Red Black Card Game and Generalized Catalan Numbers

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Abstract

A link between Catalan numbers and a simple gambling card game is shown. The Catalan numbers are generalized in order to form the complete statistics to the card game. My discovery of the statistics of the game was done by a progression from brute force to smarter and smarter spreadsheets and C programs (because I am a programmer). It took me far too long to reach the final solution. The solution is somewhat simple and elegant, but the path to reach the solution was not clear (at least not for me). I hope that this paper will help others skip straight to the elegant solution. The paper also explains the reason why the possible "winnings" distribution closely resembles a Rayleigh distribution.

Card Game

Consider a single player card game with a standard 52 card deck. If you draw black, you win \$1. If you draw red, you lose \$1. (Think financial...black is good/positive; red is bad/negative, but if you get it flipped, it doesn't really matter.) You decide to stop drawing whenever you want. It is impossible to lose money in this game; if you have drawn more red cards than black cards, you can just keep drawing to the end of the deck to end up at net \$0. If you stop drawing before reaching the end of the deck, you keep your money that you had won up until that point. There is a nice explanation of the optimal strategy for this game at http://puzzles.nigelcoldwell.co.uk/fourteen.htm. This paper doesn't really investigate the optimal strategy. Instead, it investigates the statistics of the game.[1]

There are $\binom{52}{26}$ possible deck configurations (~496 trillion configurations). For the sake of simplicity, I will often instead describe a deck of 8 cards (4 red + 4 black). In that case, there are $\binom{8}{4}$ configurations (70 configurations). One way of visualizing the problem is by constructing an n x n lattice (where "n" is the number of red cards [or black cards] in the deck) and enumerating paths that start at the upper left corner and finish at the lower right corner where the only allowed moves are "right" and "down". The figure below shows the lattice.

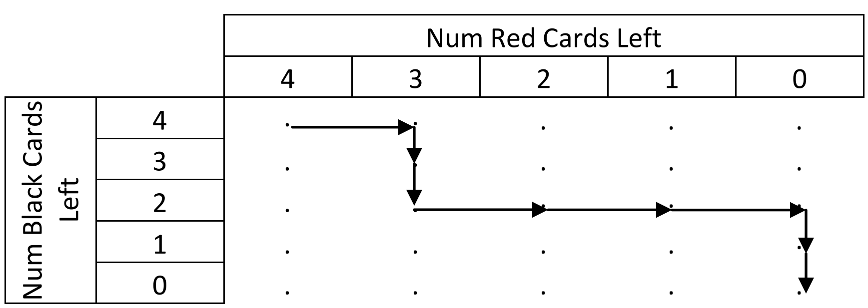

Figure 1 - The "path" problem representation of the red black card game

[1] One interesting aspect of the optimal strategy is that even if you draw 5 straight black cards right away (and you're up \$5), the strategy says you should draw *again*...even though there's only 21 black cards left and 26 red cards left. That seems very counter-intuitive because the deck is *literally* stacked against you, but the statistics of the optimal strategy say the "expected winnings" if you draw again is \$5.05 (at that one spot...not \$5.05 for every configuration), which is higher than the \$5 you currently hold, so it is to your benefit to draw again. Strange stuff.

One possible path is shown. The path sequence could be described as "RDDRRRDD", where "R" is right and "D" is down. You could also describe it as "RBBRRRBB", where "R" is red and "B" is black. Also note that there is actually a 5x5 grid of dots because it possible to have 0, 1, 2, 3, or 4 cards of a given color remaining. (But you can just as easily picture a 4x4 grid of squares and the paths travel along the edge of the squares.)

The next figure shows the number of paths that pass through each of the 25 points. At each point, you can only go right or down. It was created by starting at the upper left hand corner and counting the number of ways there are to get to each point. Note that this is a *45° rotated* Pascal's triangle.

