# THE ECONOMIC LIFE OF INDUSTRIAL EQUIPMENT

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WHEN TO REPLACE individual units of durable equipment by similar or improved units is one of the main problems, upon which the success of industrial enterprise depends. Nevertheless, no unified presentation of its many aspects appears to have been published up to the present. The principal writers refer to replacement merely incidentally, when discussing the subject of depreciation. From the theoretical point of view, such an approach really amounts to putting the cart before the horse.[1] Replacement is the basic problem, because it actually affects the composition and productivity of a plant. Calculations of depreciation are mere figures entered into books, the significance of which depends entirely on the use to which they are put. The concept of depreciation does not enter into the theory of capital value at all. In practice, on the other hand, differences in depreciation methods do to some extent influence the judgment of traders in the negotiable symbols of composite capital goods. This anomaly is due partly to defective accounting methods. A study of the replacement problem by itself must precede attempts to correct the situation.

The value aspect of replacement or "economic life" arises from the familiar phenomenon that many types of "machines" outlive their usefulness. The income stream derived from their operation gradually declines, until a more attractive alternative becomes available. The theory that the economic life of a machine is a period which makes the unit cost (plus interest) of the product a minimum, appears to have been originated by Professor J. S. Taylor.[2] His algebraic presentation was simplified and refined by Professor Harold Hotelling,[3] who employs continuous functions for the purpose. The basic formula given by the latter writer is:[4]

[1] I have done that too in "Annual Survey of Economic Theory: The Theory of Depreciation," ECONOMETRICA, Vol. 6, July, 1938, pp. 219–241. The present article is an attempt to organize and expand the comments there made on the replacement problem.

I am greatly indebted to Professors James C. Bonbright, Ragnar Frisch, and Harold Hotelling for the interest they have taken in my MS at various stages of its preparation. Professor Bonbright's unfailing readiness to discuss the main trend of reasoning helped me greatly in clarifying my ideas and their presentation. Instances of mathematical obscurity pointed out by Professor Frisch were corrected in footnotes. Professor Hotelling's extensive comments and my replies will be found in the appendix, which is submitted in lieu of revisions in the text.

[2] "A Statistical Theory of Depreciation," *Journal of the American Statistical Association*, December, 1923, pp. 1010–1023.

[3] "A General Mathematical Theory of Depreciation," *ibid.*, September, 1925, pp. 340–353.

[4] P. 343, formula (2).

(1) $$B = \int_0^T [wQ(t) - E(t)]e^{-\int_0^t i(\tau)d\tau}dt + S(T)e^{-\int_0^T i(\tau)d\tau}.$$

In this changed notation, $B$=original cost of a single machine, $T$=unknown date at which it ought to be discarded, $w$=unknown lowest unit cost (plus interest) of the product, $Q(t)$=rate of production, $E(t)$=combined rate of all expenses, except depreciation and interest, $i(t)$=rate of interest, and $S(T)$=selling price (scrap value) of the machine, when discarded.

By differentiating with respect to $T$, the unit cost may be written:

(2) $$w = \frac{E(T) + i(T)S(T) - S'(T)}{Q(T)}.$$

"This equation states that the cost of a unit of product is found by adding the operating cost $E(T)$ of the machine (at the time when it is least efficient and about to be scrapped) to interest $i(T)S(T)$ on the scrap value and the rate of depreciation $-S'(T)$ of the scrap value and dividing this sum by machine's rate of production."[5] The result will be a minimum, when $T$ is determined by substituting (2) in (1) and solving. Professor Taylor's algebraic formula corresponds to equation (1) solved for $w$, viz.:

(3) $$w = \frac{B - S(T)e^{-iT} + \int_0^T E(t)e^{-it}dt}{\int_0^T Q(t)e^{-it}dt},$$

from which he obtained the minimum by successive trials equivalent to $dw/dT=0$. Since the argument is not essentially concerned with the variability of the rate of interest, it is permissible to employ a constant rate for brevity.

Neither of the authors cited defines clearly the exact limitations, within which he considers this method valid. In a general way, both have in mind principally the static situation, where a machine will be replaced by another of identical type, operated under the same economic conditions. With respect to dynamic developments, Professor Taylor merely hints that "if replacement alternatives were changing, whether through changing operating costs or through changed service unit requirements, the optimum economic life of the machine in use would be altered and it would continue in operation for any period for which its discounted operating cost was less than the unit cost plus of the best current replacement alternative."[6] This statement apparently expresses the condition:

[5] Hotelling, p. 345.
[6] Taylor, p. 1022.

$$w_1 = \frac{E_1(T_1) + iS_1(T_1) - S_1'(T_1)}{Q_1(T_1)} \tag{4}$$
$$= \frac{E_2(T_2) + iS_2(T_2) - S_2'(T_2)}{Q_2(T_2)} = w_2$$

where the entire economic life $T_2$ of the replacement alternative must first be calculated from (1) and (2), or (3). The unexpired life $T_1$ of the machine now in service is then allegedly determined.

Closer examination shows that the problem of economic life is not quite so simple as Professor Taylor's sketch implies. Some of the principal ramifications worthy of study may be classified under three headings:

*A. Scope*

1. A single machine;
2. A finite chain of replacements;
3. An infinite chain;
4. A number of parallel chains, whose replacement dates are evenly staggered;
5. A large plant continuously renewed in accordance with natural variations in the behavior of similar machines.

*B. Limitations*

1. Scarcity of new machines available for replacement;
2. Scarcity of various operating facilities or ingredients of production;
3. Scarcity of demand for product;
4. Scarcity of capital;
5. Regulation of profit by law.

*C. Economic conditions*

1. The static case, where only variations due to the age of the machine are considered;
2. Variations due to the number of co-operating machines;
3. Change in ownership and outlook;
4. Change in the type of machine used (obsolescence);
5. The general dynamic case, embracing extraneous influences as well.

It would be difficult to exhaust the implications of this triple classification. All that can be done is to outline the general trend of reasoning by building up the main problem gradually.

I

When a single machine will not be replaced, it becomes immediately apparent that economic life depends, not on the unit cost, but on the market price of the product. If the capital value of a single-machine enterprise shall be a maximum, the equations to be solved simultaneously are:

$$V = \int_0^T [zQ(t) - E(t)]e^{-it}dt + Se^{-iT} \tag{5}$$

and

$$dV/dT = 0 = zQ(T) - E(T) - iS. \tag{6}$$

This idea is analogous to (1) and (2), except that the unknown capital value $V$ was substituted for the known original cost $B$ and the known market price $z$ of the product for its unknown "unit cost plus" $w$. The variability of the scrap value is also disregarded here, but only for brevity. The most lucrative life-span $T$ of the machine can now be found from (6) alone and inserted in (5) to find $V$.

Partisans of the Taylor theory invariably point out that no machine can have a capital value greater than replacement cost. Any goodwill in excess of that amount must be due, not to the machine, but to extraneous and intangible advantages. This argument is valid enough for a number of purposes, but the calculation of economic life is not among them. Since no income whatever can be had without the machine, the entire value of the enterprise must be imputed to it to determine the proper date of scrapping.[7]

A simple example is presented in Figure 1. The data are $z = \$10$, $Q(t) = 1 - 0.04t$, $E(t) = 1 + 0.2t$, $B = \$19.5016$, $i = 0.05$, and $S = \$4$. The correct economic life is accordingly $T = 14.\dot{6}\dot{6}$ years, whereas the result from the Taylor-Hotelling calculation would be $w = \$5$ and $T = 9.5$ years.

The approach (5), (6) is valid regardless of what the Taylorean unit cost $w$ may be. But replacement is justified only if $z > w$. If the supply of replacements limits the size of the enterprise, a machine will be acquired whenever possible and operated until the date determined by (6). To facilitate reference, I shall call this rule the *individual rule*, since each machine stands only on its own merits.

The situation is altogether different, when only a single machine can be kept in operation at any given time. Starting with a finite chain of replacement, we may write $V = B + G$, where $G$ means goodwill. The capital value of this chain consists of the capital value of the machine

[7] **But see appendix, points 3 and 5.**

in service, plus the present worth of the aggregate goodwill of all replacements:

$$G_j = \int_0^{T_j} [zQ(t) - E(t)]e^{-it}dt + (S + G_{j-1})e^{-iT_j} - B. \tag{7}$$

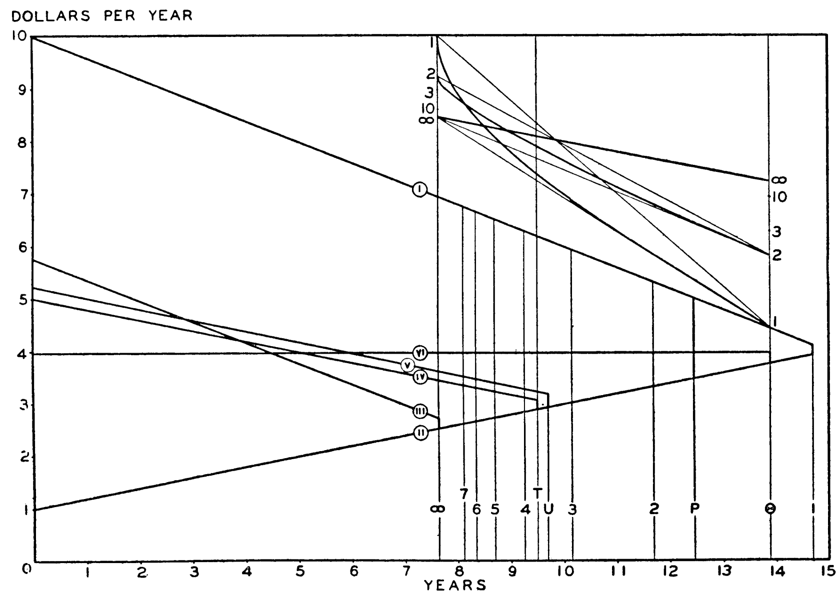


FIGURE 1.—A single machine. Cost $B = \$19.5016$, scrap value $S = \$4$; gross income (sales) (I) $= zQ(t) = \$10(1-0.04t)$; operating expenses (II) $= E(t) = \$(1+0.2t)$.

