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A Steepest-Ascent Method for Solving Optimum Programming Problems

A systematic and rapid steepest-ascent numerical procedure is described for solving two-point boundary-value problems in the calculus of variations for systems governed by a set of nonlinear ordinary differential equations. Numerical examples are presented for minimum time-to-climb and maximum altitude paths for a supersonic interceptor and maximum-range paths for an orbital glider.

1 Summary

A systematic and rapid steepest-ascent numerical procedure is described for determining optimum programs for nonlinear systems with terminal constraints. The procedure uses the concept of local linearization around a nominal (non-optimum) path. The effect on the terminal conditions of a small change in the control variable program is determined by numerical integration of the adjoint differential equations for small perturbations about the nominal path. Having these adjoint (or influence) functions, it is then possible to determine the change in the control variable program that gives maximum increase in the pay-off function for a given mean-square perturbation of the control variable program while simultaneously changing the terminal quantities by desired amounts. By repeating this process in small steps, a control variable program that minimizes one quantity and yields specified values of other terminal quantities can be approached as closely as desired. Three numerical examples are presented: (*a*) The angle-of-attack program for a typical supersonic interceptor to climb to altitude in minimum time is determined with and without specified terminal velocity and heading. (*b*) The angle-of-attack program for the same interceptor to climb to maximum altitude is determined. (*c*) The angle-of-attack program is determined for a hypersonic orbital glider to obtain maximum surface range starting from satellite speed at 300,000 ft altitude.

2 Introduction

Optimum programming problems arise in connection with processes developing in time or space, in which one or more control variables must be programmed to achieve certain terminal conditions. The problem is to determine, out of all possible programs for the control variables, the one program that maximizes (or minimizes) one terminal quantity while simultaneously yielding specified values of certain other terminal quantities.

The calculus of variations is the classical tool for solving such problems. However, until quite recently, only rather simple problems had been solved with this tool owing to computational difficulties. Even with a high-speed digital computer these problems are quite difficult because, in the classical formulation, they are two-point boundary-value problems for a set of nonlinear ordinary differential equations. Numerical solution requires guessing the missing boundary conditions at the initial point, integrating the differential equations numerically to the terminal point, finding how badly the specified terminal boundary conditions are missed, and then attempting to improve the guess of the unspecified initial conditions. This process must be repeated over and over until all terminal conditions are satisfied. This process is not only tedious, expensive, and frustrating, it sometimes does not seem to work at all [1].[1] It is remarkably sensitive to small changes in the initial conditions; however, it can be made to work through great patience, good guessing, and second-order multiple interpolation [2, 3, 4].

Recently Kelley [5, 6] and the authors with several coworkers [7, 8] have revived a little-known procedure which offers a practical, straightforward method for finding numerical solutions to even the most complicated optimum programming problems. It is essentially a steepest-ascent method and it requires the use of a high-speed digital computer.

3 A Maximum Problem in Ordinary Calculus

In order to explain the steepest-ascent method it is helpful to consider its use in a simpler problem first; namely, the problem of finding the maximum of a nonlinear function of many variables subject to nonlinear constraints on these variables. This is a problem in the ordinary calculus. A quite general problem of this type can be stated as follows:

Determine α so as to maximize

$$\phi = \phi(\mathbf{x}), \tag{1}$$

subject to the constraints

$$\boldsymbol{\psi} = \boldsymbol{\psi}(\mathbf{x}) = 0, \tag{2}$$

$$\mathbf{f}(\mathbf{x}, \boldsymbol{\alpha}) + \mathbf{f}_0 = 0 \tag{3}$$

where

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_m \end{bmatrix}, \text{ an } m \times 1 \text{ matrix of } \textit{control variables}, \text{ which we are free to choose,} \tag{4}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \text{ an } n \times 1 \text{ matrix of } \textit{state variables}, \text{ which result from the choice of } \boldsymbol{\alpha}, \tag{5}$$

$$\boldsymbol{\psi} = \begin{bmatrix} \psi_1 \\ \cdot \\ \cdot \\ \cdot \\ \psi_p \end{bmatrix}, \text{ a } p \times 1 \text{ matrix of } \textit{constraint functions}, \text{ each a known function of } \mathbf{x}, \tag{6}$$

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[1] Numbers in brackets designate References at end of paper.

ϕ is the *pay-off function*, a known function of $\mathbf{x}$, (7)

$$\mathbf{f} = \begin{bmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{bmatrix}, \text{ an } n \times 1 \text{ matrix of known functions of } \mathbf{x} \text{ and } \boldsymbol{\alpha}, \quad (8)$$

$$\mathbf{f}_0 = \begin{bmatrix} f_{10} \\ \cdot \\ \cdot \\ \cdot \\ f_{n0} \end{bmatrix}, \text{ an } n \times 1 \text{ matrix of constants} \quad (9)$$

4 Steepest-Ascent Method in Ordinary Calculus

The maximum problem stated in the preceding section can be solved systematically and rapidly on a high-speed digital computer using the steepest-ascent method. This method starts with a nominal control variable matrix $\boldsymbol{\alpha}^*$, and then improves this estimate by determining the direction of steepest ascent in the $\boldsymbol{\alpha}$ hyperspace; this determination is made by a linearization about the nominal point in the $\boldsymbol{\alpha}$ hyperspace. The method proceeds as follows:

(*a*) Guess some reasonable control variables $\boldsymbol{\alpha}^*$, and use them in equations (3) to calculate numerically the state variables $\mathbf{x}^*$ that correspond to this choice. In general, this nominal "point" will *not* satisfy the constraint conditions $\boldsymbol{\psi} = 0$, or yield the maximum value of ϕ.

(*b*) Consider small perturbations $d\boldsymbol{\alpha}$, about the nominal control variable point where

$$d\boldsymbol{\alpha} = \boldsymbol{\alpha} - \boldsymbol{\alpha}^* \quad (10)$$

These perturbations will cause perturbations $d\mathbf{x}$ in the state variables, where

$$d\mathbf{x} = \mathbf{x} - \mathbf{x}^* \quad (11)$$

Taking differentials of equations (3) we obtain to first order in the perturbations the linear set of equations for dx,

$$\mathbf{F}d\mathbf{x} + \mathbf{G}d\boldsymbol{\alpha} = 0 \quad (12)$$

where

$$\mathbf{F} = \begin{bmatrix} \left(\frac{\partial f_1}{\partial x_1}\right)^*, \ldots\ldots \left(\frac{\partial f_1}{\partial x_n}\right)^* \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \left(\frac{\partial f_n}{\partial x_1}\right)^*, \ldots\ldots \left(\frac{\partial f_n}{\partial x_n}\right)^* \end{bmatrix} \quad (13)$$

$$\mathbf{G} = \begin{bmatrix} \left(\frac{\partial f_1}{\partial \alpha_1}\right)^*, \ldots\ldots \left(\frac{\partial f_1}{\partial \alpha_m}\right)^* \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \left(\frac{\partial f_n}{\partial \alpha_1}\right)^*, \ldots\ldots \left(\frac{\partial f_n}{\partial \alpha_m}\right)^* \end{bmatrix} \quad (14)$$

and ( )* indicates that the partial derivatives are evaluated at the nominal point. Using Lagrange multipliers (Appendix, section 1), we may write

$$d\phi = \boldsymbol{\lambda}_\phi'\mathbf{G}d\boldsymbol{\alpha} + \boldsymbol{\lambda}_\phi'd\mathbf{f}_0 \quad (15)$$

$$d\boldsymbol{\psi} = \boldsymbol{\lambda}_\psi'\mathbf{G}d\boldsymbol{\alpha} + \boldsymbol{\lambda}_\psi'd\mathbf{f}_0 \quad (16)$$

where the λ matrices are determined by the linear equations

$$\left(\frac{\partial \phi}{\partial \mathbf{x}}\right)^* + \boldsymbol{\lambda}_\phi'\mathbf{F} = 0, \quad (17)$$

$$\left(\frac{\partial \boldsymbol{\psi}}{\partial \mathbf{x}}\right)^* + \boldsymbol{\lambda}_\psi'\mathbf{F} = 0, \quad (18)$$

where

$$\frac{\partial \phi}{\partial \mathbf{x}} = \left[\frac{\partial \phi}{\partial x_1}, \ldots \frac{\partial \phi}{\partial x_n}\right], \quad \frac{\partial \boldsymbol{\psi}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial \psi_1}{\partial x_1}, \ldots\ldots \frac{\partial \psi_1}{\partial x_n} \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \frac{\partial \psi_p}{\partial x_1}, \ldots\ldots \frac{\partial \psi_p}{\partial x_n} \end{bmatrix}, \quad (19)$$

and ( )′ indicates the transpose of ( ); i.e., rows and columns are interchanged.

Note that the $\boldsymbol{\lambda}$ are *influence numbers* since they tell how much ϕ or $\boldsymbol{\psi}$ is changed by small changes in the constraint levels $\mathbf{f}_0$.

For steepest ascent, we wish to find the $d\boldsymbol{\alpha}$ matrix that maximizes $d\phi$ in equation (15) for a given value of the positive definite quadratic form

$$(dP)^2 = d\boldsymbol{\alpha}'\mathbf{W}d\boldsymbol{\alpha}, \quad (20)$$

and given values of $d\boldsymbol{\psi}$ in equations (16). The values of $d\boldsymbol{\psi}$ are chosen to bring the nominal solution closer to the specified constraints, $\boldsymbol{\psi} = 0$. Choice of dP is made to insure that the perturbations $d\boldsymbol{\alpha}$ will be small enough for the linearization leading to equations (12) to be reasonable. $\mathbf{W}$ is an arbitrary non-negative definite $m \times m$ matrix of weighting numbers, essentially a metric in the $\boldsymbol{\alpha}$ hyperspace; it is at the disposal of the "optimizer" to improve convergence of the procedure.