Equals 1 + 1 (sum of highlighted values)

		Num Red Cards Left				
		4	3	2	1	0
Num Black Cards Left	4	1	1	1	1	1
	3	1	2	3	4	5
	2	1	3	6	10	15
	1	1	4	10	20	35
	0	1	5	15	35	70

Figure 2 - The number of possible paths that pass through each point in the lattice. Notice that each value is the sum of the "up" and "left" neighbors since you can only travel "right" and "down".

Note that the bottom right number is 70. That means there are 70 possible paths. That also means that there are 70 different deck configurations (for an 8-card deck).

Pascal's triangle is shown below for reference so you can see the similarity with the above "path" problem.

```
                1
              1   1
            1   2   1
          1   3   3   1
        1   4   6   4   1
      1   5  10  10   5   1
    1   6  15  20  15   6   1
  1   7  21  35  35  21   7   1
1   8  28  56  70  56  28   8   1
```

Figure 3 - Pascal's Triangle

Catalan Numbers

Catalan numbers have many uses in combinatorial mathematics. We will only be focusing on one use of them in this paper. They were first discovered by Euler as a solution to the number of ways of dividing a polygon into triangles. The Catalan sequence is named after Eugène Charles Catalan, who discovered the connection to parenthesized expressions during his exploration of the Towers of Hanoi puzzle.[2] Catalan numbers are defined as

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} \; for \; n \geq 0$$

[2] From Wikipedia - http://en.wikipedia.org/wiki/Catalan_number

The first Catalan numbers for n = 1, 2, 3, 4 are 1, 2, 5, and 14. You can visualize the Catalan numbers by looking at the figure below.

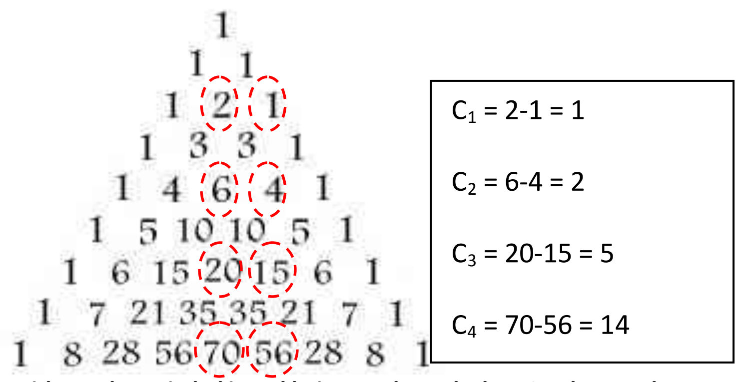

Figure 4 - Pascal's Triangle with numbers circled in red being used to calculate Catalan numbers

How do Catalan numbers relate to the aforementioned red black card game? Catalan numbers are the number of paths that can be taken from the upper left corner to the lower right *without crossing below the main diagonal line*. For example, in our 4x4 lattice, there are 70 total paths, but only C_4 = 14 of them will stay above the main diagonal line throughout the whole path. That means that 70 - 14 = 56 = $\binom{8}{5}$ will *at some point* cross below the main diagonal line. The lattice grid below shows the diagonal line along with a sample path that does not cross the diagonal.

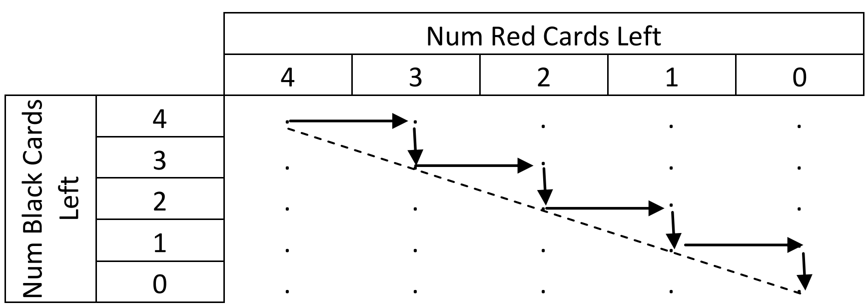

Figure 5 - Lattice grid shown with a sample possible path that does not ever pass below the diagonal line. The number of such paths can be calculated using Catalan numbers.