*Economic life:* (1) individual rule; (2) · · · (7) · · · (∞) chain rule, depending upon how many successive replacements will occur; ($T$) Taylor rule; ($P$) profit rule (discount rate $p_{\max.} = 38.31555\%$ per annum); ($U$) public-utility rule = Taylor rule, when discount rate $p = 7\%$ per annum; (Θ) Taylor rule = chain rule, when sales are constant. Discount rate used for ($T$) and (Θ) was 5% per annum.

*Cost of sales $wQ(t)$:* (I) profit rule; (III) infinite chain rule; (IV) Taylor rule; (V) public-utility rule; (VI) Taylor rule = chain rule for constant sales.

Upper right shows auxiliary construction of regression lines of Figure 3. Economic life varies between maximum (Θ) and minimum (∞). Numbering along those ordinates refers to number of evenly staggered parallel chains. See note 11.

The subscript $j = 1, 2, 3, \cdots$, indicates the number of links in the chain. The second relation needed for solution is again the derivative $dG_j/dT_j = 0$, viz.:

$$G_{j-1} = \frac{zQ(T_j) - E(T_j)}{i} - S. \tag{8}$$

Beginning with $G_0 = 0$, the latter equation determines $T_1$, the economic life of the last machine in the chain, which is the same as that obtained from (6). This result, inserted in (7), yields $G_1$, which is in turn substituted in (8) to find $T_2$. Returning to (7) we now obtain $G_2$ and so forth. Each machine will have a longer life than its predecessor and a shorter life than its successor in the chronological chain. The life-spans of the last seven links in a chain of replacements are shown in Figure 1, numbered backward from the end. The data assumed lead to $T_1=14.66$, $T_2=11.687$, $T_3=10.142$, $T_4=9.247$, $T_5=8.696$, $T_6=8.345$, and $T_7=8.116$ years.

As the chain is lengthened, a limit eventually emerges, where $G_j=G_{j-1}$ and $T_j=T_{j-1}$. That is the case of the infinite chain, in which the economic life of all replacements is the same, viz., $T_\infty=7.6461$ years in Figure 1. This answer can be obtained directly by omitting all subscripts and substituting $G$ from (8) for both $G$'s in (7). The equation for the calculation of the standard period $T$ may then be rearranged to read:

$$(9)\quad z\left[Q(T)-Q(0)-\int_0^T Q'(t)e^{-it}dt\right] = E(T)-E(0)-\int_0^T E'(t)e^{-it}dt - i(B-S).$$

To simplify the expression a bit, the integrations of (7) were performed by parts. Accordingly, $Q'(t)$ and $E'(t)$ represent the rates of change of production and expenses respectively. The criteria of replacement for a single infinite chain and static conditions now become apparent from a generalized graphic solution of (9).

Figure 2 compares the behavior of three types of machines. They are similar in all respects, excepting only their rates of production. The output of the *first* is constant, that of the *second* decreases with growing age (as assumed in Figure 1), and that of the *third* increases with age. Depending upon the functions chosen, the curves will differ in details, but their general behavior is quite uniform. For $T>0$, the left side of equation (9) is always *zero*, when $Q_1'(t)=0$; *negative*, i.e., declining from a maximum of zero, when $Q_2'(t)<0$; and *positive*, i.e., rising from a minimum of zero, when $Q_3'(t)>0$. As for the right side, there is only one curve, rising from the minimum $-i(B-S)$, since the same expense function growing with age serves for all examples.

Upon considering the three types of behavior in turn, we find *first* that, when the rate of production is constant, the market price of the product has no bearing on the date of replacement which is given by the intercept $T_1$ of the right side of (9) on the axis of abscissae. In such circumstances, formula (9) leads to Taylor's date of replacement.

Curves II and III correspond to the *second* and *third* type and were drawn for the special case, where the product happens to be sold at its exact minimal unit cost $w$, as determined by formula (2). The two sides of equation (9) then intersect at points, the abscissae of which could also have been computed by the Taylor-Hotelling method.

When the product is sold at more than its unit cost, i.e., when $z > w$, curves II and III will evidently be deflected toward the axis of ordinates, thereby accelerating replacement for Type II and retarding it for Type III. The two broken curves indicate this development. For

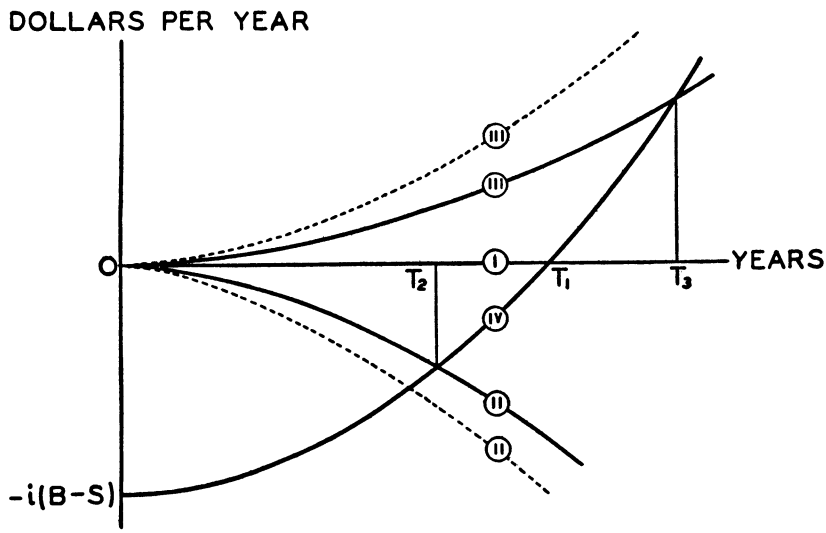


FIGURE 2.—Graphic solution of formula (9). A single infinite chain consisting alternatively of three different types of machines. Type I has constant output; Type II has output declining with age; and Type III has output growing with age. All other characteristics are identical and expressed by curve IV.

Solid curves were drawn for case when output is sold at its exact unit cost $z = w$. Broken curves refer to case $z > w$, showing that increase in profits leaves the economic life of Type I unchanged, but shortens life of Type II and lengthens life of Type III. The graph is not applicable to case $z < w$, which forbids replacement.

the extreme $z = \infty$, Type II would have to be discarded and replaced instantaneously whereas, beyond the limit $zQ_3'(t) = E'(t)$, Type III no longer presents any problem of economic life. Replacement then depends on physical life alone.

Neither formula (9) nor its graph is valid for $z < w$. In such a case, replacement would be uneconomic, but the machine already in service should be operated until the date determined from (6). The lower the selling price, the sooner the enterprise must be abandoned.

The results obtained from (9) can be duplicated by the Taylor technique by pretending that sales above or below a certain constant volume of output $P$ are deductions from, or additions to operating expenses. The net rental $zQ(t)-E(t)$ may be expressed alternatively as $zP-F(t)$, when $F(t)=E(t)+z[P-Q(t)]$. The economic life $T$ of any machine in an infinite chain is then given by

$$(10) \qquad w = \frac{B - Se^{-iT} + \int_0^T F(t)e^{-it}dt}{\frac{P}{i}(1 - e^{-iT})} = \frac{F(T) + iS}{P}$$

regardless of the value assigned to $P$. This equation is preferable to (9) for general purposes. It seems appropriate to call it the *chain rule.*

Before leaving the single chain of replacements, the complexity of its general dynamic aspect may be outlined. Let $T_1$ be the unexpired remainder of the economic life of the machine now in service and $T_2, \cdots, T_h, \cdots, T_\omega$ the entire economic lives of the successive replacements. The dates of replacement will then be $D_h = \sum_{j=1}^{h} T_j$, so that equation (7) may be generalized in the form:

$$(11) \qquad \begin{aligned} G_{\omega-h+1}(D_{h-1}) = \int_{D_{h-1}}^{D_h} R_h(t)e^{-\int_{D_{h-1}}^{t} i(\tau)d\tau}dt \\ + [S_h(D_h) + G_{\omega-h}(D_h)]e^{-\int_{D_{h-1}}^{D_h} i(\tau)d\tau} - B_h(D_{h-1}), \end{aligned}$$

where $R_h(t)=z(t)Q_h(t)-E_h(t)$. The index $h$ means that all functions differ not only from machine to machine by virtue of changes in type, but also according to the economic conditions which happen to prevail during the periods $T_h$, which begin at the unknown dates $D_{h-1}$.

The derivative of (11) for $T_h$ is:

$$(12) \qquad G_{\omega-h}(D_h) = \frac{R_h(D_h) + S_h'(D_h) + G_{\omega-h}'(D_h)}{i(D_h)} - S_h(D_h).$$

Beginning now with $h=\omega$ and $G_0=0$ as in the static case, it is theoretically possible to express from (12) the economic life $T_\omega$ of the last machine in terms of known functions of the unknown date $D_{\omega-1}$. The result $G_1$ from (11) is then substituted in (12), placing $h=\omega-1$ to find $T_{\omega-1}$ in terms of $D_{\omega-2}$. By repeating the process, the aggregate goodwill of all future machines will eventually be expressed as a function of the unexpired life $T_1$ of the machine now in service. The final step is then to solve (11) and (12) for $T_1$, when $h=1$, but omitting the last term $B_1(D_0)$ on the right of (11).

This presentation makes allowance for all replacement problems which a so-called "one-horse outfit" can encounter in a dynamic economy. The only requirement is that known functions be substituted for the symbols treated as known. The effect of various degrees of competition or monopoly, etc., can be readily studied within the framework (11) by using suitable definitions. But if such complications are introduced too soon, the main point may be overlooked. Thus, Dr. C. F. Roos[8] devotes considerable attention to the market price of the product, but considers only a single replacement, i.e., several co-operating chains of only two links each. For longer chains it will not be true, even under static assumptions that "the analogue of the Hotelling hypothesis . . . would require for the replacement problem that the operator endeavor to maximize the sum $V_1(t_1)+V_2(t_1)$."[9] Neither of these two capital values can be found without first knowing those of all subsequent replacements.

From a practical viewpoint, the foregoing demonstration amounts to a *reductio ad absurdum,* for it is obviously impossible to predict how machines, which have not even been invented as yet, will behave under economic conditions prevailing in the dim future. The best that can be done is to estimate somehow the aggregate goodwill of all future machines and then to determine the life of the machine now in service in such a manner that its capital value, plus the present worth of the estimated future goodwill, shall be a maximum. For this purpose it is also reasonable to assume that the goodwill estimate remains unchanged within the limited range of doubt surrounding the date of replacement of the present machine.