The proper choice of $d\boldsymbol{\alpha}$ is derived in section 2 of the Appendix and the result is

$$d\boldsymbol{\alpha} = \pm\mathbf{W}^{-1}\mathbf{G}'(\boldsymbol{\lambda}_\phi - \boldsymbol{\lambda}_\psi\mathbf{I}_{\psi\psi}^{-1}\mathbf{I}_{\psi\phi})\left[\frac{(dP)^2 - d\boldsymbol{\beta}'\mathbf{I}_{\psi\psi}^{-1}d\boldsymbol{\beta}}{I_{\phi\phi} - \mathbf{I}_{\psi\phi}'\mathbf{I}_{\psi\psi}^{-1}\mathbf{I}_{\psi\phi}}\right]^{1/2} + \mathbf{W}^{-1}\mathbf{G}'\boldsymbol{\lambda}_\psi\mathbf{I}_{\psi\psi}^{-1}d\boldsymbol{\beta} \quad (21)$$

where

$$\begin{aligned} d\boldsymbol{\beta} &= d\boldsymbol{\psi} - \boldsymbol{\lambda}_\psi'd\mathbf{f}_0, \\ \mathbf{I}_{\psi\psi} &= \boldsymbol{\lambda}_\psi'\mathbf{G}\mathbf{W}^{-1}\mathbf{G}'\boldsymbol{\lambda}_\psi, \\ \mathbf{I}_{\psi\phi} &= \boldsymbol{\lambda}_\psi'\mathbf{G}\mathbf{W}^{-1}\mathbf{G}'\boldsymbol{\lambda}_\phi, \\ I_{\phi\phi} &= \boldsymbol{\lambda}_\phi'\mathbf{G}\mathbf{W}^{-1}\mathbf{G}'\boldsymbol{\lambda}_\phi, \end{aligned} \quad (22)$$

$( )^{-1}$ indicates inverse matrix, and the + sign is used if ϕ is to be increased, the − sign if ϕ is to be decreased. Note that the numerator under the square root in equation (21) can become negative if $d\boldsymbol{\beta}$ is chosen too large; thus there is a limit to the size of $d\boldsymbol{\beta}$ for a given dP. Since dP is chosen to insure valid linearization, the $d\boldsymbol{\beta}$ asked for must also be limited. The predicted change in ϕ for the change in control variables of equation (21) is

$$d\phi = \pm[((dP)^2 - d\boldsymbol{\beta}'\mathbf{I}_{\psi\psi}^{-1}d\boldsymbol{\beta})(I_{\phi\phi} - \mathbf{I}_{\psi\phi}'\mathbf{I}_{\psi\psi}^{-1}\mathbf{I}_{\psi\phi})]^{1/2} + \mathbf{I}_{\psi\phi}'\mathbf{I}_{\psi\psi}^{-1}d\boldsymbol{\beta} + \boldsymbol{\lambda}_\phi'd\mathbf{f}_0. \quad (23)$$

Notice if $d\boldsymbol{\psi} = 0$, $d\mathbf{f}_0 = 0$, equation (23) becomes

$$\frac{d\phi}{dP} = \pm(I_{\phi\phi} - \mathbf{I}_{\psi\phi}'\mathbf{I}_{\psi\psi}^{-1}\mathbf{I}_{\psi\phi})^{1/2} \quad (24)$$

which is the magnitude of the gradient in the $\boldsymbol{\alpha}$ hyperspace, since dP is the length of the step in the $\boldsymbol{\alpha}$ hyperspace. As the maximum is approached and the constraints are met ($d\boldsymbol{\psi} = 0$), this gradient must tend to zero, which results in

$$d\phi = I_{\psi\phi}' I_{\psi\psi}^{-1} d\psi + (\lambda_\phi' - I_{\psi\phi}' I_{\psi\psi}^{-1} \lambda_\psi') df_0 \qquad (25)$$

This relation shows how much the maximum pay-off function changes for small changes in the constraint levels.

(*c*) A new control variable point is obtained as

$$\boldsymbol{\alpha}_{\text{NEW}} = \boldsymbol{\alpha}_{\text{OLD}} + d\boldsymbol{\alpha}$$

where $d\boldsymbol{\alpha}$ is obtained from equation (21). $\boldsymbol{\alpha}_{\text{NEW}}$ is used in the original nonlinear equations (3) and the whole process is repeated several times until the $\psi = 0$ constraints are met and the gradient is nearly zero in equation (24). The maximum value of ϕ has then been obtained.

This process can be likened to climbing a mountain in a dense fog. We cannot see the top but we ought to be able to get there by always climbing in the direction of steepest ascent. If we do this in steps, climbing in one direction until we have traveled a certain horizontal distance, then reassessing the direction of steepest ascent, climbing in that direction, and so on, this is the exact analog of the procedure suggested here in a space of m-dimensions where ϕ is altitude and α_1, α_2 are co-ordinates in the horizontal plane, Fig. 1. There is, of course, a risk here in that we may climb a secondary peak and, in the fog, never become aware of our mistake.

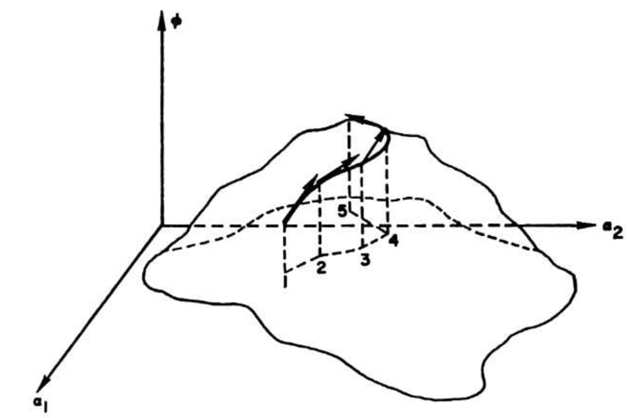

Fig. 1 Finding maximum of a function of two variables by steepest-ascent method

5 Optimum Programming, a Problem in the Calculus of Variations

An optimum programming problem of considerable generality can be stated as follows, Fig. 2:

Determine $\boldsymbol{\alpha}(t)$ in the interval $t_0 \leq t \leq T$, so as to maximize

$$\phi = \phi(\mathbf{x}(T), T), \qquad (26)^2$$

subject to the constraints,

$$\psi = \psi(\mathbf{x}(T), T) = 0, \qquad (27)$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}(t), \boldsymbol{\alpha}(t), t), \qquad (28)$$

$$t_0 \text{ and } \mathbf{x}(t_0) \text{ given}, \qquad (29)^3$$

$$T \text{ determined by } \Omega = \Omega(\mathbf{x}(T), T) = 0 \qquad (30)$$

The nomenclature of the problem is as follows:

$$\boldsymbol{\alpha}(t) = \begin{bmatrix} \alpha_1(t) \\ \cdot \\ \cdot \\ \cdot \\ \alpha_m(t) \end{bmatrix}, \text{ an } m \times 1 \text{ matrix of \textit{control variable programs}, which we are free to choose,} \qquad (31)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix}, \text{ an } n \times 1 \text{ matrix of \textit{state variable programs}, which result from a choice of } \boldsymbol{\alpha}(t) \text{ and given values of } \mathbf{x}(t_0), \qquad (32)$$

$$\psi = \begin{bmatrix} \psi_1 \\ \cdot \\ \cdot \\ \cdot \\ \psi_p \end{bmatrix}, \text{ a } p \times 1 \text{ matrix of \textit{terminal constraint functions}, each of which is a known function of } \mathbf{x}(T) \text{ and } T, \qquad (33)$$

[2] If an integral is to be maximized, simply introduce an additional state variable x_q and an additional differential equation $\dot{x}_q = q(\mathbf{x}, \alpha, t)$ where q is the integrand of the integral. $x_q(T)$ is then maximized with $x_q(t_0) = 0$.

[3] In some problems not all of the state variables are specified initially; in this case the unspecified state variables may be determined along with $\alpha(t)$ to maximize ϕ.

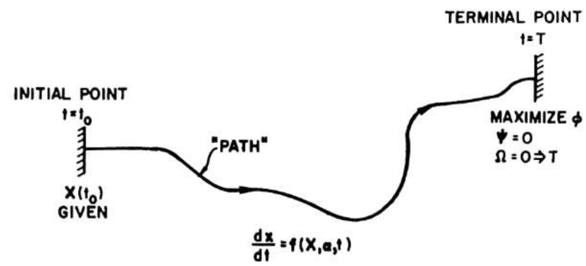

Fig. 2 Symbolic sketch of optimum programming problem

ϕ is the *pay-off function* and is known function of $\mathbf{x}(T)$ and T, (34)

$$\mathbf{f} = \begin{bmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{bmatrix}, \text{ an } n \times 1 \text{ matrix of known functions of } \mathbf{x}(t), \boldsymbol{\alpha}(t), \text{ and } t, \qquad (35)$$

$\Omega = 0$ is the *stopping condition that determines final time* T, and is a known function of $\mathbf{x}(T)$ and T (36)

The formulation of the necessary conditions for an extremal solution to this problem has been given by Breakwell [2] with the added complexity of inequality constraints on the control variables. The present paper is concerned with the efficient and rapid solution of such problems using a steepest-ascent procedure.

6 Steepest-Ascent Method in Calculus of Variations

The optimum programming problem stated in the preceding section can be solved systematically and rapidly on a high-speed digital computer using the steepest-ascent technique. This technique starts with a nominal control variable program $\boldsymbol{\alpha}^*(t)$, and then improves this program in steps, using information obtained by a mathematical diagnosis of the program for the previous step. Conceptually it is a process of local linearization around the path of the previous step. The method proceeds as follows:

(*a*) Guess some reasonable control variable programs $\boldsymbol{\alpha}^*(t)$, and use them with the initial conditions (29) and the differential equations (28) to calculate, numerically, the state variable programs $\mathbf{x}^*(t)$ until $\Omega = 0$. In general, this nominal "path" will *not* satisfy the terminal conditions $\psi = 0$, or yield the maximum possible value of ϕ.

