A Research Note on Deriving the Square-Cube Law of Formal Organizations from the Theory of Time-Minimization*

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There are relatively few empirical laws in sociology, at least in the sense in which that word is applied in the physical sciences: empirically supported equations which precisely describe relations between variables. Most of us are satisfied to find reasonably strong associations between variables. It is therefore a matter of some interest when an exact law is proposed. It becomes a matter of some importance to derive such a law theoretically since the more general theoretical statement may permit derivation of still other laws. Several empirical laws—the size-density law, the rank-size rule, the urban density law, the gravity model, and the urban area-population law—have been reported in the ecological or social-demographic literature. They have also been derived from the theory of time-minimization (Stephan).

The purpose of this paper is to examine a nonecological law, one developed from the study of formal organizations, and to derive that law from the theory of time-minimization. The law is Mason Haire's "square-cube law," a law which has stirred considerable interest and controversy since its introduction. Haire examined longitudinal data from four firms. He divided the employees of these firms into "external employees," those who interact with others outside the firm, and "internal employees," those who interact only with others inside the firm. His finding was that, over time, the cube-root of the number of internal employees was directly proportional to the square-root of the number of external employees. The scatter diagrams he presented (286–7) show regression lines of the form

$$I^{1/3} = a + bE^{1/2} \quad (1)$$

where I and E are the number of internal and external employees and a and b are the intercept and slope of the regression line (see Figure 1 for an example). His explanation of the square-cube law is based on certain

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mathematical properties of physical objects, extended to an explanation of biological form and analogically applied by Haire to the shape of formal organizations. For a given physical object, say a cube, an increase in the length of a side results in an increase of the surface area and also of the volume. If the new length is ten times the old, the area will be 10^2 or 100 times the old, and the new volume will be 10^3 or 1000 times the old. Thus, the cube-root of the volume will be proportional to the square-root of the surface area.

D'Arcy Thompson applied this physical model in accounting for various properties of living organisms. Of particular interest to Haire is Thompson's explanation (see Haire, 273–4) of why Jack the Giant Killer had nothing to fear from the giant. The giant was supposedly ten times as large as a man and proportioned exactly like one. If this were so, Thompson argued, the cross-section of the leg bones of the giant would only be 100 times larger than those of a man while his volume, and hence his mass,