In the card game, consider that every deck has a "maximum black disparity" (MBD) value. This is the maximum possible winnings for a given deck. I.e. if you were able to inspect the deck before playing the card game, you would know exactly what the maximum winnings for that deck are by calculating the maximum black disparity as you are looking through the deck. If you knew what the MBD was for a given deck, you would *always* win that amount. What is the probability distribution of the MBDs for all possible decks? (Knowing this might help you understand the possible payouts for the game, but as mentioned previously, this paper is not really about finding the *optimal* solution.)

So how do the Catalan numbers help calculate the probabilities of the various MBDs? In the 8 card deck, C_4 = 14 tells you how many paths stay above the diagonal line. In other words, the MBD for all those paths is 0 because you *never* have more black cards than red cards in your hand. You would be forced to draw to the end of the deck. The probability that MBD = 0 is then given by

$$P_8(MBD = 0) = P_8(MBD \leq 0) = \frac{C_4}{\binom{8}{4}} = \frac{14}{70} = 20\%$$

Note that since MBD can't be less than 0, $P_8(MBD = 0)$ does in fact equal $P_8(MBD \leq 0)$.

Let's verify this for an 8-card deck. As before, the number at each grid point is the number of possible paths that pass through that point, but the figure has been updated to eliminate the paths that would go below the diagonal line.

		Num Red Cards Left				
		4	3	2	1	0
Num Black Cards Left	4	1	1	1	1	1
	3	·	1	2	3	4
	2	·	·	2	5	9
	1	·	·	·	5	14
	0	·	·	·	·	14

Equals C_4.

Figure 6 - Lattice grid with number of paths shown at each possible point. The paths must stay above the diagonal line.

Expanding to a 52-card deck, we see the probability that MBD = 0 is given by

$$P_{52}(MBD = 0) = P_{52}(MBD \leq 0) = \frac{C_{26}}{\binom{52}{26}} \approx \frac{1.84 \times 10^{13}}{4.59 \times 10^{14}} \approx 3.7\%$$

That means that in 3.7% of the games, you are doomed to draw to the end of the deck because you will never be "net > 0" throughout the game.

So this helps us calculate P(MBD=0), but how do we calculate the probabilities for the other MBDs? If we extend the Catalan number definition to include another term, k, where k describes the diagonal line that we do not wish to cross. The conventional Catalan numbers that we have described thus far have k = 0. The expanded Catalan numbers are denoted by C_n^k.

Definition: C_n^k is the number of (right, down) paths that go from the upper right corner to the lower left that do not go below the diagonal lines shown below.

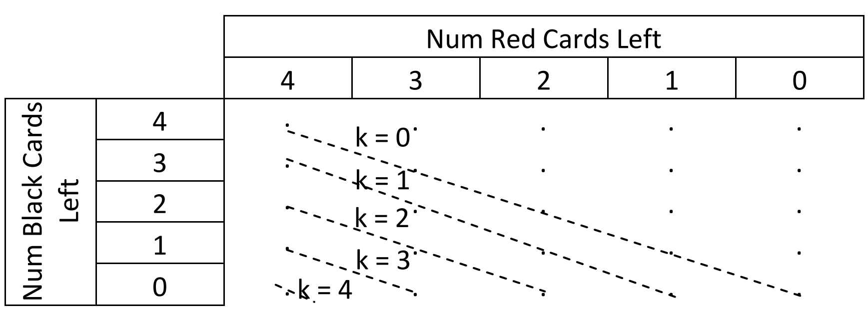

Figure 7 - Lattice grid with different diagonal lines shown at different k values. k = 0 corresponds to the typical Catalan diagonal.