To maintain the cost of the product at a variable minimum consistent with dynamic changes is equally impossible, though the underlying theory of public welfare could be readily reviewed by placing $G=0$ and substituting the weighted average unit cost $\overline{w}_{\omega-h+1}$ of all future machines for the market price $z(t)$ in (11) and (12). Returning to Professor Taylor's suggestion, interpreted by equation (4), it may be seen that he considers the unit cost $w_2$ of the replacement alternative as permanent. In strict theory $\overline{w}_{\omega-1}(D_1)$, an absurdly complicated and untrustworthy function of all future events, would have to be substituted for $w_2$.

## II

Let us now consider the static case, where more than a single chain can be operated, the number being limited only by the demand for the product. For this purpose we may rewrite equation (7) in the form:

$$(13) \qquad G = k\,\frac{\int_0^T [z(\lambda k)Q(t) - E(t)]e^{-it}dt + Se^{-iT} - B}{1 - e^{-iT}}\,.$$

[8] "A Mathematical Theory of Depreciation and Replacement," *American Journal of Mathematics*, January, 1928, pp. 147–157.

[9] *Ibid.*, p. 156. See also his final formula (6)

The goodwill of $k$ chains equals $k$ times the goodwill of a single chain, regardless of how the chains are staggered as to replacement dates, because the rule found valid for a single chain distributes the stream of excess profits evenly over the life of a machine (see Figure 1). The nature of the demand function (in which $\lambda$ is the parameter of scarcity) and the method of staggering may introduce summations of successive integrals in lieu of a single integral. In any event, however, the *general rule* consists of applying the theory of maxima and minima with respect to both $T$ and $k$. To avoid cumbersome formulae, let the numerator of equation (3) be denoted by $W$ [i.e., all terms in the numerator of (13) which are independent of $k$ by $-W$], the denominator of (3) [i.e., the factor of $z(\lambda k)$ in (13)] by $Q$, and the denominator of (13) by $Ii$. If derivatives are indicated by primes, the two operations lead to the results:

$$(14)^{10} \qquad dG/dk = 0, \qquad z(\lambda k) + kz'(\lambda, k) = \frac{W}{Q},$$

and

$$(15) \qquad dG/dT = 0, \qquad z(\lambda k) = \frac{WI' - IW'}{QI' - IQ'} \cdot$$

When the division of (15) is performed, subtraction of (15) from (14) gives, upon rearrangement:

$$(16) \qquad \frac{W}{Q} = \frac{W' - kz'(\lambda, k)(QI'/I - Q')}{Q'} \cdot$$

Since the number of machines is a discrete variable, the unadjusted result of differentiation for $k$ implies $k = \infty$ and $\lambda = 0$. On the other hand, $W/Q = W'/Q'$ is the essence of the Taylor method. It follows that this rule is a limit of the general rule, but that it can be valid only when the second term in the numerator vanishes. Apart from the two special cases mentioned in Part I, that can happen only when the market price of the product is independent of the output of an infinitely large number of machines. In addition, all other limitations on the size of the enterprise must be ineffective.

To illustrate this aspect of the Taylor rule, it is necessary to assume that the market can absorb only a given output $Pk$ per unit of time.

[10] Professor Frisch suggested that I clarify the meaning of my symbol $z'(\lambda, k)$ thus: Let $\lambda k = v$. Then, $dz(v)/dv = z'(v) = z'(\lambda k)$, whereas $z'(\lambda, k) = dz(v)/dk = \lambda z'(v)$. The important point is that both $z(\lambda k)$ and $kz'(\lambda, k)$ are functions of the product $\lambda k$ and therefore $\lambda$ and $k$ occur in the final solution only in the form of that product and not singly.

If so, the optimum number of evenly staggered chains and their renewal period $T$ can be found from:

$$(17)\quad \begin{aligned} G = \frac{k}{1-e^{-iT}}\Bigg\{ z \sum_{j=0}^{k-1}\left[\int_0^x P + \int_x^{T/k} Q(t+jT/k)\right] e^{-i(t+jT/k)}dt \\ - \int_0^T E(t)e^{-it}dt + Se^{-iT} - B\Bigg\}. \end{aligned}$$

The symbol $x$ represents the recurrent fractional periods, for which the combined productive capacity of evenly staggered machines exceeds the salable output $Pk = \sum_{j=0}^{k-1} Q(x+jT/k)$. Between $x$ and $T/k$ the productive capacity is deficient and sales are lost. To any volume of demand $Pk$ there will correspond a definite number $k$ of parallel chains and a definite replacement period, which makes the aggregate goodwill a maximum. Formula (17), as written, applies to productive capacity declining with age. Should the reverse be true, $P$ and $Q(t+jT/k)$ change places.

The regression lines of Figure 3 are obtainable from the equivalent of (15) by placing $k=1, 2, 3, \cdots$, and varying $P$ in each instance within such limits that $x$ will vary from 0 to $T/k$. The actual task would be quite tedious, but the principles of a simple graphic solution soon emerge.[11] As the addition of successive chains declines in relative importance, the oscillations of economic life are gradually damped until they merge with the axis $T=9.5$ years, which is the Taylor-Hotelling solution for the data underlying both Figures 1 and 3.

The special expression (13) can now be generalized by letting all functions depend in various ways (e.g., *via* the output) on the product $\lambda k$, where $\lambda$ = a common parameter of many different kinds of elastic scarcity. If $dG/dT=0$ and $dG/dk=0$ are then solved simultaneously,[12]

[11] The initial and terminal points of each regression line, the initial tangents on the right, and the levels of indifference are given by the relation of $P$, $k$, and $x$ to each other. The final tangents on the left must be vertical. Only the curvatures may be slightly inaccurate, except for the first and first two chains, which I have calculated. The technique is clarified by an auxiliary diagram, in which the ordinates *per chain* of all terminal points are plotted and connected by straight lines. See upper right of Figure 1. For this purpose, ten units of the scale of ordinates represent one unit of annual demand per machine.

[12] Professor Frisch was particularly interested in "the way in which $\lambda$ and $k$ approach their respective limits," fearing that the results might depend thereon. In the course of correspondence on this point, it developed that my procedure must be explained in greater detail.

In equations (13), (14), and (15), let us assume that $k$ is a continuous variable and that the necessarily discrete number of parallel chains is $\kappa$ instead. Differentiation with respect to $k$ and $T$ then yields the solution $\lambda k = c$, where $\lambda$ is a given constant, so that $k = c/\lambda$ will not be an integer, except by accident. If it is

$T$ = axis of convergence and $1/\lambda k$ = distance of successive indifference levels from each other. These levels should be counted from the floor level defined by $G=0$, below which it would not be profitable to replace a machine.

It can thus be shown that the general rule may lead to innumerable different limits, of which the Taylor rule is not the most likely one. To calculate a plausible example would be a task of some magnitude. I have therefore illustrated only a single change in the data of Figure 1, namely $z(\lambda k) = \$10(1-\lambda k)$. As seen in Figure 4, economic life converges rapidly toward an axis of $T=8.42221$ years. A slightly better example of elastic demand would be $z(\lambda k) = \$10\left[1-\lambda\sum_{j=0}^{k-1}Q(t+jT/k)\right]$, but it requires the summation of separate integrals having the successive intervals $jT/k \leqq t \leqq (j+1)T/k$. I have calculated only the limit for $k=\infty$, i.e., the axis $T=11.8324$ years. The Taylor method would give 9.5 years in both instances. The respective goodwills are $\lambda G_{\text{max.}} = 0.2487075 \times \$79.7042 = \$19.8230$ as compared to the Taylorean $\lambda G = 0.25 \times \$78.3735 = \$19.5934$ for Figure 4 and $\lambda G_{\text{max.}} = 0.327534 \times \$38.6264 = \$12.6515$ as against $\lambda G = 0.2996193 \times \$41.2412 = \$12.3566$

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not, I take the two nearest integral values $\kappa = k-a$ and $\kappa+1 = k+b$, where $a+b=1$. For the two alternative arguments $\lambda\kappa$ and $\lambda(\kappa+1)$, formula (15) furnishes the two corresponding periods $T_a$ and $T_b$, one of which is longer and the other shorter than the abscissa of the axis of convergence $T$ derived from the simultaneous equations (14) and (15). Similarly, formula (13) will then have two solutions, viz., $G_a$ for $\lambda\kappa$ and $T_a$, and $G_b$ for $\lambda(\kappa+1)$ and $T_b$. Both $G_a$ and $G_b$ must be smaller than the impossible maximum $G$ for $\lambda k$ and the axis $T$. In the circumstances, the best answer obtainable is either $G_a$ or $G_b$, whichever be greater. The correct number of parallel chains and the economic life of the machines is accordingly either $\kappa$ and $T_a$, or $\kappa+1$ and $T_b$, as the case may be.

The parameter of elastic scarcity $\lambda$ in Figure 4 (or the salable output $P$ per machine in Figure 3) is not a variable in this presentation. Changes in the value assigned to it create, not a dynamic problem, but merely so many separate static ones. The value of $\lambda k = c$ is unique for a given set of functions $W$, $Q$, and $I$, so that $k$ is always defined rigidly by $\lambda$. In place of $k$, however, only integral values of $\kappa$ are available, of which the best must be selected. If the solutions of successive problems are connected by regression lines, Figures 3 and 4 result. To the extreme $\lambda=0$ there will correspond the extreme $k=\kappa+a=\kappa+1-b=\infty$. The difference between a continuous and a discrete variable then disappears and economic life can not differ from the abscissa of the axis of convergence.

The problem could be further clarified graphically in a plane having the coordinates $\lambda$ and $k$. A rectangular hyperbola will define all rectangles of equal area $\lambda k = c$. By raising ordinates from the two integral values of $\kappa$ nearest to any abscissa $k$, two rectangles are formed, one of which is too large and the other too small. Variation of $\lambda$ will then lead to a zigzag line oscillating around the hyperbola. But the lower the value assigned to $\lambda$, the less the two curves can differ from each other, until both merge at the limit $\lambda=0$, $k=\infty$. The opposite limit of $\lambda$ is that value $\Lambda$, for which $z(\Lambda) = w$ = Taylorean unit cost. Any further infinitesimal increase in $\lambda$ causes $G<0$ and breaks the last chain of replacements.

