Are These the Most Beautiful?

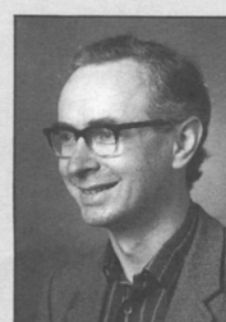
David Wells

In the Fall 1988 *Mathematical Intelligencer* (vol. 10, no. 4) readers were asked to evaluate 24 theorems, on a scale from 0 to 10, for beauty. I received 76 completed questionnaires, including 11 from a preliminary version (plus 10 extra, noted below.)

One person assigned each theorem a score of 0, with the comment, "Maths is a tool. Art has beauty"; that response was excluded from the averages listed below, as was another that awarded very many zeros, four who left many blanks, and two who awarded numerous 10s.

The 24 theorems are listed below, ordered by their average score from the remaining 68 responses.

Rank	Theorem	Average
(1)	$e^{i\pi} = -1$	7.7
(2)	Euler's formula for a polyhedron: $V + F = E + 2$	7.5
(3)	The number of primes is infinite.	7.5
(4)	There are 5 regular polyhedra.	7.0
(5)	$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \pi^2/6.$	7.0
(6)	A continuous mapping of the closed unit disk into itself has a fixed point.	6.8
(7)	There is no rational number whose square is 2.	6.7
(8)	π is transcendental.	6.5
(9)	Every plane map can be coloured with 4 colours.	6.2
(10)	Every prime number of the form $4n + 1$ is the sum of two integral squares in exactly one way.	6.0
(11)	The order of a subgroup divides the order of the group.	5.3
(12)	Any square matrix satisfies its characteristic equation.	5.2
(13)	A regular icosahedron inscribed in a regular octahedron divides the edges in the Golden Ratio.	5.0
(14)	$\frac{1}{2 \times 3 \times 4} - \frac{1}{4 \times 5 \times 6} + \frac{1}{6 \times 7 \times 8} - \ldots = \frac{\pi - 3}{4}.$	4.8
(15)	If the points of the plane are each coloured red, yellow, or blue,	4.7



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	there is a pair of points of the same colour of mutual distance unity.	
(16)	The number of partitions of an integer into odd integers is equal to the number of partitions into distinct integers.	4.7
(17)	Every number greater than 77 is the sum of integers, the sum of whose reciprocals is 1.	4.7
(18)	The number of representations of an odd number as the sum of 4 squares is 8 times the sum of its divisors; of an even number, 24 times the sum of its odd divisors.	4.7
(19)	There is no equilateral triangle whose vertices are plane lattice points.	4.7
(20)	At any party, there is a pair of people who have the same number of friends present.	4.7
(21)	Write down the multiples of root 2, ignoring fractional parts, and underneath write the numbers missing from the first sequence. 1 2 4 5 7 8 9 11 12 3 6 10 13 17 20 23 27 30 The difference is $2n$ in the nth place.	4.2
(22)	The word problem for groups is unsolvable.	4.1
(23)	The maximum area of a quadrilateral with sides a,b,c,d is $[(s-a)(s-b)(s-c)(s-d)]^{1/2}$, where s is half the perimeter.	3.9
(24)	$\frac{5[(1-x^5)(1-x^{10})(1-x^{15})\ldots]^5}{[(1-x)(1-x^2)(1-x^3)(1-x^4)\ldots]^6} = p(4)+p(9)x+p(14)x^2+\ldots,$ where $p(n)$ is the number of partitions of n.	3.9

The following comments are divided into themes. Unattributed quotes are from respondents.

Theme 1: Are Theorems Beautiful?

Tony Gardiner argued that "Theorems aren't usually 'beautiful'. It's the ideas and *proofs* that appeal," and remarked of the theorems he had not scored, "The rest are hard to score—either because they aren't really beautiful, however important, or because the formulation given gets in the way. . . ." Several respondents disliked judging theorems. (How many readers did not reply for such reasons?)

Benno Artmann wrote "for me it is impossible to judge a 'pure fact' "; this is consistent with his interest in Bourbaki and the axiomatic development of structures.

Thomas Drucker: "One does not have to be a Russellian to feel that much of mathematics has to do with deriving consequences from assumptions. As a result, any 'theorem' cannot be isolated from the assumptions under which it is derived."

Gerhard Domanski: "Sometimes I find a problem more beautiful than its solution. I find also beauty in mathematical ideas or constructions, such as the Turing machine, fractals, twistors, and so on. . . . The ordering of a whole field, like the work of Bourbaki . . . is of great beauty to me."

R. P. Lewis writes, '(1) . . . I award 10 points not so much for the equation itself as for Complex Analysis as a whole.' To what extent was the good score for (4) a vote for the beauty of the Platonic solids themselves?