How does this relate to MBDs? By examining the paths, you can see that (as previously discussed) when k = 0, that means the MBD = 0, or put another way, MBD ≤ 0. When k = 1, that means MBD ≤ 1. So in general, to find out the number of paths with MBD ≤ k, find C_n^k. To find out the probability that MBD ≤ k, use the following formula.

$$P_{52}(MBD \le k) = \frac{C_{26}^k}{\binom{52}{26}}$$

It can be shown that

$$C_n^k = \binom{2n}{n} - \binom{2n}{n+k+1} \; for\; n \ge 0, 0 \le k \le n-1$$

Thus, we have our cumulative distribution function, $P_{52}(MBD \le k)$. It is shown below.

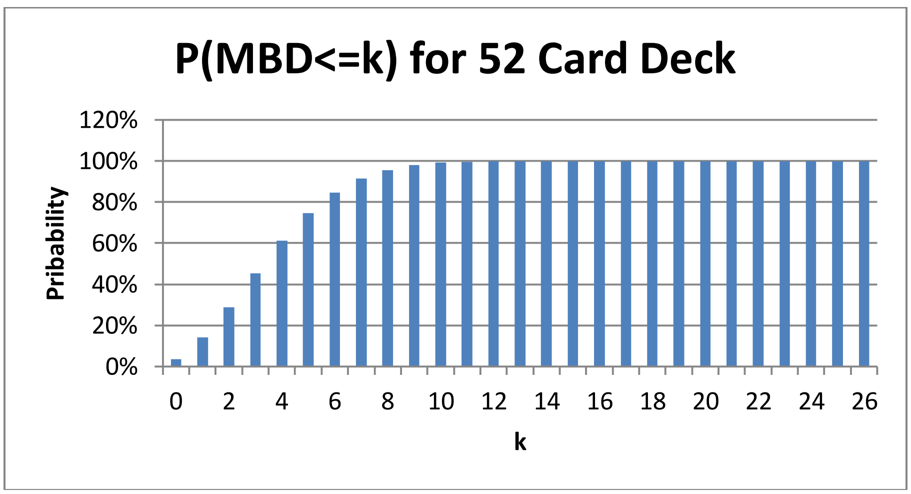

Figure 8 - The Cumulative Distribution Function (CDF) for the Maximum Black Disparities in a 52-card deck

The probability mass function (or probability distribution function) is given by taking the numerical derivative.

$$P_{52}(MBD = k) \quad = P_{52}(MBD \le k) - P_{52}(MBD \le k-1)$$

$$= \frac{C_{26}^k - C_{26}^{k-1}}{\binom{52}{26}}, where\; C_{26}^{-1} = 0$$

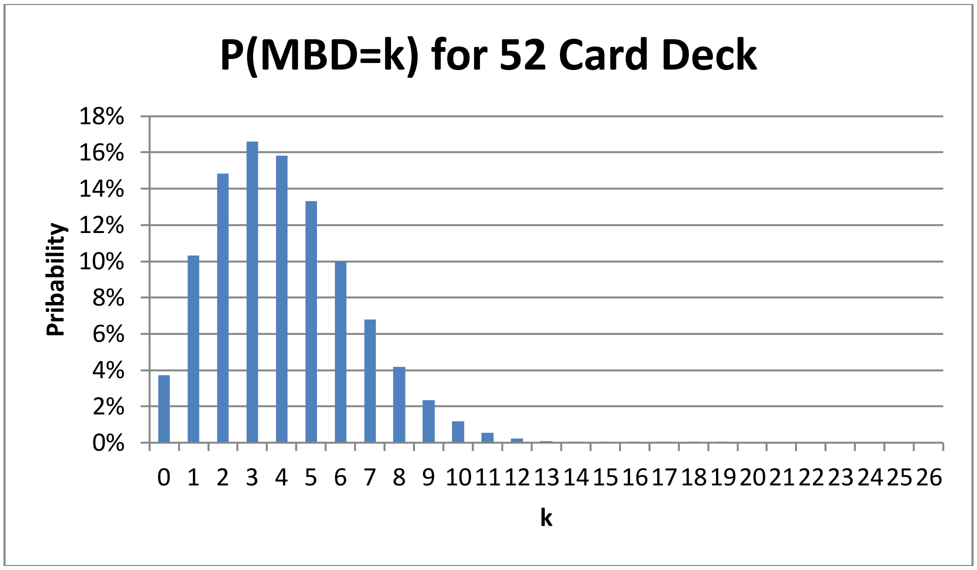

Figure 9 - The Probability Distribution Function (PDF) for the Maximum Black Disparities in a 52-card deck