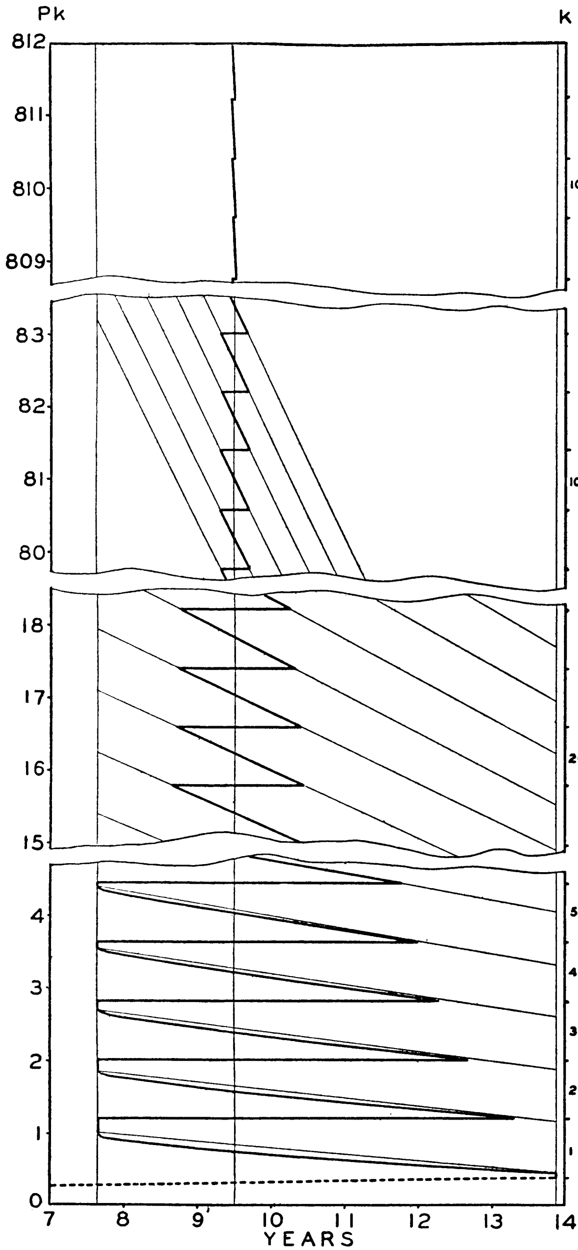


FIGURE 3

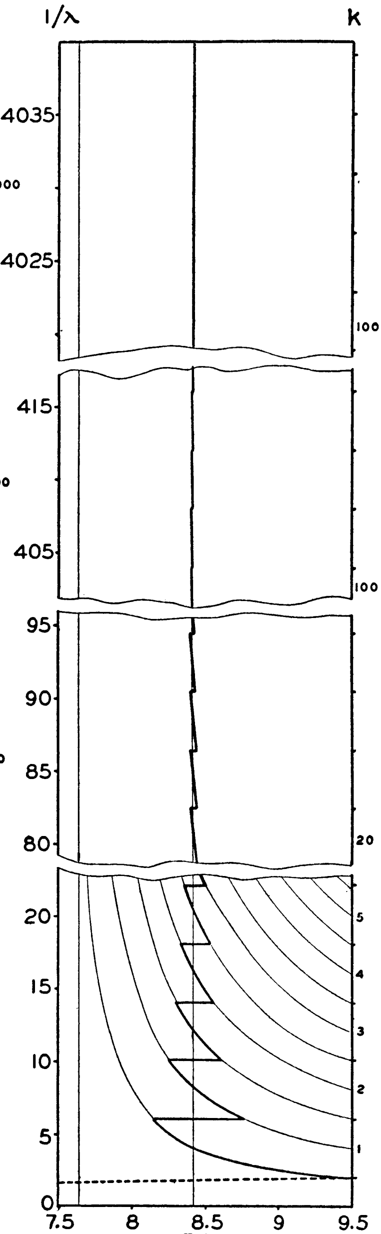


FIGURE 4

(For description see page 25.)

for the second example. The coefficients are the various values of $\lambda k$, when $\lambda = 0$ and $k = \infty$. A comparison of the relative magnitudes of infinitely large amounts is thus possible.

The scope of the individual rule must also be mentioned. *First*, it is always valid when $G < 0$, i.e., when replacement would be uneconomic. It minimizes the loss on machines already on hand (see broken lines at bottom of Figures 3 and 4). *Second*, it is a limit of the general rule, when a scarcity of renewals is the only effective limitation. This may be shown by restoring $z$ for $z(\lambda k)$ and substituting instead $B(\lambda k)$ for $B$ in formula (13). Operations analogous to (14), (15), (16) furnish the necessary condition $kB'(\lambda, k) = 0$. Limited volume at a fixed price, i.e., a rigid scarcity of replacements, must prevail, just as for the Taylor rule with respect to demand. *Third*, the individual rule will affect the problem, when $k \neq \infty$. If the stream of replacements is limited only elastically by rising costs, it is not proper to discard a machine upon acquisition of another. This situation may be stated with slight oversimplifications as follows:

$$G = \frac{k}{1 - e^{-iT}} \left\{ \sum_{j=0}^{k-1} \left[ \int_0^x R(k+1, t+j\theta) + \int_x^\theta R(k, t+j\theta) \right] e^{-i(t+j\theta)} dt + Se^{-iT} - B(\theta) \right\}, \tag{18}$$

where $R(k, t) = z(k)Q(k, t) - E(k, t)$ and $\theta$ = interval between consecutive purchases of machines. The loose relationship of $k$, $x$, $\theta$, and $T$ to each other can be stated only by stipulating $x = T - k\theta$, when $k\theta < T < (k+1)\theta$. As $\theta$ is decreased continuously, $k$ increases by units and recurrent odd periods $x$ are left over, during which there will be one more machine in service. Ultimately $k = \infty$, $\theta = 0$, and $x = 0$,

---

FIGURE 3.—Economic life of a plant consisting of several evenly staggered infinite chains, when the annual demand $Pk$ is rigidly limited in volume, but the market price ($z = \$10$) is independent of that volume. The greater the absorptive capacity of such an artificial market is, the less can economic life differ from the Taylor-Hotelling solution $T = 9.5$ years. Machines are of type illustrated in Figure 1. When $Pk = P < 0.397756$, the broken line (individual rule) defines the age at which the machine in service, and with it the enterprise, must be scrapped.

FIGURE 4.—Economic life of a plant consisting of several evenly staggered infinite chains, when the market is elastically limited by prices linearly, but inversely related to the number of parallel chains employed. Convergence occurs toward a limit differing from the Taylor-Hotelling solution. Machines are of type illustrated in Figure 1, but market price is $z(\lambda k) = \$10(1 - \lambda k)$, where $\frac{1}{2} \geqq \lambda \geqq 0$ is the parameter of scarcity and $k = 1, 2, 3, \cdots, \infty$ the number of cooperating chains. When $\lambda > \frac{1}{2}$, i.e., $1/\lambda < 2$, the broken line (individual rule) defines the age at which the machine in service, and with it the enterprise, must be scrapped.

whereupon the influence of the individual rule disappears and the simplified approach based on the chain rule becomes correct. Lack of space prevents further consideration of this complication, but its existence should be noted.

III

Up to this point, explicit reference was made only to the first three types of limitation listed in the introduction. Tacitly, however, the principle of the scarcity of capital was also observed in its conventional formulation, according to which additional funds are available only up to the marginal dose, which earns barely the rate of interest. This limitation will now be examined for its own sake.

The capital value of a plant consisting of $k$ machines of all ages from 0 to $\Theta$ may be written:

$$(19)\qquad Vk = \frac{\int_0^{\Theta/k} \sum_{j=0}^{k-1} R(t + j\Theta/k)e^{-it}dt - (B - S)e^{-i\Theta/k}}{1 - e^{-i\Theta/k}}.$$

When $k=1$, the formula is evidently equivalent to (7) without subscripts. Upon passing to the limit $k=\infty$, we obtain:

$$(20)\qquad V\Theta = \frac{\int_0^{\Theta} R(t)dt - b}{i}, \qquad b = B - S.$$

The capital value of a static composite plant equals the present worth of a perpetual income stream of constant intensity. The factor $\Theta$ on the left indicates that the number of machines in service is linearly related to the renewal period.

Since the statement is often heard that the owner will do everything in his power to make "enterprise capital value" a maximum, it should be noted that this is true only in a rather technical sense. Literal interpretation would suggest such operations as $d(V\Theta)/d\Theta=0$ or $d[V\Theta/\int_0^{\Theta} Q(t)dt]/d\Theta=0$, which would indeed make capital value a maximum attainable under the corresponding exclusive and rigid conditions of scarcity. Such a procedure, however, would be distinctly detrimental to the owner. The correct attitude is that of a purchaser, who has just acquired the plant in question at a cost $C$. He will want to know, whether or not there exists a different renewal period $T$, which would make his purchase more, i.e., most profitable. The search is simplest in the case, where the size of the enterprise is limited only by the stream of replacements available:

$$(21)\qquad G = \frac{\int_0^{T} R(t)dt - b}{i} + (\Theta - T)S - C.$$

When the present renewal period is too long, it is best to sell the oldest machines for scrap and replace only the remainder. Differentiation for $T$ leads immediately to the individual rule. Conversely, when the present renewal period is too short, (21) must be rewritten:

$$(22) \qquad G = \frac{\int_0^T R(t)dt - b}{i} - \int_0^{T-\Theta} e^{-it}\left[\int_{\Theta+t}^{T} R(\tau)d\tau + S\right] dt - C.$$

The second term on the right is the capital value of machines between the ages $\Theta$ and $T$, which are nonexistent at the moment, when the lengthening of the life-span is decided upon. Differentiation for $T$ again leads to the same rule. The proof by the Taylor rule is similar in principle, except that the formulae corresponding to (21) and (22) must be so set up as to maintain constant the old composite output $\int_0^\Theta Q(t)dt$. This entails a change in both the renewal rate and the number of machines in service.