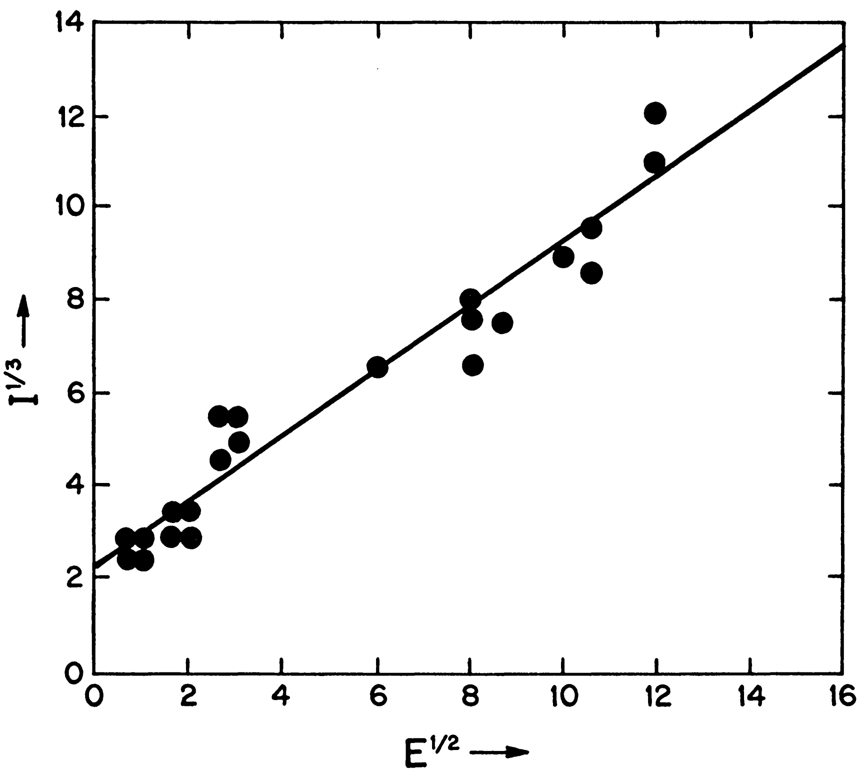

Figure 1. THE RELATIONSHIP BETWEEN NUMBER OF INTERNAL EMPLOYEES, I, AND NUMBER OF EXTERNAL EMPLOYEES, E, OVER TIME, FOR THE ORGANIZATION REFERRED TO BY HAIRE AS "COMPANY B" (ADAPTED FROM HAIRE, 286)

would be 1,000 times larger. His legs would break if he ever stood up. Haire, with the evidence of a similar square-cube relationship before him, argues that formal organizations must be similarly constrained with regard to the relations between their "volumes" (the internal employees) and their "surface areas" (Their external employees).

The analogy isn't quite exact. If Equation 1 were to describe the relation between the volume of a cube and its surface area, then the intercept would have to equal zero and the slope would have to equal unity. In fact, the intercept in all of the cases Haire studied is slightly greater than zero, and the slopes are all somewhat less than unity. Nevertheless, there clearly does seem to be a strict linear relation between the two variables in Equation 1, reflected in the very high correlation coefficients reported by Haire for each case, with none lower than .95 (Haire, 285).

Subsequent Tests and Criticisms

Levy and Donhowe tested Haire's law with cross-sectional data for 62 firms in eight industries. They conclude that the square-cube law "is a reasonable and consistent description of the industrial organizational composition among firms of varying size in different industries" (342). A second study, by Draper and Strother, examined data for a single educational organization over a 45-year period. They showed that regression analysis of the *untransformed* data produced nearly as good a fit as did the square-cube transformation in Equation 1. They argued that, since their "more parsimonious model" of simple proportional growth gave nearly as good a fit as Haire's square-cube one, the latter was superfluous.

McWhinney reanalyzed the data employed by Haire, Levy–Donhowe, and Draper–Strother. In addition to retesting Equation 1 and testing the untransformed linear relation suggested by Draper and Strother, McWhinney tests a logarithmic transformation of the relation between E and I derived from Equation 1 as follows. He begins with what he says is Haire's hypothesized relation:

$$E^{1/2} = a + bI^{1/3} \qquad (2)$$

Haire himself did not write out either Equation 1 or Equation 2. He did present scatter diagrams and regression lines on graphs which suggest that $E^{1/2}$ was his intended independent variable. McWhinney's equation, Equation 2, reverses the order of dependent and independent variables, a practice which may or may not involve problems of statistical analysis (the regression coefficient b_{xy} does not even imply its own inverse, b_{yx}, except under conditions of perfect correlation). However, since Haire's own argument from analogy puts him in the position of assuming that $a = 0$ and

$b = 1$ in either Equation 1 or Equation 2, the difference between the two equations becomes irrelevant since both reduce to

$$E^{1/2} = I^{1/3} \tag{3}$$

McWhinney, for the purpose of testing, squares both sides of Equation 3 and inserts a proportionality constant b, to obtain

$$E = bI^{2/3} \tag{4}$$

which he logarithmically transforms to

$$\log E = \log b + c(\log I) \tag{5}$$

If Haire were correct, *log b* should equal zero and c should equal $^2/_3$. One principal advantage of Equation 5, according to McWhinney, is that it does not force a particular value for c beforehand.