(*b*) Consider small perturbations $\delta\boldsymbol{\alpha}(t)$ about the nominal control variable programs, where

$$\delta \boldsymbol{\alpha} = \boldsymbol{\alpha}(t) - \boldsymbol{\alpha}^*(t) \tag{37}$$

These perturbations will cause perturbations in the state variable programs $\delta \mathbf{x}(t)$, where

$$\delta \mathbf{x} = \mathbf{x}(t) - \mathbf{x}^*(t) \tag{38}$$

Substituting these relations into the differential equations (28) we obtain, to first order in the perturbations, the linear differential equations for $\delta \mathbf{x}$,

$$\frac{d}{dt}(\delta \mathbf{x}) = \mathbf{F}(t)\delta \mathbf{x} + \mathbf{G}(t)\delta \boldsymbol{\alpha}, \tag{39}$$

where

$$\mathbf{F}(t) = \begin{bmatrix} \left(\frac{\partial f_1}{\partial x_1}\right)^*, \ldots\ldots\ldots \left(\frac{\partial f_1}{\partial x_n}\right)^* \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \left(\frac{\partial f_n}{\partial x_1}\right)^*, \ldots\ldots\ldots \left(\frac{\partial f_n}{\partial x_n}\right)^* \end{bmatrix} \tag{40}$$

$$\mathbf{G}(t) = \begin{bmatrix} \left(\frac{\partial f_1}{\partial \alpha_1}\right)^*, \ldots\ldots\ldots \left(\frac{\partial f_1}{\partial \alpha_m}\right)^* \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \left(\frac{\partial f_n}{\partial \alpha_1}\right)^*, \ldots\ldots\ldots \left(\frac{\partial f_n}{\partial \alpha_m}\right)^* \end{bmatrix} \tag{41}$$

and $(\ )^*$ indicates that the partial derivatives are evaluated along the nominal path. From the theory of adjoint equations, section 3 of the Appendix, we may write

$$d\phi = \int_{t_0}^{T} \boldsymbol{\lambda}_\phi'(t)\mathbf{G}(t)\delta\boldsymbol{\alpha}(t)dt + \boldsymbol{\lambda}_\phi'(t_0)\delta\mathbf{x}(t_0) + \dot{\phi}dT, \tag{42}$$

$$d\boldsymbol{\psi} = \int_{t_0}^{T} \boldsymbol{\lambda}_\psi'(t)\mathbf{G}(t)\delta\boldsymbol{\alpha}(t)dt + \boldsymbol{\lambda}_\psi'(t_0)\delta\mathbf{x}(t_0) + \dot{\boldsymbol{\psi}}dT, \tag{43}$$

$$d\Omega = \int_{t_0}^{T} \boldsymbol{\lambda}'_\Omega(t)\mathbf{G}(t)\delta\boldsymbol{\alpha}(t)dt + \boldsymbol{\lambda}_\Omega'(t_0)\delta\mathbf{x}(t_0) + \dot{\Omega}dT, \tag{44}$$

where the elements of the $\boldsymbol{\lambda}$ matrices are determined by numerical integration of the differential equations adjoint to equations (39); namely,

$$\frac{d\boldsymbol{\lambda}}{dt} = -\mathbf{F}'(t)\boldsymbol{\lambda} \tag{45}$$

with boundary conditions

$$\boldsymbol{\lambda}_\phi'(T) = \left(\frac{\partial\phi}{\partial\mathbf{x}}\right)^*_{t=T}, \quad \boldsymbol{\lambda}_\psi'(T) = \left(\frac{\partial\boldsymbol{\psi}}{\partial\mathbf{x}}\right)^*_{t=T}, \quad \boldsymbol{\lambda}_\Omega'(T) = \left(\frac{\partial\Omega}{\partial\mathbf{x}}\right)^*_{t=T} \tag{46}$$

where

$$\frac{\partial\phi}{\partial\mathbf{x}} = \left[\frac{\partial\phi}{\partial x_1}, \ldots \frac{\partial\phi}{\partial x_n}\right], \quad \frac{\partial\boldsymbol{\psi}}{\partial\mathbf{x}} = \begin{bmatrix} \frac{\partial\psi_1}{\partial x_1} \cdots \frac{\partial\psi_1}{\partial x_n} \\ \cdot \qquad \cdot \\ \cdot \qquad \cdot \\ \cdot \qquad \cdot \\ \frac{\partial\psi_p}{\partial x_1} \cdots \frac{\partial\psi_p}{\partial x_n} \end{bmatrix},$$

$$\frac{\partial\Omega}{\partial\mathbf{x}} = \left[\frac{\partial\Omega}{\partial x_1}, \ldots \frac{\partial\Omega}{\partial x_n}\right], \tag{47}$$

and

$$\dot{\phi} = \left(\frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial\mathbf{x}}\mathbf{f}\right)^*_{t=T}, \quad \dot{\boldsymbol{\psi}} = \left(\frac{\partial\boldsymbol{\psi}}{\partial t} + \frac{\partial\boldsymbol{\psi}}{\partial\mathbf{x}}\mathbf{f}\right)^*_{t=T},$$

$$\dot{\Omega} = \left(\frac{\partial\Omega}{\partial t} + \frac{\partial\Omega}{\partial\mathbf{x}}\mathbf{f}\right)^*_{t=T}, \tag{48}$$

$(\ )'$ indicates the transpose of $(\ )$; i.e., rows and columns are interchanged.

Note that the $\boldsymbol{\lambda}$ are *influence functions* since they tell how much a certain terminal condition is changed by a small change in some initial state terminal variable. Note also that the adjoint equations (47) must be integrated *backward* since the boundary conditions are given at the terminal point, $t = T$.

For steepest ascent we wish to find the $\delta\boldsymbol{\alpha}(t)$ programs that maximize $d\phi$ in equation (42) for a given value of the integral

$$(dP)^2 = \int_{t_0}^{T} \delta\boldsymbol{\alpha}'(t)\mathbf{W}(t)\delta\boldsymbol{\alpha}(t)dt, \tag{49}$$

given values of $d\boldsymbol{\psi}$ in equations (43) and $d\Omega = 0$ in equation (44). The values of $d\boldsymbol{\psi}$ are chosen to bring the nominal solution closer to the desired terminal constraints, $\boldsymbol{\psi} = 0$. Choice of dP is made to insure that the perturbations $\delta\boldsymbol{\alpha}(t)$ will be small enough for the linearization leading to equations (46) to be reasonable. $\mathbf{W}(t)$ is an arbitrary symmetric $m \times m$ matrix of weighting functions chosen to improve convergence of the steepest-ascent procedure; in some problems it is desirable to subdue $\delta\boldsymbol{\alpha}$ in certain highly sensitive regions in favor of larger $\delta\boldsymbol{\alpha}$ in the less sensitive regions.

The proper choice of $\delta\boldsymbol{\alpha}(t)$ is derived in section 4 of the Appendix and the result is

$$\delta\boldsymbol{\alpha}(t) = \pm\mathbf{W}^{-1}\mathbf{G}'(\boldsymbol{\lambda}_{\phi\Omega} - \boldsymbol{\lambda}_{\psi\Omega}\mathbf{I}_{\psi\psi}^{-1}\mathbf{I}_{\psi\phi})\left[\frac{(dP)^2 - d\boldsymbol{\beta}'\mathbf{I}_{\psi\psi}^{-1}d\boldsymbol{\beta}}{I_{\phi\phi} - \mathbf{I}_{\psi\phi}'\mathbf{I}_{\psi\psi}^{-1}\mathbf{I}_{\psi\phi}}\right]^{1/2} + \mathbf{W}^{-1}\mathbf{G}'\boldsymbol{\lambda}_{\psi\Omega}\mathbf{I}_{\psi\psi}^{-1}d\boldsymbol{\beta}, \tag{50}$$

where

$$\begin{aligned} d\boldsymbol{\beta} &= d\boldsymbol{\psi} - \boldsymbol{\lambda}_{\psi\Omega}'(t_0)\delta\mathbf{x}(t_0), \\ \boldsymbol{\lambda}_{\phi\Omega} &= \boldsymbol{\lambda}_\phi - \frac{\dot{\phi}}{\dot{\Omega}}\boldsymbol{\lambda}_\Omega, \\ \boldsymbol{\lambda}_{\psi\Omega} &= \boldsymbol{\lambda}_\psi - \boldsymbol{\lambda}_\Omega\frac{\dot{\boldsymbol{\psi}}'}{\dot{\Omega}} \\ \mathbf{I}_{\psi\psi} &= \int_{t_0}^{T} \boldsymbol{\lambda}_{\psi\Omega}'\mathbf{G}\mathbf{W}^{-1}\mathbf{G}'\boldsymbol{\lambda}_{\psi\Omega}dt, \\ \mathbf{I}_{\psi\phi} &= \int_{t_0}^{T} \boldsymbol{\lambda}_{\psi\Omega}'\mathbf{G}\mathbf{W}^{-1}\mathbf{G}'\boldsymbol{\lambda}_{\phi\Omega}dt, \\ I_{\phi\phi} &= \int_{t_0}^{T} \boldsymbol{\lambda}_{\phi\Omega}'\mathbf{G}\mathbf{W}^{-1}\mathbf{G}'\boldsymbol{\lambda}_{\phi\Omega}dt, \end{aligned} \tag{51}$$