It can thus be demonstrated in many different ways that "enterprise capital value" must be made a maximum, not for any mature plant, but for one in the process of transition to the optimum renewal period. In terms of the data underlying Figure 1, the greatest capital value obtainable from formula (20) would correspond to $T=15$ years. Nevertheless the economic life under the individual rule is $14.\dot{6}\dot{6}$ because, during the change to the shorter life-span, more capital can be extracted from the plant than the amount by which its capital value will ultimately drop. Similarly, if the Taylor rule should be applicable, $T=9.5$ is a better answer than the maximum at $\Theta=9.072$. The conventional concept of the scarcity of capital is thus illustrated. The conclusion may also be drawn immediately that *economic life is independent of the prices at which the plant is bought and sold.* Whatever the so-called investment $C$ may be, it always disappears in the differentiation process leading to the optimum.

Further scrutiny of the Taylor rule shows that it may also be looked upon as a method of maximizing the gross income per dollar of outgo. By making $w$ from formula (3) a minimum, its reciprocal $1/w$ automatically becomes a maximum. This has also been hailed as proof of universal validity even though, in the general case, it clearly implies a greater scarcity of capital than indicated by the market rate of interest. Under the conventional assumption, it will pay to use additional doses beyond the maximal rate of return, down to the margin.

The "rate of return" maximized by the Taylor rule is an unusual concept, to say the least. When a business man has occasion to use the term, he means something else, namely the annual amount of profit divided by his investment. Operating expenses must be paid out of current income to keep going, so that he has but little choice in the matter. The need for timely replacement, on the other hand, is less apparent and therefore often underrated. Such neglect enhances the scarcity value or productivity of renewal expenditures beyond those of operating costs. Many plants are in a run-down condition, because the resultant rise in the rate of profit hides the more significant decline in its amount. The limit is reached, when the operator has "spread himself so thin" that he can no longer obtain a loan. A replacement policy then arises, which aims to maximize the rate of return on the investment. The static theoretical solution can be found from any one of the special rules given so far, by substituting the unknown rate of profit $p$ for the rate of interest $i$. The original cost of a machine always equals the net rental and the scrap value, discounted at the rate of profit. Since the unit selling price of the product accordingly equals the unit cost (plus profit), all approaches lead to the same rule $G(p)=0$, which will be called the *profit rule.* For the data of Figure 1, the answer is $p_{\text{max.}}=38.31555$ per cent per annum (compounded instantaneously) at $T=12.4563$ years.

The profit rule is another limit of the general rule. The substitution of $p_{\text{max.}}$ for $i$ will be theoretically proper, whenever the hire of additional capital would exceed the maximal rate of profit. The owner of several enterprises must equalize the scarcity of capital for all by considering the lowest of the various rates of profit as his private rate of interest. Only the least lucrative enterprise will then be governed by the profit rule; to the rest, the general rule will apply, as determined by all elastic scarcities. As soon as an outsider is willing to lend at a lower, though perhaps still exorbitant rate, that new rate supersedes the former private scarcity rate in all the enterprises owned.

This reasoning leads directly to the correct principle of public-utility regulation. By fixing the "fair rate of return" by law, a rigid condition of scarcity is created, which automatically cancels the influence of all other elastic (or less stringent rigid) scarcities upon economic life. For corroboration, see formula (14), where $G=0$ is a sufficient condition for $kz'(\lambda, k)=0$. It follows that *the profit rule reduces the consumer's costs to a minimum.* Provided only that the fair rate of return be used in lieu of the rate of interest, this rule may be called the Taylor rule, the individual rule, or the chain rule, as preferred.

The theory of public-utility regulation is thus quite definite and leaves no room for equivocation. Most of the difficulties created by dynamic changes could also be overcome by actually fixing, i.e., also guaranteeing the rate of return. A "consumers' surplus (deficit)" account, serving as a temporary reservoir of all differences, would furnish the necessary guidance for "rate" regulation, if its level and rate of change were currently observed. From an accounting viewpoint, the problem is simple enough, but the clerical cost of calculating all adjustments by hindsight might well be prohibitive. Perhaps sampling methods could be employed.[13] Whether or not it would be sound public policy to guarantee the return is beyond the scope of this article, but the present obstacle race between inflated rate bases and inadequate returns thereon is certainly no solution.

## IV

An assumption used throughout the two preceding sections must now be revised. For the purpose of gradual transition from the single-machine problem to that of a large plant, it may be permissible to postulate identical economic lives for co-operating machines of identical type, but observation shows that that is never the case in practice. Some machines drop out very soon, while others continue to render useful service far beyond the average life of a large group.

Statisticians have published voluminous data on the behavior of different types of equipment, but have made no attempt to justify scrapping by any value theory of economic life. They merely assembled the observed facts, namely, how many machines of a given type were actually scrapped at what ages out of what total number.[14] This information, compiled in the form of a histogram, can be normalized and fitted with a frequency distribution $f(t)$, which vanishes for all values of $t$, except those within the interval $0 \leqq t \leqq n$, where $n$ = maximal age. The frequency distribution cumulated backward is the mortality curve $M(t)$ and the area enclosed by the latter and the co-ordinate axes is the average life $a$:

$$M(t) = \int_t^n f(\tau)d\tau, \quad M(0) = 1, \quad M(n) = 0, \quad \int_0^n M(t)dt = a. \tag{23}$$

The procedure must now be connected with the value theory of economic life, as outlined in the preceding sections. If the mortality curve is the result of scrapping each machine separately in accordance with the general rule, it follows that the operating expenses and in general also the rate of production differ from machine to machine, despite the identity in type. The symbols of the net rental $R(t) = zQ(t) - E(t)$ should therefore be expanded to $R(y, t) = zQ(y, t) - E(y, t)$, where $y$ expresses the variation from machine to machine. The mor-

[13] For general suggestions, see my articles, "The Principles of Public Utility Depreciation," *Accounting Review*, June, 1938, pp. 149–165 and "The Practice of Depreciation," ECONOMETRICA, Vol. 7, July, 1939, p. 259 and p. 262, point 5.

[14] Cf. Edwin B. Kurtz, *Life Expectancy of Physical Property*, Ronald Press Co., New York, 1930.

tality curve is then defined in terms of $T = M^{-1}(y)$, i.e., by the individual economic lives of many machines acquired at the same time, when those lives are extended horizontally and arrayed from top to bottom, from the shortest to the longest. The inversion $y = M(T)$ discloses, how many per centum survive any age $T$.

The mechanics of renewal are defined by a variable-limit integral equation of the closed-cycle type, here written in the form of a Volterra equation of the first kind:[15]

$$e^{\int_0^t x(\tau)d\tau} = X(t) + \int_0^t u(\tau)X(t-\tau)d\tau, \qquad 0 \leqq t \leqq n. \tag{24}$$

The term on the left is an index of size or volume, when $x(t)$ = rate of growth. On the right, $X$ represents any function of limited variation, which differs from zero only within a range $n$. The renewal rate is denoted by $u$. Whenever $n \neq \infty$, equation (24) must be applied to subsequent intervals of $n$ years in the form:

$$e^{\int_0^t x(\tau)d\tau} = \int_{t-n}^{nj} u_j(\tau)X(t-\tau)d\tau + \int_{nj}^{t} u_{j+1}(\tau)X(t-\tau)d\tau, \qquad nj \leqq t \leqq n(j+1). \tag{25}$$

The relationship of renewals to mortality and growth may be expressed in any number of ways, for instance:

I. The *independent renewal rate* $u_{\text{I}}$ may be given by the probable future scarcity of new machines available as replacements. Upon placing $X(\alpha) = M(\alpha)$ = mortality under the corresponding limit of the general rule, the right side of (24) and (25) is known and determines an index of the optimum number of machines, which should compose the plant at any given time $t$.

II. The *number renewal rate* corresponds to situations where it seems easiest to guess the future in terms of the number of machines employable. The left side of (24) and (25) is known and $X(\alpha) = M(\alpha)$ determines $u_{\text{II}}$. This entails a serial calculation for successive values

[15] For the derivation and solution see *op. cit.* in note 1, pp. 221 *et seq.* Since publication of that paper, my attention was called to A. J. Lotka's numerous articles dealing in the main with similar population problems, but including also an excursion into "Industrial Replacement" (*Skandinavisk Aktuarietidskrift*, 1933, pp. 51–63). He employs a technique developed by Paul Hertz in "Die Bewegung eines Elektrons" (*Mathematische Annalen*, 1908, pp. 84–86). This approach consists of substituting a generalized Fourier series for the real solution and gives very poor results during the early years to which foresight can possibly extend. For a demonstration see my paper, "The Theory of Industrial Replacement," *Skandinavisk Aktuarietidskrift*, 1939, pp. 1–9.

of $j$, so that $u_{\text{II},j}(t)$ is always known from the preceding step, whereas $u_{\text{II},j+1}(t)$ is the function sought.

III. The *output renewal rate* will be obtained, if it is found preferable to guess $x(t)$ = future rate of growth in output. In that case, $X(\alpha) = \int_0^{M(\alpha)} Q(y, \alpha)dy$ and equations (24) and (25) must be solved for $u_{\text{III}}$ as before.

IV. The *capital renewal rate* is suggested by the profit rule. The left side of (24) and (25) expresses the capital available at various times and therefore on the right $X(\alpha) = \int_\alpha^n e^{-p\nu} [\int_0^{M(\nu)} R(y, \nu)dy + Sf(\nu)]d\nu$. The task of solving for $u_{\text{IV}}$ must again be undertaken.

Carrying theory to extremes, it might be held that the choice of the renewal rate is governed by that one of the innumerable composite limits of the general rule, which happens to be in force in a particular case. Obviously, however, the capital value of the plant must be the same for any kind of renewal rate, if the same rule of economic life was employed in its calculation. Only the unit of measurement will differ, in terms of which the capital value is expressed. Such units are for instance the intensity of the flow of new machines at $t = 0$, the originally installed number of machines, the original volume of output, or the original capital. All rates of renewal and all rates of growth are thus convertible into terms of one another.