Such algebraic manipulations can be troublesome. If we were to assume that c in Equation 5 did equal $^2/_3$, so that Equations 4 and 5 were mathematically equivalent, that would not necessarily imply that an empirical test of Equation 5 would be a test of Equation 4. As suggested above, mathematical equivalence implies statistical equivalence only under conditions of perfect correlation. The problem lies in the fact that statistical equations involve error terms while the mathematical equations do not. Square-root or logarithmic transformations also transform the error terms, and this may or may not affect the results (see Hays and Winkler, 651–4).

McWhinney objects to both the square-cube model employed by Haire and Levy–Donhowe and to the linear model employed by Draper–Strother, since the former ignores both the intercept and the slope suggested by Equation 1 (or 2) while the latter ignores the intercept. He argues that, had either model been tested with a zero intercept (0,0) thrown into the data—expected from the physical or biological analogy—the regression coefficients reported would have been much lower. "As performed, (these) regression analyses cannot be interpreted as tests of their models" (349). He finds that his own general exponential model, Equation 5, fits all the data better than either of the others do (349), which ought to be expected, of course, since—unlike Haire's original model, even in logarithmic transformation—McWhinney's does not impose the particular $^2/_3$ value for c.

Carlisle analyzed data for seven school districts using both the square-cube transformations and the raw data. He found, supporting Draper–Strother, that the correlation coefficients were about equally good under the two tests. He did not test McWhinney's logarithmic transformation, though McWhinney was cited.

Derivation of the Square-Cube Law

The purpose of this paper is not to present arguments favoring either the Haire–Levy–Donhowe square-cube test, the Draper–Strother–Carlisle linear test, or the McWhinney logarithmic test. As is often the case, the same data may provide extremely close fits to many models. Put another way, many models are so flexible that real data sets do not provide convincing ways of arguing that one model is to be accepted while others are to be rejected. As McWhinney's own scatter diagram shows (345), all three fit the data fairly well. Under such conditions, when the data themselves do not provide conclusive evidence favoring one model over another, the best criterion is often a logical one: Can one of the models be derived from some general theory?

The trouble with Haire's analogy, his Jack the Giant Killer explanation of the square-cube law, is that it is only that. An analogy is not a theory; it is an interpretation, however poetic, and nothing more. Draper–Strother and McWhinney are each highly critical of Haire's analogy, McWhinney writes: ". . . the very concentration on the generalized geometric interpretation perpetuates a tradition that organizations can appropriately be described by archetypical objects in a three-dimensional space" (349–50). He goes on to say, ". . . these and other social scientists display a Pythagorean devotion to numbers, *diverting their attention from the underlying processes* either in the biological world they choose for illustrations or in the organizational world they describe" (350; italics added). A Pythagorean devotion to numbers, it might be argued, is actually central to scientific work and hence should not be criticized. Be that as it may, McWhinney is certainly correct that Haire's reasoning by analogy has diverted us from attending to the underlying organizational processes which might have resulted in producing a square-cube law.

We can go further than this. None of these investigators—Haire, Levy–Donhowe, Draper–Strother, McWhinney, Carlisle—has provided genuine *theoretical* derivations of the relationship between external and internal employees. Aside from Haire's original attempt at reasoning or explaining by analogy, the work to date has consisted of an unresolvable debate in curve-fitting. We now proceed to suggest a theoretical derivation of the square-cube law, not by analogy but by a direct consideration of the underlying processes involved. The general theory from which the derivation will proceed is the theory of time-minimization mentioned above (Stephan). Its central assumption is that social structures evolve in such a way as to minimize the time which must be expended in their operation.

Assume a firm specified by a boundary which separates it from its environment, and which includes people who spend some of their time as its employees. Assume two measurements made on the firm, measurements which produce the numbers E (the number of "external employees," those who interact with others outside the firm) and I (the number of "internal employees," those who interact only with others inside the firm). Finally, from the general theory of time-minimization, assume that social structures, including the firm, evolve in such a way as to minimize the time which must be expended in their operation.