As you can see, this tends to follow a Rayleigh distribution.[3] The reason for this is interesting. As you go deeper and deeper into Pascal's triangle, the rows tend to become more and more shaped like a Gaussian function. The reason they tend more and more toward a Gaussian function is because each row is formed by convolving [1,1] with the previous row, and due to the Central Limit Theorem, if you do this enough times, you end up with a Gaussian function. Given that the CDF can be rewritten as

$$P_{2n}(MBD \leq k) = \frac{C_n^k}{\binom{2n}{n}} = 1 - \frac{\binom{2n}{n+k}}{\binom{2n}{n}}$$

And the CDF of a Rayleigh distribution is given by

$$CDF_{Rayleigh} = 1 - e^{-k^2/2\sigma^2}\ where\ k \geq 0$$

You can see that since the rows of Pascal's triangle are shaped like a Gaussian function, the MBD probability distribution will be shaped like a Rayleigh distribution (especially as there are more and more cards in the deck). It's also interesting to note (but not proven here) that $\sigma \approx \sqrt{\frac{n}{2}} \approx \sqrt{13}$ for a 52-card deck. The average winnings is approximated by $\sqrt{\frac{\pi}{2}}\sigma - 0.5 \approx 4.02$ for a 52-card deck (using Rayleigh distribution formulas).

The expected MBD for a 52-card deck is *actually* 4.04. That means that if you "cheated" and knew the MBD of every deck, you would win an average of $4.04 per game. Also (not covered here in detail), if you set an arbitrary stopping rule of stopping once you hit net $4, you would "win" 54.6% of the time, making your average winnings $4 * 0.546 = $2.18 per game.

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[3] A real Rayleigh distribution has value 0 at k = 0, but it matches extremely well if you mentally add 0.5 to the k value before plugging it into the real Rayleigh formula. I'm fairly certain this has something to do with taking the numerical derivative of discrete points.

Numerical Stability Notes

It turns out that the most numerically stable way (and probably quickest way) to calculate the probability distribution is to calculate $P_{2n}(MBD \geq k)$ as follows [noting that $P_{2n}(MBD \geq k) = 1 - P_{2n}(MBD < k)$]...

$$P_{2n}(MBD \geq 0) = 1$$
$$P_{2n}(MBD \geq k) = \frac{n-k+1}{n+k} \cdot P_{2n}(MBD \geq k-1), for\ 1 \leq k \leq n$$

This makes it unnecessary to calculate any large factorials (that would otherwise be susceptible to overflows). (This can also be used to numerically calculate a Gaussian curve without using the "exp" function.) Then calculate $P_{2n}(MBD = k)$as follows:

$$P_{2n}(MBD = k) = P_{2n}(MBD \geq k) - P_{2n}(MBD \geq k+1)$$

Conclusion

So I hope that this short paper has shown how the statistics for the red black card game relate to an expanded definition of Catalan numbers. (If there is an even simpler interpretation of the final formula, I have missed it.) It is unknown whether or not this expanded definition of Catalan numbers would be useful in other combinatorial problems, but they were certainly helpful in this particular problem. I suspect they could be used in calculating the statistics of a modified ballot problem that helped bring about Catalan numbers' popularity in which the modified problem is: how likely is it that while counting 2n ballots (dealing with 2 candidates), candidate A never falls behind candidate B by more than "k" votes. (Original ballot problem: http://en.wikipedia.org/wiki/Bertrand's_ballot_theorem). If you've made it this far, you now know how to calculate the statistics for this card game. That will most likely never help you in life. Sorry, maybe I should have put that in the abstract!