$(\ )^{-1}$ indicates inverse matrix, and the $+$ sign is used if ϕ is to be increased, the $-$ sign is used if ϕ is to be decreased. Note that the numerator under the square root in equation (50) can become negative if $d\boldsymbol{\beta}$ is chosen too large; thus there is a limit to the size of $d\boldsymbol{\beta}$ for a given dP. Since dP is chosen to insure valid linearization, the $d\boldsymbol{\beta}$ asked for must also be limited. The predicted change in ϕ for the change in control variable program (50) is

$$d\phi = \pm\left[((dP)^2 - d\boldsymbol{\beta}'\mathbf{I}_{\psi\psi}^{-1}d\boldsymbol{\beta})(I_{\phi\phi} - \mathbf{I}_{\psi\phi}'\mathbf{I}_{\psi\psi}^{-1}\mathbf{I}_{\psi\phi})\right]^{1/2} + \mathbf{I}_{\psi\phi}'\mathbf{I}_{\psi\psi}^{-1}d\boldsymbol{\beta} + \boldsymbol{\lambda}_{\phi\Omega}'(t_0)\delta\mathbf{x}(t_0) \tag{52}$$

Notice that if $d\boldsymbol{\psi} = 0$, $\delta\mathbf{x}(t_0) = 0$, equation (52) becomes

$$\frac{d\phi}{dP} = \pm(I_{\phi\phi} - \mathbf{I}_{\psi\phi}'\mathbf{I}_{\psi\psi}^{-1}\mathbf{I}_{\psi\phi})^{1/2} \tag{53}$$

which is a "gradient" in function space, since dP is the "length" of the step in the control variable programs. As the optimum program is approached and the terminal constraints are met ($d\boldsymbol{\psi} = 0$), this gradient must tend to zero, which results in

$$d\phi = \mathbf{I}_{\psi\phi}'\mathbf{I}_{\psi\psi}^{-1}d\boldsymbol{\psi} + [\boldsymbol{\lambda}_{\phi\Omega}'(t_0) - \mathbf{I}_{\psi\phi}'\mathbf{I}_{\psi\psi}^{-1}\boldsymbol{\lambda}_{\psi\Omega}'(t_0)]\delta\mathbf{x}(t_0) \quad (54)$$

This relation shows how much the maximum pay-off function changes for small changes in the terminal constraints and for small changes in the initial conditions.

(c) New control variable programs are obtained as

$$\boldsymbol{\alpha}_{\text{NEW}}(t) = \boldsymbol{\alpha}_{\text{OLD}}(t) + \delta\boldsymbol{\alpha}(t) \quad (55)$$

where $\delta\boldsymbol{\alpha}$ is given in equation (50). $\boldsymbol{\alpha}_{\text{NEW}}(t)$ is used in the original nonlinear differential equations (28) and the whole process is repeated several times until the terminal constraints are met and the gradient is nearly zero in equation (53). The optimum program has then been obtained.

Note that *minimizing final time T* fits into this pattern conveniently. For this case we let $\phi = -t$ which implies $\dot{\phi} = -1$, $\boldsymbol{\lambda}_\phi = 0$, and hence

$$\boldsymbol{\lambda}_{\phi\Omega} = \frac{1}{\dot{\Omega}}\boldsymbol{\lambda}_\Omega$$

$$d\phi = -dT = \int_{t_0}^{T}\frac{1}{\dot{\Omega}}\boldsymbol{\lambda}_\Omega'\mathbf{G}\delta\boldsymbol{\alpha}dt + \frac{1}{\dot{\Omega}}\boldsymbol{\lambda}_\Omega'(t_0)\delta\mathbf{x}(t_0) \quad (56)$$

7 Computing Procedures

The computing procedures evolved over a period of time in solving many types of problems are summarized here. The computer used for all these problems was an IBM 704.

(a) Compute the nominal path by integrating the nonlinear physical differential equations with a nominal control variable program and appropriate (here assumed fixed) initial conditions and store the solution on tape.

(b) Compute the $\boldsymbol{\lambda}_\phi, \boldsymbol{\lambda}_{\psi\,1}, \boldsymbol{\lambda}_{\psi\,2}, \ldots \boldsymbol{\lambda}_{\psi\,p}, \boldsymbol{\lambda}_\Omega$ functions *all at the same time* by integrating the adjoint differential equations backward, evaluating the partial derivatives on the nominal path by reference to the tape in (a).

(c) Simultaneously with (b), calculate the quantities $\boldsymbol{\lambda}_{\phi\Omega}, \boldsymbol{\lambda}_{\psi 1\Omega}, \ldots \boldsymbol{\lambda}_{\psi_p\Omega}$ and store $\boldsymbol{\lambda}_{\phi\Omega}'\mathbf{G}$ and $\boldsymbol{\lambda}_{\psi\Omega}'\mathbf{G}$ on tape.

(d) Also simultaneously with (b) and (c) perform the integrations (backward) leading to the numbers $I_{\phi\phi}$, $\mathbf{I}_{\psi\phi}$, $\mathbf{I}_{\psi\psi}$.

(e) Print out the values of $\phi, \psi_1, \psi_2, \ldots \psi_p$ achieved by the nominal path.

(f) Select desired terminal condition changes $d\psi_1, d\psi_2, \ldots d\psi_p$ to bring the next solution closer to the specified values $\boldsymbol{\psi} = 0$ than were achieved by the nominal path.

(g) Select a reasonable value of $(dP)^2/(T - t_0)$, which is a mean-square deviation of the control variable programs from the nominal to the next step.

(h) Use the values of $d\boldsymbol{\psi}$ and dP to calculate $(dP)^2 - d\boldsymbol{\psi}'\mathbf{I}_{\psi\psi}^{-1}d\psi$; if this quantity is negative, automatically scale down $d\psi$ to make this quantity vanish. If the quantity is positive, leave it as is.

(i) Using the values of dP and $d\boldsymbol{\psi}$ (modified by (h) if necessary) calculate $\delta\boldsymbol{\alpha}(t)$ from equation (50), ($\delta\mathbf{x}(t_0) = 0$).

(j) If final time, T, is not specified or being extremalized, compute the predicted change, dT, for the next step:

$$dT = -\frac{1}{\dot{\Omega}}\int_{t_0}^{T}\boldsymbol{\lambda}_\Omega'\mathbf{G}\delta\boldsymbol{\alpha}dt$$

If $|dT|$ is greater than a preselected maximum allowable value, scale down $\delta\boldsymbol{\alpha}(t)$ to achieve this maximum value.

(k) Obtain a new nominal path by using $\boldsymbol{\alpha}_{\text{NEW}} = \boldsymbol{\alpha}_{\text{OLD}} + \delta\boldsymbol{\alpha}$ and repeat processes (a) through (k) until the terminal constraints $\boldsymbol{\psi} = 0$ are satisfied and the square of the gradient, $I_{\phi\phi} - \mathbf{I}'_{\psi\phi}\mathbf{I}_{\psi\psi}^{-1}\mathbf{I}_{\psi\phi}$, tends to zero.

8 Example 1

Angle-of-Attack Program for a Supersonic Interceptor to Climb From Sea Level to a Given Altitude in Least Time. A typical supersonic interceptor is considered with lift, drag, and thrust characteristics as shown in Fig. 4. The vehicle is considered as a mass point (short period pitching motions are thus neglected) and the nomenclature used is shown in Fig. 3. The problem is to find the angle-of-attack program $\alpha(t)$, using maximum thrust, that takes the interceptor to a given final altitude in the least time, starting just after take-off at $M = 0.22$, $\gamma = 0$, at sea level ($h = 0$). First the problem is solved with no terminal constraints, then with $M = 0.9$ specified at the terminal point, then with $M = 0.9$ and $\gamma = 0$ at the terminal point. The differential equations for the interceptor path are

$$\dot{V} = \frac{F(h, M)}{m}\cos\alpha - \frac{D(h, M, \alpha)}{m} - g\sin\gamma \quad (57)$$

$$\dot{\gamma} = \frac{L(h, M, \alpha)}{mV} + \frac{F(h, M)}{mV}\sin\alpha - \frac{g\cos\gamma}{V} \quad (58)$$

$$\dot{h} = V\sin\gamma \quad (59)$$

$$\dot{x} = V\cos\gamma \quad (60)$$

$$\dot{m} = \dot{m}(h, M) \quad (61)$$

where $(\dot{\ })$ means $\frac{d}{dt}(\ )$, and

$F = F(h, M)$ is thrust, given as a tabular function, Fig. 4(a)

$D = C_D(\alpha, M)\dfrac{\rho V^2 S}{2}$ is drag

$L = C_L(\alpha, M)\dfrac{\rho V^2 S}{2}$ is lift

$C_D(\alpha, M)$, $C_L(\alpha, M)$ are given as tabular functions, Fig. 4(b)

$\rho = \rho(h)$ is air density, given as a tabular function

g = acceleration due to gravity (taken as constant here)

$M = \dfrac{V}{a}$ is Mach number

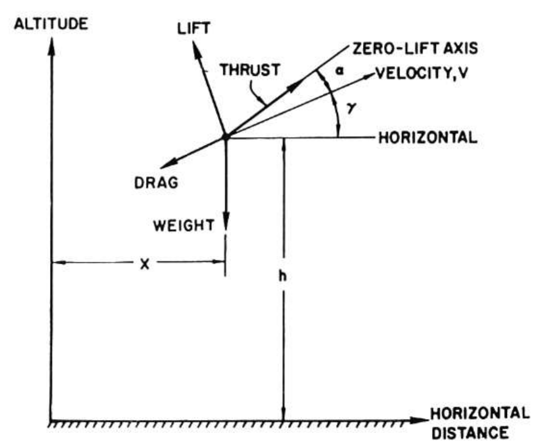

Fig. 3 Nomenclature used in analysis of supersonic interceptor

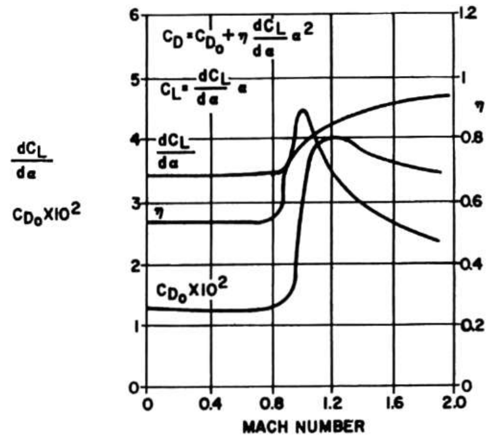

Fig. 4(a) Supersonic interceptor—lift and drag coefficient versus Mach number and angle-of-attack