To find the capital value of the plant, the net rentals of all machines composing it at various times are first added up. This involves two steps, namely summation for each separate age group and totalling for all such groups:

$$(26) \qquad r(t) = \int_0^{M(t)} R(y, t)dy + \int_0^t u(\tau) \int_0^{M(t-\tau)} R(y, t - \tau)dyd\tau, \qquad 0 \leqq t \leqq n.$$

The first term on the right refers to the machines originally installed and hence vanishes when $t > n$. The second term should then be subdivided as in (25). Purely for brevity of notation, however, it is permissible to designate the renewal rate by $u(t)$ for the entire interval $0 \leqq t \leqq \infty$. We may accordingly write:

$$(27) \qquad r(t) = \int_{t-n}^t u(\tau) \int_0^{M(t-\tau)} R(y, t - \tau)dyd\tau, \quad n \leqq t \leqq \infty.$$

For a constant rate of growth $x(t) = x \gtreqless 0$, the rate of renewal converges more or less rapidly toward its asymptote:

$$(28) \qquad u(\infty)e^{-x\infty} = \frac{1}{\int_0^n X(\tau)e^{-x\tau}d\tau} = U,$$

which can be readily derived from formula (25). The ultimate net rental function then becomes:

$$
(29)\qquad
\begin{aligned}
r(\infty)e^{-x\infty} &= U\int_0^n e^{-x\tau}\int_0^{M(\tau)} R(y,\,\tau)dyd\tau \\
&= U\int_0^1\int_0^T R(y,\,\tau)e^{-x\tau}d\tau dy, \qquad T = M^{-1}(y),
\end{aligned}
$$

and leads to the capital value of a mature plant:

$$
(30)\qquad V(\zeta) = \frac{U_1 e^{x\zeta}}{i-x}\left[\int_0^1\int_0^T R(y,\,\tau)e^{-x\tau}d\tau dy - b - \frac{Sx}{U_2}\right], \qquad i > x,\ \zeta = t - \infty \geqq 0.
$$

In this formula, $U_1$ = asymptote of any kind of renewal rate, whereas $U_2$ = asymptote of number renewal rate, since the scrap values sunk into a growing plant can be related to no other. No similar distinction need be drawn for the rates of growth, as they will all be equal in the end. The formula differs from (20) only by the introduction of individual characteristics and a rate of growth. If those two refinements are deleted, only the constant factor $U_1$ remains to denote the unit of measurement. The integration with respect to $y$ is merely an averaging process, so that *all conclusions drawn in Sections II and III hold good in terms of the average machine.* The rate of growth does not influence that average because all renewal lots, though unlike in size, are considered large enough to show a similar distribution of individual characteristics.

Let us now consider the problem of obsolescence. With the passage of time, we shall reach the successive dates $D_h$ (where $h = 1, 2, 3, \cdots, \omega$), when replacement by an improved type of machine becomes possible. To simplify the final formula, let us also include immediately the items of income and outgo arising in addition to the net rental. These are the renewal rate and the rate of scrap sales. For machines of type $h$, we thus obtain the triple expression:

$$
(31)\qquad
\begin{aligned}
P_h(t) &= \int_{D_{h-1}}^t u_h(\tau)\left[\int_0^{M_h(t-\tau)} R_h(y,\,t-\tau)dy + S_h f_h(t-\tau)\right]d\tau - B_h u_h(t), \\
&\qquad\qquad D_{h-1} \leqq t \leqq D_{h-1} + n_{h-1}, \\
&= \int_{t-n_h}^t u_h(\tau)\left[\int_0^{M_h(t-\tau)} R_h(y,\,t-\tau)dy + S_h f_h(t-\tau)\right]d\tau - B_h u_h(t), \\
&\qquad\qquad D_{h-1} + n_{h-1} \leqq t \leqq D_h, \\
&= \int_{t-\nu_h}^{D_h} u_h(\tau)\left[\int_0^{\mu_h(t-\tau)} R_h(y,\,t-\tau)dy + S_h \phi_h(t-\tau)\right]d\tau, \\
&\qquad\qquad D_h \leqq t \leqq D_h + \nu_h.
\end{aligned}
$$

How the first of these equations must be amended for $h=1$, may be seen by providing all component elements of (26) with that subscript. Equation (27) is equivalent to the corresponding term of the second form of $P_1(t)$.

At the dates $D_h$, the replacement of type $h$ stops and the installation of type $h+1$ begins. During the last of the three intervals (31), therefore, the flow of net income is $P_h(t)+P_{h+1}(t)$, where the latter function takes its first form. It should also be noted that although, in the absence of a change in type, forecasts of $R_h$, $B_h$, and $S_h$ determine $M_h$ by means of the governing limit of the general rule of economic life, the same relations do not hold with respect to the third variety of $P_h(t)$. The final mortality curve $\mu_h$, its derivative $-\phi_h$, and $\nu_h=\mu_h^{-1}(0)$ are forms of an unknown function to be determined in such a manner that the capital value

$$(32) \qquad V(D_h) = \int_{D_h}^{H} P(\tau)e^{i(D_h-\tau)}d\tau + L(H)e^{i(D_h-H)}$$

shall become a maximum. In this formula, $P(\tau)$ represents the partly overlapping series $P_h(\tau)$ for all consecutive values of $h$. The other new symbols are $H=D_\omega$=horizon, i.e., limit beyond which it is not expected that operations will continue,[16] and $L(H)$=liquidating value, best defined as the capital value after abandonment of further renewals.

The determination of the number renewal rate $u_{h+1}$ may vary according to circumstances. If perfect co-operation between machines of types $h$ and $h+1$ is possible, its first segment will be defined by the equation

$$(33) \qquad e^{\int_0^t x(\tau)d\tau} = \int_{t-\nu_h}^{D_h} u_h(\tau)\mu_h(t-\tau)d\tau + \int_{D_h}^{t} u_{h+1}(\tau)M_{h+1}(t-\tau)d\tau, \quad D_h \leqq t \leqq D_h+\nu_h,$$

which contains the same two unknown functions as the corresponding segment of $P(\tau)$ in (32). Multiplication of $u_{h+1}$ by an appropriate constant will be in order, when old and new machines are not exchangeable

[16] Whenever the rate of growth equals or exceeds the rate of interest, the counterpart of formula (30) leads to an infinite capital value. This impossible result calls attention to the element of risk, usually expressed by the addition of an insurance premium to the pure money rate. In my opinion, such a two-in-one tool of analysis is not always flexible enough and therefore I have elsewhere suggested the use of a horizon as a measure of risk in the valuation of common stocks. Cf. my book *The Nature of Dividends*, New York, 1935, p. 10 and mathematical appendix, where two separate horizons are employed to approximate the risks of perpetual earnings and expansion respectively. For present purposes a single horizon will do, so long as the rate of growth is considered variable.

at par. It is also possible, however, that the whole plant of type $h$ must be scrapped immediately,[17] because the two types can not be used side by side. In that case, $u_{h+1}$ is defined by $M_{h+1}$ alone. Many other cases may arise between these two extremes. Depending upon the relationship between the average age of the machines composing the plant at the time $D_h$ and the net proceeds from their forced sale, it may or may not be worth while to operate the abandoned type $h$ for a short time and to curtail the early purchases of type $h+1$ accordingly. The special problem of economic life during a process of liquidation will also emerge at the final date $H$. The details of (32) will thus vary according to the specific case under consideration. Ordinarily, it is reasonable to assume that only one change in type can be foreseen at any given time, so that the index $h$ can have only the value $\omega-1$. If that should not be the case, solution must begin at the end of the composite chain and proceed backward in a manner reminiscent of single-chain dynamics.

Although this description of the composite chain considers only obsolescence, the general dynamic situation can also be readily visualized. The subscript $h$ identifies not only the type of replacements, but also the successive periods, during which one type is preferred to all others. Its meaning may therefore be expanded to include a reference to all extraneous changes in economic conditions as well. All functions then obtain an additional argument to denote that they are also subject to change with the passage of time.[18]

The index $h$ can also be used to express successive changes in ownership and outlook. For otherwise static conditions, $R_h = R_{h+1}$, $B_h = B_{h+1}$, and $S_h = S_{h+1}$, but $M_h \neq M_{h+1}$ and $u_h \neq u_{h+1}$, when the new owner can improve upon the replacement policy followed by his predecessor. In that case $M_{h+1}$ is given by the new forecast of future conditions, including the nature of the governing rule, while, within the period of transition, $M_h$ is once more an unknown function to be determined so as to make $V(D_h)$ a maximum. This generalizes the simpler discussion based on formulae (20), (21), and (22).

The main structure of replacement theory may be completed by

[17] This problem is discussed in simpler terms by Professor P. O. Pedersen, "On the Depreciation of Public Utilities," *Ingeniørvidenskabelige Skrifter B Nr. 12*, Dansk Ingeniørforening, Copenhagen, 1934, pp. 69–99.

[18] Such a presentation is of course subject to Professor Hotelling's observation that it considers "time as a passive parameter, carrying along the gradually changing influences of a mass of unspecified sources of variation" (cf. "The Work of Henry Schultz," ECONOMETRICA, Vol. 7, April, 1939, p. 99). For purposes of a general theory of economic life, however, it is sufficient to note that, if specifications are available, they can be readily utilized within the framework of formula (32).

mentioning the *liquidating rule* derived from (32) by the operation $dV(t)/dH=0$. This rule is analogous to the individual rule (6) and states that liquidation should take place, when the difference between income and outgo drops below the rate of interest on the liquidating or break-up value of the enterprise. This value is of course the present worth, at the date $H$, of all proceeds collected, less expenses incurred thereafter. As in (6), the amount of investment shown by the books has no bearing on the optimum.

## V

To summarize the conclusions reached, it may be said that the theory of economic life is essentially a theory of scarcity. Successful enterprise has many tangible and intangible ingredients, each of which may be limited either rigidly as to volume, or elastically by price movements. Starting with a total lack of elasticity, it is evident that the relatively greatest scarcity alone determines the size of the plant. All others pass unnoticed. As shown, Professor Taylor's theory presupposes that a rigid demand for the product is the least abundant ingredient. The individual rule and the profit rule are counterparts with respect to the supply of new machines and capital. Analogous rules can be readily developed for rent, labor, fuel, etc.