By definition, any benefits which accrue to the firm from its environment must be obtained through the time-expenditures of its external employees. If each external employee generates a particular amount of benefit, then the total benefits received will equal the average benefit productivity—which in the short run we may assume to be constant—times the number of external employees.

The factors determining the average level of benefit productivity need not concern us here (they would include such employee factors as skill, motivation, and job assignment, along with such environmental factors as resource abundance and the activity level of competing firms). We assume a given set of environmental conditions and a fixed level of average benefit productivity. In other words, we assume a given value for E and ask, how many internal employees will there be, what is the expected value of I, under the theory of time-minimization?

All the employees of the firm must be supported or compensated from the total pool of benefits held within the firm. Since this pool of benefits is brought in through the time-expenditures of the external employees, we may say that they in effect support themselves. At least on average, a portion of what they bring in is consumed by them. In contrast, the internal employees represent a special time-cost to the firm. The internal employees, by definition, do not bring the means of their own support into the firm. They must be supported, ultimately, through the time-expenditures of the external employees. The average support time will be directly proportional to the number of internal employees and inversely proportional to the number of external employees. Thus

$$T_s = aI/E \tag{6}$$

where a is the constant of proportionality.

If the internal employees thus appear parasitical, as a cost factor, they also contribute to reducing other costs of the firm. The benefit factor is that internal employees contribute by coordinating the work of the external employees. If there were no internal structure, if the external employees had to spend time coordinating their own activities by themselves, the amount of time spent would detract from the time they could spend at their primary assignment, bringing resources into the firm. How much time would be spent in coordination? Assuming that each one potentially could interact with all others, the time spent should be proportional to $E(E-1)/2$, the number of pairwise interactions in a group of E individuals; thus, as E becomes modestly large, the coordination time should be proportional to E^2. Since this work is actually done by the internal employees, we have an average coordination time which is directly proportional to E^2 and inversely proportional to I. Thus,

$$T_c = bE^2/I \tag{7}$$

where b is the constant of proportionality.

These two cost/benefit ratios represent the time expenditures of the internal and the external employees relative to one another. Their sum should give the overall time expenditure, the expenditure which the theory of time-minimization says will be minimized. Combining Equations 6 and 7 we obtain

$$T = aI/E + bE^2/I \tag{8}$$

To determine the relation between E and I when time is minimized, we differentiate T with respect to I (E is given) to obtain

$$dT/dI = a/E - b(E/I)^2 \tag{9}$$

which, set equal to zero, gives us

$$aI^2 = bE^3 \tag{10}$$

The values of E and I can never be negative, so the second derivative must be positive; Equation 10 therefore represents the condition when T is a minimum. Rearranging terms, and taking the sixth root of both sides, we obtain

$$I^{1/3} = kE^{1/2} \tag{11}$$

where $k = (b/a)^{1/6}$. Finally, since there is nothing in any of the above to prevent us from assuming some minimum value for I—in fact, it makes sense to assume that there must be some minimum amount of internal organization prior to any effective external activity, some start-up cost—we can insert this minimum value m to obtain

$$I^{1/3} = m + kE^{1/2} \tag{12}$$

an equation which is identical in form to Equation 1. It was developed from a consideration of the processes which would occur within the theoretical firm rather than from an analogy with physical objects or biological organisms. It does not presume a zero intercept; it assumes the positive intercept actually observed. It does not assume a slope equal to unity; the value of k depends entirely on the proportionality factors a and b which we have not constrained arbitrarily. Finally, Equation 12 does preserve the proportionality between E and I which Haire observed and which we have argued has not been satisfactorily rejected by the subsequent empirical work of others.

We conclude, first, that Haire's square-cube law has been successfully derived from theory and, second, that time-minimization theory has for the first time been extended beyond its prior applications in human ecology. Considering the amount of existing formal organizations theory and research based on the assumption of rationality in organization behavior, and considering the fact that time-minimization must surely be a major factor in such rationality, we would expect to see many further such derivations in the future.

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