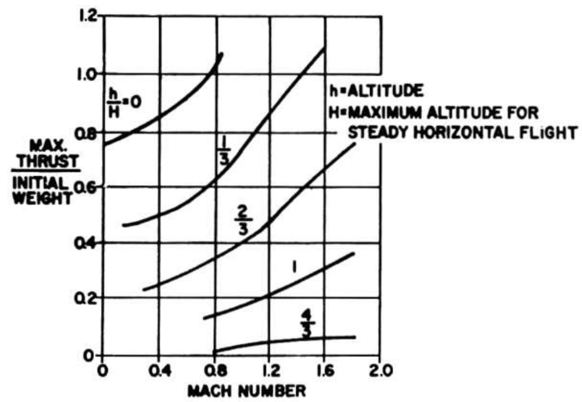

Fig. 4(b) Supersonic interceptor—thrust versus Mach number and altitude

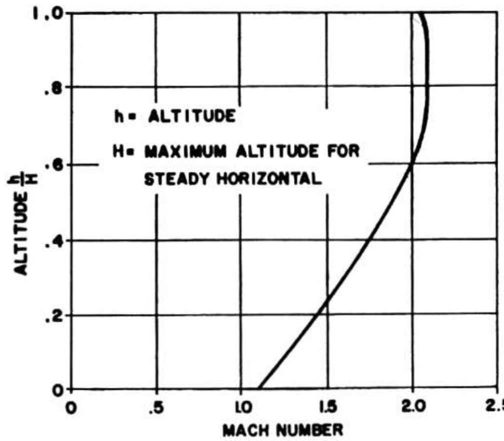

Fig. 4(c) Supersonic interceptor—altitude versus Mach number for steady level flight

$a = a(h)$ is speed of sound, given as a tabular function
$\dot{m}(h, M)$ is fuel consumption, given as a tabular function
m is mass of vehicle

The adjoint differential equations are:

$$\dot{\lambda}_V + \lambda_V\left(\frac{\cos\alpha}{ma}\frac{\partial F}{\partial M} - \frac{2D}{mV} - \frac{D}{ma}\frac{1}{C_D}\frac{\partial C_D}{\partial M}\right) + \lambda_\gamma\left(\frac{g\cos\gamma}{V^2} + \frac{L}{mV^2} + \frac{L}{mVa}\frac{1}{C_L}\frac{\partial C_L}{\partial M} - \frac{F\sin\alpha}{mV^2} + \frac{\sin\alpha}{mVa}\frac{\partial F}{\partial M}\right) + \lambda_h\sin\gamma + \lambda_x\cos\gamma + \lambda_m\frac{1}{a}\frac{\partial\dot{m}}{\partial M} = 0 \quad (62)$$

$$\dot{\lambda}_\gamma + \lambda_V(-g\cos\gamma) + \lambda_\gamma\left(\frac{g\sin\gamma}{V}\right) + \lambda_h V\cos\gamma - \lambda_x V\sin\gamma = 0 \quad (63)$$

$$\dot{\lambda}_h + \lambda_V\left(-\frac{D}{m}\frac{1}{\rho}\frac{d\rho}{dh} + \frac{D}{m}\frac{1}{C_D}\frac{\partial C_D}{\partial M}\frac{M}{a}\frac{da}{dh} + \frac{\cos\alpha}{m}\frac{\partial F}{\partial h}\right) + \lambda_\gamma\left(\frac{L}{mV}\frac{1}{\rho}\frac{d\rho}{dh} - \frac{L}{mV}\frac{1}{C_L}\frac{\partial C_L}{\partial M}\frac{M}{a}\frac{da}{dh} + \frac{\sin\alpha}{mV}\frac{\partial F}{\partial h}\right) + \lambda_m\frac{\partial\dot{m}}{\partial h} = 0 \quad (64)$$

$$\dot{\lambda}_x = 0 \quad (65)$$

$$\dot{\lambda}_m + \lambda_V\left(\frac{D}{m^2} - \frac{F\cos\alpha}{m^2}\right) + \lambda_\gamma\left(-\frac{L}{m^2V} - \frac{F\sin\alpha}{m^2V}\right) = 0 \quad (66)$$

In the nomenclature of Sections 5 and 6, three cases were calculated:

Case	ϕ	ψ_1	ψ_2	Ω	
1	$-t$,	$-$,	$-$,	$h - H$	
2	$-t$,	$M - 0.9$,	$-$,	$h - H$	(67)
3	$-t$,	$M - 0.9$,	γ,	$h - H$	

where H is the final altitude desired, taken in this case as the maximum altitude for steady horizontal flight (the service ceiling). Initial conditions were

$$M = 0.22,\quad \gamma = 0,\quad h = 0,\quad x = 0,\quad m = \frac{W_0}{g} \quad (68)$$

where W_0 is the initial weight.

The results are shown in Fig. 5. The trajectories all show the "zoom" characteristic found by other investigators [9]; the airplane first climbs, then dives accelerating through sonic speed, then pulls up, trading kinetic for potential energy in the earth's gravitational field. This characteristic appears to be caused by the sharp transonic drag rise, the rapid thrust attenuation with altitude, and the thrust increase with Mach number. These surprising trajectories show the danger in using classical performance methods on high-performance airplanes. The quasi-steady analysis to determine local maximum rate-of-climb shows an almost constant slightly subsonic Mach number to be the best for climb. If the airplane is flown this way, it takes nearly *twice as long* to reach the altitude H as it does with the trajectories shown in Fig. 5. This quasi-steady flight path was used as the nominal path in the steepest-ascent method and the optimum path was reached in six or seven "steps."

Note that the addition of the $M = 0.9$ terminal constraint increased the minimum time to climb by 8 per cent. Further addition of the $\gamma = 0$ terminal constraint increased the minimum time to climb less than $^1/_2$ per cent. The angle-of-attack programs are not at all unusual and should not be difficult to approximate for actual flight situations.

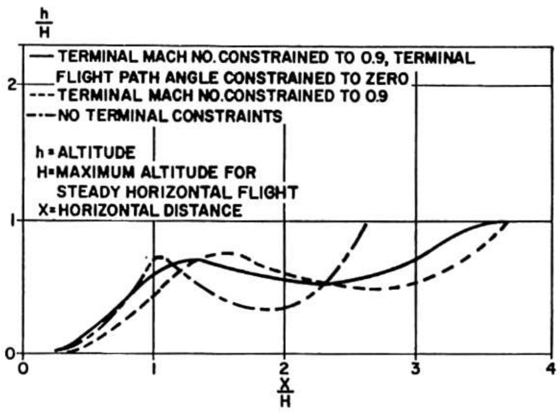

Fig. 5(a) Supersonic interceptor—flight path for minimum time to climb to service ceiling from sea level—altitude versus range

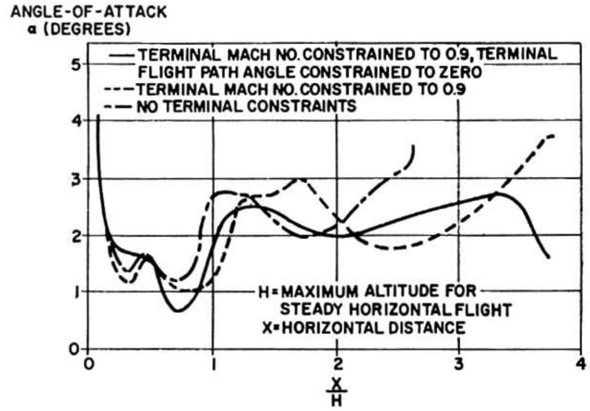

Fig. 5(c) Supersonic interceptor—flight path for minimum time to climb to service ceiling from sea level—angle-of-attack versus range

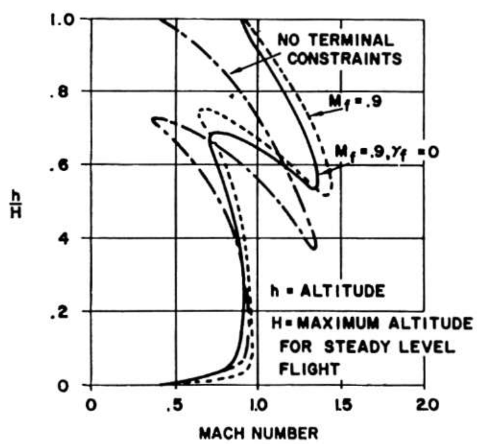

Fig. 5(b) Supersonic interceptor—flight path for minimum time to climb to service ceiling from sea level—altitude versus Mach number

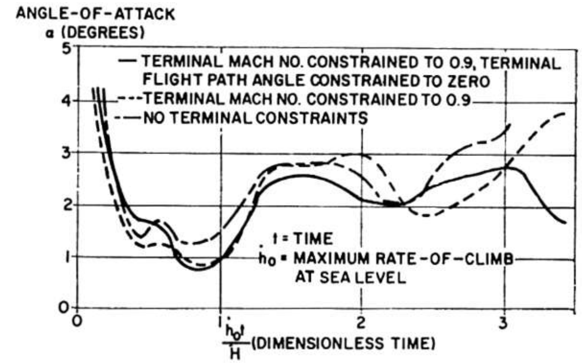

Fig. 5(d) Supersonic interceptor—flight path for minimum time to climb to service ceiling from sea level—angle-of-attack versus time

9 Example 2

Angle-of-Attack Program for a Supersonic Interceptor to Climb to Maximum Altitude. The same airplane considered in Section 8 is used here with the pay-off function, ϕ, being altitude, h. The stopping condition is $\Omega = \gamma = 0$; i.e., flight-path angle zero. The initial condition was taken as the maximum energy condition for steady horizontal flight, where energy per unit mass is $(V^2/2) + gh$; i.e., kinetic plus potential energy. This maximum energy condition occurred at

$$\frac{h}{H} = 0.81$$

$$\gamma = 0$$

$$M = 2.1$$

In this case no terminal constraints were used. The physical and adjoint equations are the same as those in Section 8.

