K(\tilde{A},0)) = 0$ for $\tilde{A}$ injective and $p \neq 0$. It follows that the $\pi_*\mathbb{H}^{\cdot}_Y(X;K(\ ,0))$ are the derived functors of $H^0_Y(X;\ )$, i.e., they are the $H^*_Y(X;\ )$.

This proves D.4.3 for all $\tilde{A}$ and all p, as required.

D.5. (Optional). Since [Th1] was written, André Joyal has shown that the category of simplicial objects in any topos has the structure of a Quillen closed model category [Q2]. Jardine has extended the result to provide a closed model structure on the category of presheaves of spectra on a site, and on that of sheaves of spectra in the topos [Ja].

This allows a more flexible and simple construction of $\mathbb{H}^{\cdot}_Y(X;\ )$ and $\mathbb{H}^{\cdot}(X;\ )$ than the canonical Godement resolution construction of [Th1]. Now to form $\mathbb{H}^{\cdot}(X;G)$ of a presheaf of spectra G, one takes any homotopy equivalent $G^{\sim}$ that is fibrant for the model structure, and takes global sections $\Gamma(X;G^{\sim}) \cong \mathbb{H}^{\cdot}(X;G)$. To make a version of $\mathbb{H}^{\cdot}(X;G)$ that is strictly functorial in the topos X, one works in the model category of the huge topos (cf. [SGA 4] IV 2.5, 4.10) that contains every topos in the universe.

Appendix E

The Nisnevich topology

E.1. *Definition.* The Nisnevich site of a scheme X is the category of schemes étale (hence finitely presented) over X, $U \to X$, with the following Grothendieck pretopology: A family $\{V_\alpha \to U\}$ in the site is cover if for all points $x \in U$, there is an α and a point $y_\alpha \in V_\alpha$ such that $V_\alpha \to U$ sends y_α to x and induces an isomorphism of residue fields $k(x) \xrightarrow{\cong} k(y_\alpha)$.

E.2. Clearly a map $f : X \to X'$ of schemes induces a preimage functor f^{-1} that determines a map of Nisnevich sites and topoi $f : X_{\mathrm{Nis}} \to X'_{\mathrm{Nis}}$ ([SGA 4] III 1.6, IV 4.9).

There are obvious natural morphisms of sites and topoi $X_{\mathrm{et}} \to X_{\mathrm{Nis}} \to X_{\mathrm{Zar}}$, as the Nisnevich topology is coarser than the étale topology, but finer than the Zariski topology.

E.3. An integral (en français "entier," pas "intègre") radicial surjective map $X' \to X$, and in particular the closed immersion $X_{\mathrm{red}} \to X$, induces an equivalence of Nisnevich sites and topoi $X'_{\mathrm{Nis}} \to X_{\mathrm{Nis}}$.

Proof. By [SGA 4] VIII 1.1, the map induces an equivalence of étale sites. Under this equivalence, corresponding objects $U' \to U$ of the sites have isomorphic residue fields at corresponding points. Hence Nisnevich covers correspond under the equivalence, and the result follows.

E.4. *Example.* Let k be a field. The Nisnevich site of k consists of all finite products of fields étale over k, $\mathrm{Spec}(\Pi k'_i) = \cup\ \mathrm{Spec}(k_i) \to\ \mathrm{Spec}(k)$. A family of fields covers a field k' exactly when a member of the family is isomorphic to k' by the given map. Thus the Nisnevich topos of a field k consists of copies of the trivial Zariski topoi of all fields k' étale over k, but with the copies related by a map of topoi for every map $\mathrm{Spec}(k') \to \mathrm{Spec}(k'')$ over $\mathrm{Spec}(k)$. Thus the Nisnevich topos of the field k is sort of a bigger Zariski topos of k, cf. [SGA 4] IV 4.10. Indeed there is a map of topoi $i : (k')_{\mathrm{Zar}} \to (k)_{\mathrm{Nis}}$ for k' étale over k, with i^* given be restriction of a sheaf to the Zariski topos of k'.