The general rule of replacement, which is simply the theory of maxima and minima, has a separate solution for every kind of rigid scarcity and for every volume of the supply so limited. When the volume required by a single machine becomes insignificant in comparison to the total, the problem is simplified into making the excess profit (goodwill) per unit of that ingredient a maximum. In the case of demand, that means making the cost per unit of demand (output) a minimum. In all other instances, the limitation operates at the other end of the productive process and therefore the first description applies. The excess profit per new machine, per square foot of space, per hour of labor, per ton of fuel, etc., must be made a maximum, depending upon where the shortage is felt.

Elastic scarcities introduced singly lead to another set of rules. Their relationship to the Taylor-Hotelling theory is best disclosed by copying the Taylor technique exemplified by formula (3). A glance at equation (16) shows that the correct *demand rule* may be written in the abbreviated notation:

$$(34) \qquad w = \frac{W}{Q} e^{-kz'(\lambda,k)\int(QI'/IW-Q'/W)dT}, \qquad dw/dT = 0.$$

This formula can also be used to express the limit of the general rule for any other case of isolated elastic scarcity, for instance a *labor rule*, a *tax rule*, etc. Only the meaning of the symbols must be appropriately generalized. In the numerator of formula (13), let some term other than the sales be dependent on the number of machines and let that term (or subterm) be rewritten in the form "price times rate of consumption." Then let $-W=$ all terms independent of $k$, $Q=$ discounted volume of the ingredient consumed during the life of the machine, and $z(\lambda k)=$ purchase price per unit of that ingredient. It follows that the counterpart of the Taylorean "unit cost plus" $w$ is that price of the only scarce ingredient, which would cause the goodwill to vanish. By maximizing it with respect to $T$, the difference $w-z(\lambda k)=$ excess profit per unit of the ingredient will also become a maximum. All operations are the same as for the demand rule, only the terminology and the signs are reversed.[19]

The last step in analysis consists of combining all scarcities. Since all are ordinarily elastic, none may be neglected. Some are no doubt negative, in which case the law of increasing returns must be offset against the law of diminishing returns. The theoretical assumption $k=\infty$ implies that the plant has reached a size, where the latter predominates. The final conclusion is therefore that *excess profits must be made a maximum in terms of a composite index of productive activity, not with reference to any single ingredient*, such as demand.

If all prices, whether paid or received, are denoted by $z_j(\lambda k)$ or $z_j$ for short and all discounted volumes consumed or produced by $\int_0^T Q_j(t)e^{-it}dt$ or $Q_j$ for short, the static replacement problem can be summarized by the symbolic formulae:

$$(35)\qquad \begin{cases} dG/dk = 0 = \sum_{j=1}^{\infty} (z_j + z_j')Q_j, \\ dG/dT = 0 = \sum_{j=1}^{\infty} (IQ_j' - Q_jI')z_j, \end{cases}$$

where $z_j'=kdz(\lambda k)/dk$, $Q_j'=dQ_j/dT$, $I=(1-e^{-iT})/i$, and $I'=e^{-iT}$. Prices received are positive and prices paid are negative. This notation covers scrap sales and renewal costs also, if $Q_S=e^{-iT}$, $z_S=S$, $Q_B=1$, and $z_B=-B$. For general dynamic conditions the twin formulae (35) will fail and a guessing process idealized by (32) must take their place.

[19] It is worth repeating that all such special rules can be valid only for $k=\infty$. The expression $kz'(\lambda, k)$ does not thereby become infinite, since it also contains the parameter of scarcity $\lambda$, the limit of which is zero. Neither should it be overlooked that the presentation in the form (34) serves only for the calculation of the unit costs. To find the limits of $T$ and $\lambda k$, equations analogous to (14) and (15) must be solved simultaneously, using the same generalized definitions as in (34).

All rules of economic life are also rules of depreciation, since each suggests the apparently most logical way (out of innumerable other possibilities conforming to the terminal condition) in which costs ought to be distributed in the corresponding circumstances. For instance, if the fuel supply were rigidly limited, all costs should be written off in proportion to the fuel consumption. To exaggerate for the sake of emphasis, it might then be said that the investment consists entirely of fuel. Similarly, it may consist of anything else. In general, it contains a bit of everything. The original cost of a machine could thus be analyzed into its components:

$$B(0) = \sum_{j=1}^{\infty} w_j Q_j, \tag{36}$$

subject to the proviso $j \neq B$, meaning that the term $w_B Q_B = -B(0)$ has been removed from the summation and transferred to the left side of the equation.

In addition to an infinite number of different sets of unit costs varying with the age of the machine, which would satisfy (36), if the $w_j$ had been left under the integral signs included in $Q_j$, we can also find an infinite number of satisfactory sets of constant unit costs. The most logical of all these sets is that which can be had by considering each ingredient in turn as the only scarce one, i.e., performing all calculations (34), after the limit of $\lambda k$ has been determined and after $w_j$ is substituted for the corresponding prices $z_j$, which are included in the cumulative symbol $W$. Each unit cost will thus be expressed in terms of all the others. Simultaneous solution of all equations (34) is then in order. The number of conditions so given is sufficient, since $w_B = z_B = -B(0)$ for this particular purpose.

If that be considered the correct procedure, the theoretical depreciation method for a single machine operated by an unregulated enterprise would be:

$$B(t) = \sum_{j=1}^{\infty} \int_t^T w_j Q_j(\tau) e^{i(t-\tau)} d\tau, \qquad j \neq B. \tag{37}$$

This expression includes the scrap value, if $w_S Q_S(\tau)$ is defined as $iS/(e^{i(T-t)} - 1)$. The only conclusion which I shall draw from this analysis for the present is that it explains, why "unexpired cost" or "investment" must remain a vague and nebulous concept for practical purposes.

Accountants are beginning to realize, how little meaning can be ascribed to balance-sheet figures certified to "conform to generally accepted accounting principles consistently applied in the past."[20] In

[20] The phrase officially adopted by the American Institute of Accountants reads: " . . . conform to generally accepted accounting principles applied on a

view of the extreme complexity of the situation and the difficulty of obtaining the data required, not much can be done beyond facing the facts frankly and striving at least for the comparability of successively reported profits by ironing out, as far as possible, all those short-term fluctuations which tend to becloud the main issue, i.e., the problem of appraising the capital value of an enterprise.[21]

Although I believe I have fairly stated the serious theoretical limitations of the Taylor-Hotelling idea of minimizing unit costs, I nevertheless consider it the most valuable single rule of thumb which can be laid down for the general guidance of the entrepreneur, at least when the number of machines employed is very large and no radical change in type is imminent. Demand is usually the most important ingredient of successful enterprise. In addition, equipment can seldom be exploited to the full extent of its capacity. In some fields, this limit can never even be approached. As far as the machines themselves are concerned, there may thus be no reason why the output of any one should not be approximately constant for successive accounting periods. Extraneous economic conditions will, of course, introduce fluctuations in output and price or, what is more to the point, in the product of both per machine. Even then, however, the trend per machine may not differ substantially from a horizontal line. In such cases, it follows that the natural instinct to minimize unit costs can lead to no great error. In any event, the rule will serve as a useful median or point of departure for guessing the average life under moderately dynamic conditions. In what direction it is apt to err in various circumstances, I have attempted to point out at least in a general way.[22]

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basis consistent with the preceding year" (*Journal of Accountancy*, July, 1939, p. 18). This amendment robs the "certificate," "report," or "opinion" of even the last vestige of its significance. See note 21 below.

[21] For a discussion of the practical criteria governing the choice of a depreciation method see "The Practice of Depreciation," cited in note 13 above. Apart from the conclusions there reached, *long-range* consistency is the most important prerequisite of comparability. Which of the various contradictory, but nevertheless "generally accepted" methods is consistently applied, matters less in the long run. See *op. cit.* in note 1, conclusions 2 and 4 on p. 232. Also p. 240, *ibid.*

[22] These comments refer to unregulated enterprise only. In addition, the Taylor-Hotelling theory is entirely correct from the viewpoint of public welfare, at least so long as each regulated enterprise must still stand on its own feet. Neither of the two authors had considered this aspect at the time the papers cited were written. Since then, Professor Hotelling has dealt extensively with "The General Welfare in Relation to Problems of Taxation and of Railway and Utility Rates" (ECONOMETRICA, Vol. 6, July, 1938, pp. 242–269), but his pertinent conclusion is that "nothing could be more absurd than the current theory that the overhead costs of an industry must be met out of the sale of its products

The practical significance of this résumé is unfortunately impaired by the difficulty of determining the extent to which any particular machine happens to differ from the average of its type. I know of no instance, where it is considered feasible to keep individual operating records in sufficient detail to be of any real help. In addition, unforeseen accidents will happen so that, in the last analysis, physical inspection will probably remain the deciding factor. On the whole then, I am not greatly impressed by the practical merits of the theory of economic life, although it is no doubt a fascinating subject, worthy of study for the sake of its legitimate place in economics. From any other viewpoint, it seems to share the well-known peculiarity of the weather: A great deal may be said, but very little can be done about it!

*New York, N.Y.*

## APPENDIX

In recognition of Professor Harold Hotelling's priority in this field, the MS of this paper was submitted to him for criticism. The resultant exchange of thoughts is here summarized with his permission and in compliance with Professor Frisch's suggestion. The numbered remarks are his, the answers mine.

1. The idea of an infinite chain of replacements which you have developed, seems to me well worthy of consideration. It is interesting that we have in the various dates of replacement an infinite number of variables, with respect to which the discounted profit is to be a maximum. Under static economic conditions, or under economic conditions varying in a preassigned fashion, the infinite chain seems to be merely another way of looking at the problem to which I gave chief attention in my 1925 paper and to yield the same numerical results.

*Answer:* I agree that the numerical results will be the same, if "static economic conditions or economic conditions varying in a preassigned fashion" are suitably defined in one of three alternative ways, viz., when: (1) so-called perfect competition prevails, i.e., cost and market price can not differ; (2) the rate of sales $z(t)Q(t)$ is constant; and (3) the selling price is independent of the output of a very large (infinite) number of machines and no other limitations exist, which would make the employment of a smaller (infinite or finite) number more profitable.