The resulting flight path is shown in Fig. 6. It again contains a preliminary climb and dive, followed by a zoom to the maximum altitude which is 64 per cent higher than the service ceiling (maximum altitude for steady horizontal flight). If all the energy

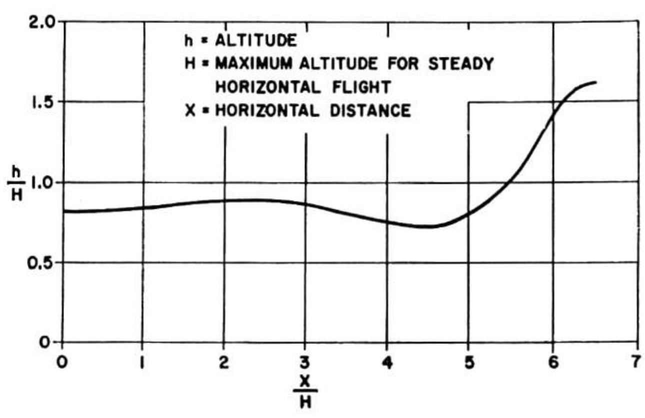

Fig. 6(a) Supersonic interceptor—flight path for maximum altitude—altitude versus range

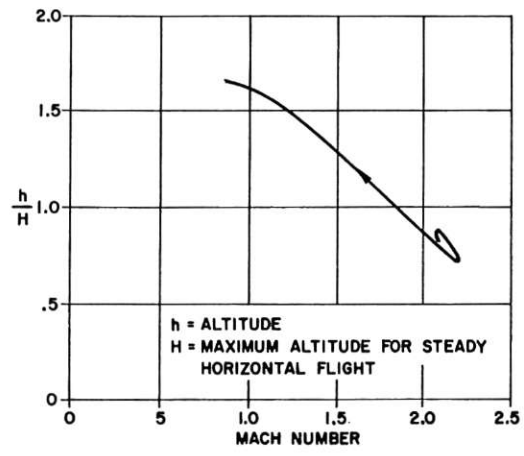

Fig. 6(b) Supersonic interceptor—flight path for maximum altitude—altitude versus Mach number

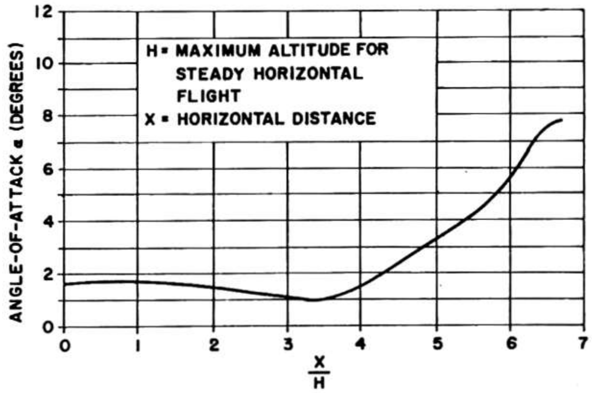

Fig. 6(c) Supersonic interceptor—flight path for maximum altitude—angle-of-attack versus range

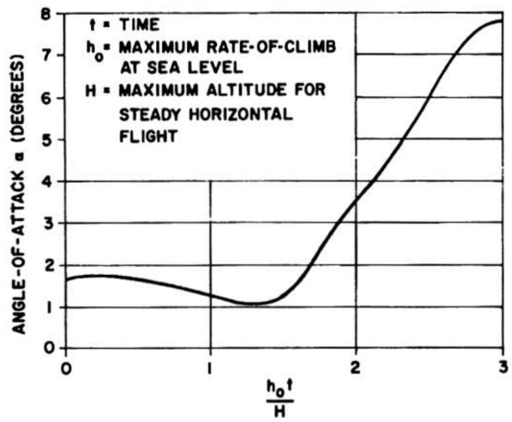

Fig. 6(d) Supersonic interceptor—flight path for maximum altitude—angle-of-attack versus time

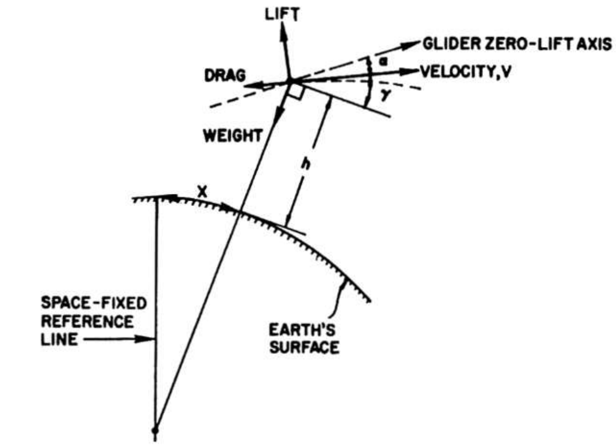

Fig. 7 Nomenclature used in analysis of hypersonic orbital glider

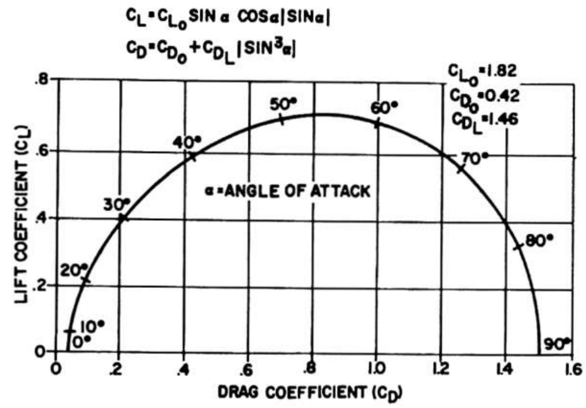

Fig. 8 Hypersonic orbital glider—lift and drag coefficients as functions of angle-of-attack

could be converted into potential energy, the maximum altitude would be 97 per cent higher than the service ceiling. Note that the thrust essentially vanishes near $h/H = 4/3$; actually the turbojet engines will "blow out" somewhere near this latter altitude.

10 Example 3

Angle-of-Attack Program for a Hypersonic Orbital Glider to Achieve Maximum Range. The problem here is to find the angle-of-attack program $\alpha(t)$ for a hypersonic glider to achieve maximum range on the surface of the earth, starting from the point where it has been injected into a low-altitude satellite orbit by a rocket booster. The nomenclature for the analysis is shown in Fig. 7, and the lift-drag characteristics are shown in Fig. 8. The wing loading used was

$$\frac{mg_0}{S} = 27.3 \text{ lb ft}^{-2}$$

where

m = mass of vehicle
g_0 = acceleration of gravity at earth's surface
S = wing plan-form area

The initial conditions used were

$$\begin{aligned} V &= 25{,}920 \text{ ft sec}^{-1} \\ \gamma &= 0.18 \text{ deg} \\ h &= 300{,}000 \text{ ft} \end{aligned} \tag{69}$$

Owing to the wide range of velocities encountered in this problem (landing speeds were around 200 ft sec^{-1}) it was found convenient to use distance along the flight path, s, as the independent variable in solving the differential equations. The physical differential equations used are:

$$\frac{dV}{ds} = -\frac{D(h, M, \alpha)}{mV} - \frac{g(h)\sin\gamma}{V} \tag{70}$$

$$\frac{d\gamma}{ds} = \frac{\cos\gamma}{R+h} + \frac{L(h, M, \alpha)}{mV^2} - \frac{g(h)\cos\gamma}{V^2} \tag{71}$$

$$\frac{dh}{ds} = \sin\gamma \tag{72}$$

$$\frac{dx}{ds} = \frac{\cos\gamma}{1 + \frac{h}{R}} \tag{73}$$

$$\frac{dt}{ds} = \frac{1}{V} \tag{74}$$

where

$g = g_0\left(\frac{R}{R+h}\right)^2$, acceleration due to gravity
R = radius of earth (3440 nautical miles)
$D = C_D(\alpha)\,\frac{1}{2}\,\rho(h)V^2S$, drag force
$L = C_L(\alpha)\,\frac{1}{2}\,\rho(h)V^2S$, lift force

$\rho = \rho(h)$, density of atmosphere, a tabular function (ARDC standard atmosphere used)

$C_D = C_{D_0} + C_{DL}|\sin^3 \alpha|$, $C_{D_0} = 0.042$, $C_{DL} = 1.46$, Newtonian drag coefficient

$C_L = C_{L_0} \sin \alpha \, |\sin \alpha| \cos \alpha$, $C_{L_0} = 1.82$, Newtonian lift coefficient

The adjoint differential equations for the influence functions are:

$$\frac{d\lambda_w}{ds} + \lambda_w \left(\frac{2g \sin \gamma}{V^2}\right) + \lambda_\gamma \left(\frac{2g \cos \gamma}{V^2}\right) + \lambda_t \left(-\frac{1}{V}\right) = 0, \quad (75)$$

$$\frac{d\lambda_\gamma}{ds} + \lambda_w \left(-\frac{g \cos \gamma}{V^2}\right) + \lambda_\gamma \left(\frac{g \sin \gamma}{V^2} - \frac{\sin \gamma}{R + h}\right) - \lambda_x \left(\frac{\sin \gamma}{1 + \frac{h}{R}}\right) + \lambda_h \cos \gamma = 0, \quad (76)$$

$$\frac{d\lambda_h}{ds} + \lambda_w \left(\frac{2g \sin \gamma}{V^2(R + h)} - \frac{D}{mV^2} \frac{1}{\rho} \frac{d\rho}{dh}\right) + \lambda_\gamma \left(\frac{2g \cos \gamma}{V^2(R + h)} - \frac{\cos \gamma}{(R + h)^2} + \frac{L}{mV^2} \frac{1}{\rho} \frac{d\rho}{dh}\right) + \lambda_x \left(- \frac{\cos \gamma}{R\left(1 + \frac{h}{R}\right)^2}\right) = 0, \quad (77)$$

$$\frac{d\lambda_x}{ds} = 0, \quad (78)$$

$$\frac{d\lambda_t}{ds} = 0, \quad (79)$$

where

$$\lambda_w = V\lambda_V, \quad (80)$$

$w = \log \frac{V}{V_0}$ (V_0 some reference velocity)

The resulting flight path is shown in Fig. 9(a). For comparison the flight path for $\alpha = 20.5$ deg is as shown; this is the angle-of-attack for maximum lift-to-drag ratio, which is in this case $(L/D)_{\max} = 2.0$. It can be seen that the optimum $\alpha(t)$ program differs from the $\alpha = 20.5$ deg path most significantly in the first 10 min of the flight; this is truly the critical part of the flight. The optimum path achieves 15 per cent more range than the maximum L/D path. Again the optimum control variable program, Fig. 9(b) should not be difficult to approximate in practice.