E.5. **Lemma** (Nisnevich [N3]). *Let X be a scheme. Then*

(a) *The Nisnevich topos X_{Nis} has enough points. In fact for every field k' étale over a residue field $k(x)$ of X, consider the map of topoi Sets $= (\mathrm{Spec}(k'))_{\mathrm{Zar}} \to (\mathrm{Spec}(k(x)))_{\mathrm{Nis}} \to X_{\mathrm{Nis}}$, where $(k')_{\mathrm{Zar}} \to (k(x))_{\mathrm{Nis}}$ is as in E.4, and $(k(x))_{\mathrm{Nis}} \to X_{\mathrm{Nis}}$ is the map induced by $\mathrm{Spec}(k(x)) \to X$. Then this family of morphisms of topoi Sets $\to X_{\mathrm{Nis}}$ is a conservative* *family of points.*

(b) *The filtering system of neighborhoods of such a point* $(k')_{\mathrm{Zar}} \to (k(x))_{\mathrm{Nis}} \to X_{\mathrm{Nis}}$ *is the system of all diagrams (E.5.1) with* $U \to X$ *étale, and* $(k') \to (k(x)) \to X$ *the given maps.*

$$\text{(E.5.1)} \qquad \begin{array}{ccc} \mathrm{Spec}(k') & \xrightarrow{y} & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k(x)) & \longrightarrow & X \end{array}$$

A cofinal subsystem is the category of such diagrams were $y : \mathrm{Spec}(k') \to U$ *induces an isomorphism of* k' *to the residue field* $k(y)$ *of* U *at the point* $y \in U$.

(c) *The stalk of the structure sheaf* $\mathcal{O}_X$ *in* X_{Nis} *at the point* $(k')_{\mathrm{Zar}} \to (k(x))_{\mathrm{Nis}} \to X$ *is the henselization of* $\mathcal{O}_{X,x}$ *at the residue field extension* $k(x) \to k'$ *([EGA] IV 18.8), or equivalently, the henselization* $\mathcal{O}^h_{U,y}$ *of the local ring* $\mathcal{O}_{U,y}$ *of any* U *in the cofinal system of (b) for which* $k' \cong k(y)$.

Proof. Part (a) follows from the criterion of [SGA 4] IV 6.5, that a set of points that distinguishes covering families from non-covering families in the site is then a conservative set of points for the topos. Part (b) follows from the definition, [SGA 4] IV 6.3. The cofinality statement results from first applying [EGA] IV 18.1 to extend $\mathrm{Spec}(k') \to \mathrm{Spec}(k(x))$ to an étale cover of the local ring $\mathrm{Spec}(\mathcal{O}_{X,x})$ by a local ring, and then extending this cover to an étale map $U \to X$ by a limit argument [EGA] IV 17.7.8. Part (c) follows from the definitions [SGA 4] IV 6.3, [EGA] IV 18.5, 18.6.

Note the analogy of this with the proofs of the corresponding results for the étale topology [SGA 4] VIII 3.5, 3.9 Section 4. See also [N3], but beware of its funny definition of point, which is not equivalent to that of [SGA 4], but is rather an acyclic category of classic points.

E.6. **Lemma** (Nisnevich, Kato-Saito). *Let* X *be a noetherian scheme. Then*

(a) The Nisnevich topos X_{Nis} *is a coherent topos, and even a noetherian topos in the sense of [SGA 4] VI 2.3, 2.11.*

(b) If $f : Y \to X$ *is a finite map, then* $f_* : Y_{\mathrm{Nis}} \to X_{\mathrm{Nis}}$ *is exact and* $R^q f_* = 0$ *for* $q > 0$.

If $Y \to X$ *is a closed immersion,* Y_{Nis} *is a closed subtopos of* X_{Nis}*, with open complement* $(X - Y)_{\mathrm{Nis}}$.

(c) If X *has finite Krull dimension* N*, the cohomological dimension of* X_{Nis} *is at most* N.

(d) If X *has finite Krull dimension,* $\mathbf{H}^{\cdot}_{\mathrm{Nis}}(X;\ )$ *preserves up to homotopy direct colimits of presheaves of spectra.*

(e) Suppose X has finite Krull dimension. Let $\{U_\alpha\}$ be an inverse system with affine bonding maps of schemes étale over X, and with inverse limit $U = \varprojlim U_\alpha$. Let F be a presheaf of spectra on the category of quasi-compact and quasi-separated schemes, which is continuous in the sense that the canonical map $\varinjlim F(X_\alpha) \xrightarrow{\sim} F(\varprojlim X_\alpha)$ is a homotopy equivalence for any inverse system with affine bonding maps of quasi-compact and quasi-separated schemes X_α (cf. [Th1] 1.42). Then the canonical map

$$\varinjlim \mathbf{H}^{\cdot}_{\mathrm{Nis}}(U_\alpha; F) \xrightarrow{\sim} \mathbf{H}^{\cdot}_{\mathrm{Nis}}(U; F)$$

is a homotopy equivalence.