2. I can see now how it is that you can in some cases arrive at a valuation of a machine that is higher than the cost. Such cases seem

or services, in order to find out whether the creation of that industry was a wise social policy" (p. 268). I don't object, but it seems to me that, pending the general adoption of such a social philosophy, the public welfare could be greatly enhanced by considering Professor Hotelling's earlier theory as a standard of public-utility regulation.

to call for a limitation in the supply of renewals. In that case, the existing machines will have a scarcity value which might enhance the value above cost. This limitation, however, violates the assumption of static economic conditions, which was made explicitly in the relevant part of my paper.

*Answer:* That is the most elementary case, used for introductory purposes. The general situation, where all scarce ingredients of enterprise contribute toward the goodwill, is reviewed in Part V. As for the underlying assumptions, a mutually satisfactory agreement could perhaps be reached by saying that the Taylor-Hotelling theory is two-dimensional, i.e., limited to a time-value plane. Within that plane, it is correct, when "static economic conditions" are defined as in the answer to comment 1 above. I have added a third dimension, namely the number of co-operating machines. In the circumstances, the two-dimensional analysis should turn out to be a special case of the three-dimensional one. As may be seen from the text, that is indeed the case.

3. The most vulnerable part of this paper is, I think, the third paragraph on page 15. As to its first sentence, I think no one would deny that the capital value of a machine might be greater than its cost new, provided economic conditions are changing sufficiently. As to the last sentence of this paragraph: "Since no income whatever can be had without the machine, the entire value of the enterprise must be imputed to it, to determine the proper date of scrapping," the argument might be applied to declare that each essential part of an enterprise has a value equal to the whole. Many fallacies of this type are cited by Bonbright in his treatise on value.

*Answer:* Taken out of its context, this passage does sound a bit vulnerable, although it is correct enough in the simple introductory case, to which it refers, i.e., where the enterprise consists only of a single machine (which will not be replaced) and an intangible differential advantage. When building up a complex problem gradually, it is sometimes hard to avoid oversimplifications at the start. Moreover, the difficulty is a purely verbal one. Whether or not it is proper to refer to the sum of the machine's cost and the goodwill (arising from the co-operation of all ingredients) as the capital value of the machine, my point is that the goodwill must not be excluded from the formula, from which the economic life of the machine is determined. In the three special cases, however, the goodwill drops out, either by definition, or in the process of solution.

4. In distinguishing between the market price $z$ and the value $w$ of a unit of product or, as I call it in my 1925 paper, "theoretical selling price," you may not have observed that in that paper I treated the case in which the market price of the product of the machine was given by

conditions beyond the control of the owner of the machine. As I pointed out near the beginning of the paper, this is a case of somewhat limited applicability, in which the entrepreneur might find himself "resting precariously on the judgment of his competitors." It seemed to me that a more generally important problem was that connected with cost determination in conjunction with the attempt to minimize cost or maximize profit. It was in connection with this more general case that I used the ideas of unit cost and unit cost plus, which had existed for a long time, but had been applied in a slightly inexact manner to this problem by Taylor.

*Answer:* I recall the passage, but am inclined to hold that the entrepreneur is always resting more or less precariously on the judgments of the consumer, the competitor, and his own. That seems to me the essence of competition. He will always sell above cost, when he can, and temporarily below cost, when he would lose more by closing his plant altogether. But if he does either of these things, the Taylor-Hotelling rule of replacement will no longer be valid, except in the two remaining special cases, both of which assume $z > w$. The attempt to maximize profit is in order, but greater generality will result from applying it to all ingredients of enterprise, not only to demand. The scarcity analysis in terms of the number of co-operating machines is essential for this purpose.

5. In many cases the market price of the product of the machine has no definite meaning because nothing is sold which is the product of one machine alone; the articles sold are the product of many machines under the same ownership, each of which is essential to the finished product. The price of the finished product may well contain an element of monopoly profit or rent in addition to special advantages which can not be assigned unambiguously to any particular physical property. This is the typical situation in industry. In such cases we cannot speak of the value of the product of a machine as determined by external market conditions alone. The "theoretical selling price" used by Taylor and myself becomes a practical tool in connection with cost accounting which should have considerable practical utility under these conditions. Value must be assigned to the service of a machine by the owner of a complicated industrial plant on the basis of the best possible alternative to that service. Under static economic and technological conditions, the best alternative to a machine is typically another machine of the same kind.

*Answer:* My omission of the typical case, where a product must pass through many plant departments, before being finished, is no doubt a major defect. The correction is not difficult, however. The problem consists of prorating the total selling price $Z$ among the plant departments in such a manner that the various economic lives thereby determined shall make the aggregate goodwill a maximum. If $j$ be the serial number of the departments, we have the unknowns $G$, $T_j$, $k_j$, and $z_j$, which can be found from an equal number of simultaneous equations $G=\sum_{j=1}^{\omega} G_j$, $dG/dT_j=0$, $dG/dk_j=0$, $z_\omega=Z-\sum_{j=1}^{\omega-1} z_j$, and $dG/dz_{j-1}=0$.

Many intangibles ranging all the way from a patent to advertising slogans behave just like machines in the sense that they not only entail original costs, but are also subject to expiration, deterioration, growing costs of maintenance, etc., and must therefore be renewed from time to time. For present purposes, they may be considered as so many additional plant departments and treated in the same manner. The goodwill is only that residual intangible, which has none of the attributes of a machine, but is a mere appendage of successful enterprise arising from monopolistic or differential advantages inherent in the ingredients of production. This goodwill can be made a maximum only by allocating it to the various plant departments in the manner outlined. Adequate elaboration must be omitted for the present, but it is nevertheless apparent enough that, if this goodwill were set up as a separate entity by determining the economic lives on the basis of the departmental Taylorean unit costs, its value would drop below the maximum attainable by the correct application of the theory of maxima and minima. Two simple numerical examples for a one-department enterprise were given in Part II. Incidentally, I do not determine the market price by external conditions alone, but assume that it is in some manner inversely related to the number of co-operating machines

6. The whole theory of value and of valuation indeed needs revision In particular, the role of marginal costs needs increased attention; thus instead of writing for the rent of the machine $R(t)=wQ(t)-E(t)$, we might well write $R(q, t)=wQ(q, t)-E(q, t)$, where $q$ is the number of units of output at time $t$, and then observe that $q$ is a function at the disposal of the owner in his attempt to maximize his aggregate profit This is a decidedly more general approach than by variation of $T$ alone

*Answer:* This fruitful lead occurs already in Professor Hotelling's 192 paper and therefore I should certainly have referred to it. As he ther states, this approach is applicable where his tentative postulate of us at full capacity is not even approximately true. A further implicatio is that a reduction of the equipment maintained is not feasible. Th idea has monopolistic aspects, which I did point out very briefly i footnote 17 of the paper cited in note 1 above. For this purpose, have considered $q$ as the unknown ratio of the optimum rate of outpu to the total capacity.

7. On page 28 you state that "the theory of public-utility regula

tion is thus quite definite and leaves no room for equivocation." This is so drastic a statement that it is quite likely to be challenged. My own feeling is that, if the statement is true, it is true only as a part of, or as an addition to current legal theory, but it seems to me that the entire basis of this current theory is extremely shaky and will ultimately have to be discarded in favor of the operation of utility plants in the genuine interest of maximum public service. As I pointed out in ECONOMETRICA, for July, 1938, this means a fundamental and drastic change from current theory and practice.

*Answer:* I agree that my statement should be interpreted only in the light of current legal theory. In fact, I realized that a qualification was in order before this comment reached me and have therefore added footnote 22, which covers the point.

8. You refer to your theory as three-dimensional, with a two-dimensional cross-section corresponding to my work. This is approximately correct but, from another point of view, the number of dimensions in dealing with an infinite chain of replacements might be regarded as infinite. In problems of the continuous variation of the rate of operations also, we have something like a function of an infinite number of variables to maximize. My work on "The Economics of Exhaustible Resources" (*Journal of Political Economy*, April, 1931) is but a special case of this. The appropriate mathematical tool for such cases is the calculus of variations. The case of an infinite chain of replacements has a special mathematical interest because it gives a problem of maximizing a function of an infinite number of discrete variables and thus, in a sense, stands between the calculus of variations and ordinary differential calculus.

*Answer:* I agree, but wish to point out that comments 5 and 6 above add two further dimensions, so that we now have five in all, in my sense of the word. As additional dimensions are introduced, each previous presentation of the problem of economic life becomes a special case of the last one.

9. It appears that we are now in agreement regarding most points. Our differences seem to be largely concerned with different cases, all of which have economic importance within their respective spheres. This whole discussion illustrates the varied possibilities of setting up a mathematical model to represent the infinite complexities of empirical reality. Different theories arise from emphasis on one or another of the many relevant considerations.

*Answer:* It seems to me that our cases differ only with respect to the number of dimensions considered in the analysis and in our respective definitions of "static economic conditions." When these differences are reconciled, all apparent contradictions disappear.

10. I suppose that the only thing that can be said about replacement that will be completely general is that an entrepreneur may be expected to try to maximize the present value of all his future net profits.[23] This general statement leaves the way open for innumerable variations regarding assumptions appropriate to special cases. The important thing in building up a theory will thus be to make perfectly clear the assumptions and definitions in each case.

*Answer:* I agree.

[23] Upon re-reading this appendix, I fear that the term "net profits," as here repeatedly employed, may be misunderstood by those, who are well aware that capital value can not be computed from what accountants call "net profit." Professor Hotelling uses the term in the sense of the Fisherian net income stream, i.e., my formulae (31). This stream includes scrap sales and purchases of new machines, although both items are considered capital transactions in accountancy. On the other hand, it ignores depreciation, although accountants must ordinarily make some such deduction, before arriving at their "net profit." The two concepts are equivalent when the "retirement method" is employed or, regardless of the method of depreciation, in the special case of a mature plant operated under static economic conditions [see formula (30) for $x=0$ and references at end of note 21 above]. Detailed definitions of the numerous "income" concepts in economics and accountancy were given in Chapter II of *op. cit.* in note 16 above. A short review and graphic clarification of the persistent verbal confusion will be found in my recent article on "Economic Theories of Goodwill," *Journal of Accountancy*, September, 1939, p. 175 *et seq.*