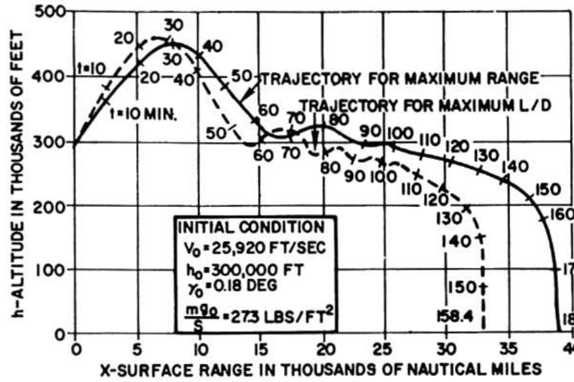

Fig. 9(a) Hypersonic orbital glider—altitude versus range

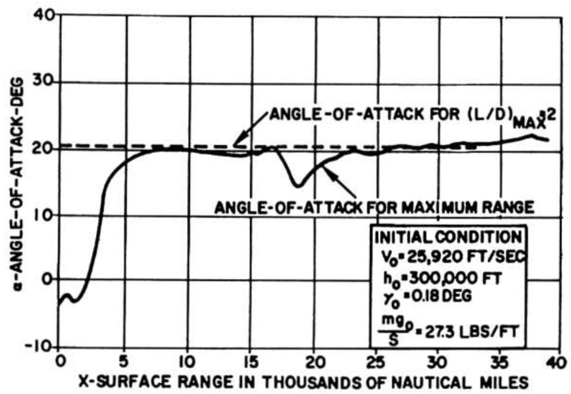

Fig. 9(b) Hypersonic orbital glider—angle-of-attack versus range

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APPENDIX

1 Use of LaGrange Multipliers for Small Perturbations About a Given Control Variable Point.

For the maximum problem stated in Section 3 of the paper, we wish to determine the changes in ϕ and ψ for a small perturbation $d\boldsymbol{\alpha}$ in the control variables from a nominal point $\boldsymbol{\alpha}^*$. To do this consider first the quantity

$$\Phi = \phi(\mathbf{x}) + \lambda_\phi'(\mathbf{f}(\mathbf{x}, \boldsymbol{\alpha}) + \mathbf{f}_0) \quad (81)$$

where λ_ϕ' is a row matrix of Lagrange multipliers to be determined for our convenience. Note that the term multiplying λ_ϕ' is zero by equation (3) so that $\Phi = \phi$. Take the differential of this quantity:

$$d\Phi = \left(\frac{\partial \phi}{\partial \mathbf{x}} + \lambda_\phi' \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right) d\mathbf{x} + \lambda_\phi' \frac{\partial \mathbf{f}}{\partial \boldsymbol{\alpha}} \, d\boldsymbol{\alpha} + \lambda_\phi' \, d\mathbf{f}_0, \quad (82)$$

and evaluate the partial derivatives at the nominal point $\boldsymbol{\alpha}^* \rightarrow \mathbf{x}^*$:

$$d\Phi = d\phi = \left(\left(\frac{\partial\phi}{\partial \mathbf{x}}\right)^* + \lambda_\phi'\mathsf{F}\right)d\mathbf{x} + \lambda_\phi'\mathsf{G}d\alpha + \lambda_\phi' d\mathsf{f}_0, \quad (83)$$

where the nomenclature is explained in Section 4 of the paper. Now let us *choose* λ_ϕ' so that the coefficient of $d\mathbf{x}$ vanishes; i.e.,

$$\left(\frac{\partial\phi}{\partial \mathbf{x}}\right)^* + \lambda_\phi'\mathsf{F} = 0, \quad (84)$$

or

$$\lambda_\phi' = -\left(\frac{\partial\phi}{\partial \mathbf{x}}\right)^* \mathsf{F}^{-1} \quad (85)$$

This reduces equation (83) to the expression

$$d\phi = \lambda_\phi'\mathsf{G}d\alpha + \lambda_\phi' d\mathsf{f}_0 \quad (86)$$

An exactly similar procedure yields

$$d\psi = \lambda_\psi'\mathsf{G}d\alpha + \lambda_\psi' d\mathsf{f}_0, \quad (87)$$

where

$$\left(\frac{\partial\psi}{\partial \mathbf{x}}\right)^* + \lambda_\psi'\mathsf{F} = 0 \quad (88)$$

2 Steepest-Ascent Method in Ordinary Calculus Using a Weighted Square Perturbation of the Control Variables to Determine Step Size.

The problem, as stated in Section 4 of the text, is to choose $d\alpha$ so as to maximize $d\phi$ for given values of $d\psi$, $d\mathsf{f}_0$ (usually zero), and dP, where

$$d\phi = \lambda_\phi'\mathsf{G}d\alpha + \lambda_\phi' d\mathsf{f}_0 \quad (89)$$

$$d\psi = \lambda_\psi' G d\alpha + \lambda_\psi' d\mathsf{f}_0 \quad (90)$$

$$(dP)^2 = d\alpha'\mathsf{W}d\alpha \quad (91)$$

We use the process of Section 1 of the Appendix again; i.e., we consider a linear combination of the three foregoing equations:

$$d\phi = \lambda_\phi'\mathsf{G}d\alpha + \lambda_\phi' d\mathsf{f}_0 + \nu'(d\psi - \lambda_\psi' d\mathsf{f}_0 - \lambda_\psi'\mathsf{G}d\alpha) + \mu((dP)^2 - d\alpha'\mathsf{W}d\alpha) \quad (92)$$

where ν' is a $1 \times p$ row matrix of constants, and μ is a constant, all to be determined for our convenience. Note that the quantities multiplying ν' and μ are both zero, by equations (90) and (91). Take the second differential of this quantity:

$$d^2\phi = (\lambda_\phi'\mathsf{G} - \nu'\lambda_\psi'\mathsf{G} - 2\mu d\alpha'\mathsf{W})d^2\alpha \quad (93)$$

Thus the maximum of $d\phi$ occurs when the coefficient of $d^2\alpha$ vanishes in equation (93). This will be the case if

$$d\alpha = \frac{1}{2\mu}\mathsf{W}^{-1}(\mathsf{G}'\lambda_\phi - \mathsf{G}'\lambda_\psi\nu) \quad (94)$$

Substituting (94) into (90) we have

$$d\beta = \frac{1}{2\mu}(\mathsf{I}_{\psi\phi} - \mathsf{I}_{\psi\psi}\nu) \quad (95)$$

where

$$\begin{aligned} d\beta &= d\psi - \lambda_\psi' d\mathsf{f}_0 \\ \mathsf{I}_{\psi\phi} &= \lambda_\psi'\mathsf{G}\mathsf{W}^{-1}\mathsf{G}'\lambda_\phi \\ \mathsf{I}_{\psi\psi} &= \lambda_\psi'\mathsf{G}\mathsf{W}^{-1}\mathsf{G}'\lambda_\psi \end{aligned} \quad (96)$$

Solving equation (95) for ν we obtain

$$\nu = -2\mu\mathsf{I}_{\psi\psi}^{-1}d\beta + \mathsf{I}_{\psi\psi}^{-1}\mathsf{I}_{\psi\phi} \quad (97)$$

Substituting (97) and (94) into (91), and solving for μ, we obtain

$$2\mu = \pm\left[\frac{I_{\phi\phi} - \mathsf{I}_{\psi\phi}'\mathsf{I}_{\psi\psi}^{-1}\mathsf{I}_{\psi\phi}}{(dP)^2 - d\beta'\mathsf{I}_{\psi\psi}^{-1}d\beta}\right]^{1/2}, \quad (98)$$

where

$$I_{\phi\phi} = \lambda_\phi'\mathsf{G}\mathsf{W}^{-1}\mathsf{G}'\lambda_\phi \quad (99)$$

Substituting (97) and (98) into (94) and (89) we obtain the results given in Sections 4 of the text in equations (21) and (23), respectively.