Proof. To prove (a), it suffices to show that for any U in the Nisnevich site of X that any Nisnevich cover $\{V_\alpha \to U\}$ of U has a finite subcover. As U is finitely presented over X, it is noetherian. So it suffices to show any Nisnevich cover of a noetherian scheme U has a finite subcover. We proceed by noetherian induction. Suppose the result is known for all closed subschemes $Y \neq U$. Let η be a generic point of U. By definition of Nisnevich cover E.1, there is a V_1 in the cover and a point $\eta' \in V_1$ such that $V_1 \to U$ induces an isomorphism $k(\eta') \to k(\eta)$. We claim that $V_1 \to U$ induces an isomorphism of an open *nbd* of η' onto an open *nbd* of η. If U and hence the étale V_1 are reduced, $k(\eta)$ and $k(\eta')$ are the local rings of U and V_1 at these generic points. The inverse isomorphism $k(\eta) \to k(\eta')$ then extends to an inverse isomorphism of some open *nbd*s by the finite presentation of $V_1 \to U$. In the general case, we apply the equivalence of sites E.3 of U and U_{red}, to reduce to the case where U is reduced. This proves our claim. So $V_1 \to U$ is a Nisnevich cover when restricted to the open *nbd* $W \subseteq U$ over which V_1 has an *nbd* isomorphic to W. As $U - W$ is a closed subspace and is not all U, the induction hypothesis shows that there is a finite set of V_β such that the induced $V_\beta \times_U (U - W) \to (U - W)$ cover $U - W$. Then $V_1 \to U$ and these $V_\beta \to U$ form a finite subcover of U, as required. This proves (a) (cf. [KS] 1.2.1).

Statement (b) follows by an argument parallel to the proof of the corresponding statements for the étale topology in [SGA 4] VII 6.3, 6.1, 5.5, replacing the descriptions of the stalks in the étale topology everywhere by E.5(b) and E.5(c). Whenever the étale case appeals to [SGA 4] VII 5.4 and 4.6, we instead use the fact that a finite extension of a hensel local ring is a hensel ring ([EGA] IV 18.5.10).

To prove (c), one proves the stronger statement that $H^q_{\mathrm{Nis}}(X; \mathcal{F}) = 0$ for all $q \geq p$ if $\mathcal{F}_y = 0$ for all $k(y)$ étale over $k(x)$ for those $x \in X$ with closure $\bar{x}$ of Krull dimension $\geq p$. This proof proceeds by induction on p and $\dim X$, using (b) and the method of [SGA 4] X 4.1. One starts by noting that if k is a field, $H^q_{\mathrm{Nis}}(k; \mathcal{F}) = 0$ for $q > 0$, as the global section functor $H^0_{\mathrm{Nis}}(k;\ )$ is isomorphic to taking the stalk at the point $Sets \to (k)_{\mathrm{Zar}} \to (k)_{\mathrm{Nis}}$ and so is exact. In general if $\dim k = 0$, k is an Artin ring, so k_{red} is a product of fields, and the result follows by E.3. This starts the induction, and one proceeds as in [SGA 4] X 4.1. Where [SGA 4] X makes an appeal to the theory of constructible sheaves, we note that these are just the coherent objects in the topos, so ([SGA 4] VI 2.14, 2.9) gives an adequate theory of constructible sheaves in any coherent topos like X_{Nis}.

For an even less detailed, hence more psychologically convincing, proof of (b) and (c), see [KS] 1.2.5. The final version of [N3] should also contain proofs.

Statements (d) and (e) follow from (a) and (c) using [Th1] 1.39, 1.41.

Appendix F

Invariance under change of universe

Let X be a scheme, and $\mathcal{U}_1 \subseteq \mathcal{U}_2$ two Grothendieck universes containing X ([SGA 4] I Appendix). We see successively for each of the categories in the following list that the change of universe functor is an equivalence of categories:

(a) category of all finitely presented $\mathcal{O}_X$-modules in the universe;

(b) category of all algebraic vector bundles on X in the universe;

(c) complicial biWaldlhausen category of all the strict perfect complexes on X in the universe;

(d) complicial biWaldhausen category of all the strict pseudo-coherent complexes on X in the universe.

For X quasi-compact and quasi-separated, we may add to this list

(e) the homotopy category of the biWaldhausen category of all perfect complexes on X in the universe;

(f) the homotopy category of the biWaldhausen category of all cohomologically bounded pseudo-coherent complexes on X in the universe.

For locally on affines of X, (e) and (f) hold by 2.3.1(d) and 2.3.1(e) and the equivalence of the homotopy categories of (c) and (d). The case of a general quasi-compact and quasi-separated X follows by the methods of 3.20.4 - 3.20.6.

Now 1.9.8 applies to show $G(X)$, $K(X)$, and hence $K^B(X)$ are invariant up to homotopy under change of universe. Similarly for $K(X \text{ on } Y)$.

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