These results have a simple geometric interpretation in the α-hyperspace. Equations (89) and (90) with $d\mathsf{f}_0 = 0$ may be written

$$d\phi = \nabla\phi d\alpha = (\nabla\phi\mathsf{W}^{-1/2})\mathsf{W}^{1/2}d\alpha \quad (1\ 0)$$

$$d\psi = \nabla\psi d\alpha = (\nabla\psi\mathsf{W}^{-1/2})\mathsf{W}^{1/2}d\alpha \quad (101)$$

i.e., $\lambda_\phi'\mathsf{G} = \nabla\phi$ is the gradient of ϕ in the α-hyperspace, and $\lambda_\psi'\mathsf{G} = \nabla\psi$ is the gradient of ψ. For the moment we will consider ψ to be a single scalar quantity rather than a column matrix. If W is the metric in the α-hyperspace then dP is the infinitesimal distance from the present nominal point to a neighboring point $\alpha + d\alpha$. We wish to choose $d\alpha$ to maximize $d\phi$, keeping $d\psi = 0$, for a given value of

$$dP = |\mathsf{W}^{-1/2}d\alpha|.$$

Hence we must subtract the component of $\nabla\phi\mathsf{W}^{-1/2}$ that is parallel to $\nabla\psi\mathsf{W}^{-1/2}$ from $\nabla\phi\mathsf{W}^{-1/2}$. Using the Pythagorean theorem we have simply then

$$\begin{aligned} \left(\frac{d\phi}{dP}\right)^2 &= (\nabla\phi\mathsf{W}^{-1/2})(\mathsf{W}^{-1/2}\nabla\phi') - \frac{[(\nabla\phi\mathsf{W}^{-1/2})(\mathsf{W}^{-1/2}\nabla\psi')]^2}{(\nabla\psi\mathsf{W}^{-1/2})(\mathsf{W}^{-1/2}\nabla\psi')} \\ &= \nabla\phi\mathsf{W}^{-1}\nabla\phi - \frac{(\nabla\phi\mathsf{W}^{-1}\nabla\psi')^2}{\nabla\psi\mathsf{W}^{-1}\nabla\psi'} \\ &= I_{\phi\phi} - I_{\psi\phi}'I_{\psi\psi}^{-}I_{\psi\phi} \end{aligned} \quad (102)$$

which gives a geometric interpretation of equation (24). Clearly this quantity is positive unless $\nabla\phi$ is parallel to $\nabla\psi$, in which case it is zero.

3 Adjoint Differential Equations for Small Perturbations About Given Control Variable Programs.

For the optimum programming problem stated in Section 6 of the text, we wish to determine the changes in ϕ, ψ, and Ω for small perturbations, $\delta\alpha(t)$, in the control variable programs about given nominal control variable programs $\alpha^*(t)$, where $\Omega = 0$ is the stopping condition. To do this we consider the linear equations (39) describing small perturbations about the nominal path:

$$\frac{d}{dt}(\delta\mathbf{x}) = \mathsf{F}(t)\delta\mathbf{x} + \mathsf{G}(t)\delta\alpha \quad (103)$$

To determine the changes in ϕ, ψ, and Ω we introduce the linear differential equations adjoint to (103) defined as

$$\frac{d\lambda}{dt} = -\mathsf{F}'(t)\lambda \quad (104)$$

where λ_ϕ is an $n \times 1$ matrix, λ_ψ is an $n \times p$ matrix, and λ_Ω is an $n \times 1$ matrix. If (103) is premultiplied by λ' and (104) is premultiplied by $(\delta\mathbf{x})'$ and the transpose of the second product is added to the first product, we have

$$\lambda'\frac{d(\delta\mathbf{x})}{dt} + \frac{d\lambda'}{dt}\delta\mathbf{x} = \lambda'\mathsf{F}\delta\mathbf{x} - \lambda'\mathsf{F}\delta\mathbf{x} + \lambda'\mathsf{G}\delta\alpha$$

or

$$\frac{d}{dt}(\lambda'\delta\mathbf{x}) = \lambda'\mathsf{G}\delta\alpha \quad (105)$$

If we integrate equations (105) from t_0 to T, the result is

$$(\lambda'\delta\mathbf{x})_{t=T} = \int_{t_0}^{T} \lambda'\mathbf{G}\delta\alpha dt + (\lambda'\delta\mathbf{x})_{t=t_0} \tag{106}$$

If we let

$$\lambda_\phi'(T) = \left(\frac{\partial\phi}{\partial\mathbf{x}}\right)^*_{t=T};\ \lambda_\psi'(T) = \left(\frac{\partial\psi}{\partial\mathbf{x}}\right)^*_{t=T},$$

$$\lambda_\Omega'(T) = \left(\frac{\partial\Omega}{\partial\mathbf{x}}\right)^* \tag{107}$$

where the nomenclature is given in equations (47), it is clear that

$$\delta\phi = (\lambda_\phi'\delta\mathbf{x})_{t=T};\ \delta\psi = (\lambda_\psi'\delta\mathbf{x})_{t=T};\ \delta\Omega = (\lambda_\Omega'\delta\mathbf{x})_{t=T} \tag{108}$$

For small perturbations the value of T will be changed only a small amount dT, so that

$$\begin{aligned} d\phi &= \delta\phi + \dot{\phi}dT \\ d\psi &= \delta\psi + \dot{\psi}dT \\ d\Omega &= \delta\Omega + \Omega dT \end{aligned} \tag{109}$$

where the nomenclature is given in equations (48).

Substituting (107) and (109) into (106) we obtain equations (42), (43), and (44) of the text.

4 Steepest-Ascent Method in Calculus of Variations Using a Weighted Mean-Square Perturbation of Control Variables to Determine Step Size.

The problem as stated in Section 6 of the text is to choose $\delta\alpha(t)$ so as to maximize $d\phi$ for given values of $d\psi$, $\delta\mathbf{x}(t_0)$ (usually zero), and dP, with $d\Omega = 0$ where

$$d\phi = \int_{t_0}^{T} \lambda_\phi'\mathbf{G}\delta\alpha dt + \lambda_\phi'(t_0)\delta\mathbf{x}(t_0) + \dot{\phi}dT \tag{110}$$

$$d\psi = \int_{t_0}^{T} \lambda_\psi'\mathbf{G}\delta\alpha dt + \lambda_\psi'(t_0)\delta\mathbf{x}(t_0) + \dot{\psi}dT \tag{111}$$

$$0 = d\Omega = \int_{t_0}^{T} \lambda_\Omega'\mathbf{G}\delta\alpha dt + \lambda_\Omega'(t_0)\delta\mathbf{x}(t_0) + \dot{\Omega}dT \tag{112}$$

$$(dP)^2 = \int_{t_0}^{T} \delta\alpha'\mathbf{W}\delta\alpha dt \tag{113}$$

We use the analog of the process of Section 2 of the Appendix with only small differences arising from the additional term in dT in the foregoing equations. The first step is to eliminate dT from equation (112):

$$dT = -\frac{1}{\dot{\Omega}}\int_{t_0}^{T} \lambda_\Omega'\mathbf{G}\delta\alpha dt - \frac{1}{\dot{\Omega}}\lambda_\Omega'(t_0)\delta\mathbf{x}(t_0) \tag{114}$$

Substituting this into equations (110) and (111), we have the relations

$$d\phi = \int_{t_0}^{T} \lambda_{\phi\Omega}'\mathbf{G}\delta\alpha dt + \lambda_{\phi\Omega}'(t_0)\delta\mathbf{x}(t_0) \tag{115}$$

$$d\psi = \int_{t_0}^{T} \lambda_{\psi\Omega}'\mathbf{G}\delta\alpha dt + \lambda_{\psi\Omega}'(t_0)\delta\mathbf{x}(t_0) \tag{116}$$

where the nomenclature is given in equations (51). Next we consider a linear combination of (116) and (113) with (115); i.e.,

$$d\phi = \int_{t_0}^{T} (\lambda_{\phi\Omega}'\mathbf{G} - \nu'\lambda_{\psi\Omega}'\mathbf{G} - \mu\delta\alpha'\mathbf{W})\delta\alpha dt$$
$$+ (\lambda_{\phi\Omega}'(t_0) - \nu'\lambda_{\psi\Omega}'(t_0))\delta\mathbf{x}(t_0) + \nu'd\psi + \mu(dP)^2 \tag{117}$$

Now we take the variation of this relation (117):

$$\delta(d\phi) = \int_{t_0}^{T} (\lambda_{\phi\Omega}'\mathbf{G} - \nu'\lambda_{\psi\Omega}'\mathbf{G} - 2\mu\delta\alpha'\mathbf{W})\delta^2\alpha dt \tag{118}$$

where $\delta\mathbf{x}(t_0)$, $d\psi$, and dP are considered constants. The maximum $d\phi$ will occur if the coefficient of $\delta^2\alpha$ in the integrand of (118) vanishes; i.e.,

$$\delta\alpha = \frac{1}{2\mu}\mathbf{W}^{-1}\mathbf{G}'(\lambda_{\phi\Omega} - \lambda_{\psi\Omega}\nu) \tag{119}$$

Substituting (119) back into (116) and then (113) we can solve for the constants ν and μ:

$$\nu = -2\mu\mathbf{I}_{\psi\psi}^{-1}d\beta + \mathbf{I}_{\psi\psi}^{-1}\mathbf{I}_{\psi\phi} \tag{120}$$

$$2\mu = \pm\left[\frac{I_{\phi\phi} - \mathbf{I}_{\psi\phi}'\mathbf{I}_{\psi\psi}^{-1}\mathbf{I}_{\psi\phi}}{(dP)^2 - d\beta'\mathbf{I}_{\psi\psi}^{-1}d\beta}\right] \tag{121}$$

where

$$\begin{aligned} d\beta &= d\psi - \lambda_{\psi\Omega}'(t_0)\delta\mathbf{x}(t_0) \\ \mathbf{I}_{\psi\psi} &= \int_{t_0}^{T} \lambda_{\psi\Omega}'\mathbf{G}\mathbf{W}^{-1}\mathbf{G}'\lambda_{\psi\Omega}dt \\ \mathbf{I}_{\psi\phi} &= \int_{t_0}^{T} \lambda_{\psi\Omega}'\mathbf{G}\mathbf{W}^{-1}\mathbf{G}'\lambda_{\phi\Omega}dt \\ I_{\phi\phi} &= \int_{t_0}^{T} \lambda_{\phi\Omega}'\mathbf{G}\mathbf{W}^{-1}\mathbf{G}'\lambda_{\phi\Omega}dt \end{aligned} \tag{122}$$

Substituting (120) and (121) into (119) and (115) we obtain the results given in Section 6 of the text in equations (50) and (51), respectively.