Theme 2: Social Factors

Might some votes have gone to (1), (3), (5), (7), and (8) because they are 'known' to be beautiful? I am suspicious that (1) received so many scores in the 7–10 range. This would surprise me, because I suspect that mathematicians are more independent than most people [13] of others' opinions. (The ten extra forms referred to above came from Eliot Jacobson's students in his number theory course that emphasises the role of beauty. I noted that they gave no zeros at all.)

Theme 3: Changes in Appreciation over Time

There was a notable number of low scores for the high rank theorems. Le Lionnais has one explanation [7]: "Euler's formula $e^{i\pi} = -1$ establishes what appeared in its time to be a fantastic connection between the most important numbers in mathematics . . . It was generally considered 'the most beautiful formula of mathematics' . . . Today the intrinsic reason for this compatibility has become so obvious that the same formula now seems, if not insipid, at least entirely natural." Le Lionnais, unfortunately, does not qualify "now seems" by asking, "to whom?"

How does judgment change with time? Burnside [1], referring to "a group which is . . . abstractly equivalent to that of the permutations of four symbols," wrote, "in the latter form the problem presented would to many minds be almost repulsive in its naked formality . . ."

Earlier [2], perspective projection was, "a process occasionally resorted to by geometers of our own country, but generally esteemed . . . to be a species of 'geometrical trickery', by which, 'our notions of elegance or geometrical purity may be violated. . . .' "

I am sympathetic to Tito Tonietti: "Beauty, even in mathematics, depends upon historical and cultural contexts, and therefore tends to elude numerical interpretation."

Compare the psychological concept of habituation. Can and do mathematicians deliberately undo such effects by placing themselves empathically in the position of the original discoverers?

Gerhard Domanski wrote out the entire questionnaire by hand, explaining, "As I wrote down the theorem I tried to remember the feelings I had when I first heard of it. In this way I gave the scores."

Theme 4: Simplicity and Brevity

No criteria are more often associated with beauty than simplicity and brevity.

M. Gunzler wished (6) had a simpler proof. David Halprin wrote "the beauty that I find in mathematics . . . is more to be found in the clever and/or succinct way it is proven." David Singmaster marked (10) down somewhat, because it does not have a simple proof. I feel that this indicates its depth and mark it up accordingly.

Are there no symphonies or epics in the world of beautiful proofs? Some chess players prefer the elegant simplicity of the endgame, others appreciate the complexity of the middle game. Either way, pleasure is derived from the reduction of complexity to simplicity, but the preferred level of complexity differs from player to player. Are mathematicians similarly varied?

Roger Penrose [10] asked whether an unadorned square grid was beautiful, or was it too simple? He concluded that he preferred his non-periodic tessellations. But the question is a good one. How simple can a beautiful entity be?

Are easy theorems less beautiful? One respondent marked down (11) and (20) for being "too easy," and (22) for being "too difficult." David Gurarie marked down (11) and (1) for being too simple, and another respondent referred to theorems that are true by virtue of the definition of their terms, which could have been a dig at (1).

Theorem (20) is extraordinarily simple but more than a quarter of the respondents scored it 7+.

Theme 5: Surprise

Yannis Haralambous wrote: "a beautiful theorem must be *surprising* and *deep*. It must provide you with a new vision of . . . mathematics," and mentioned the prime number theorem (which was by far the most popular suggestion for theorems that ought to have been included in the quiz).

R. P. Lewis: "(24) is top of my list, because it is surprising, not readily generalizable, and difficult to prove. It is also important." (12+ in the margin!)

Jonathan Watson criticised a lack of novelty, in this sense: "(24), (23), (17) . . . seem to tell us little that is new about the concepts that appear in them."

Penrose [11] qualifies Atiyah's suggestion "that elegance is more or less synonymous with simplicity" by claiming that "one should say that it has to do with *unexpected* simplicity."

Surprise and novelty are expected to provoke emotion, often pleasant, but also often negative. New styles in popular and high culture have a novelty value, albeit temporary. As usual there is a psychological connection. Human beings do not respond to just any stimulus: they do tend to respond to novelty, surprisingness, incongruity, and complexity. But what happens when the novelty wears off?

Surprise is also associated with mystery. Einstein asserted, "The most beautiful thing we can experience is the mysterious. It is the source of all true art and science." But what happens when the mystery is resolved? Is the beauty transformed into another beauty, or may it evaporate?

I included (21) and (17) because they initially mystified and surprised me. At second sight, (17) remains so, and scores quite highly, but (21) is at most pretty. (How do mathematicians tend to distinguish between beautiful and pretty?)

Theme 6: Depth

Look at theorem (24). Oh, come on now, Ladies and Gentlemen! Please! Isn't this difficult, deep, surprising, and simple relative to its subject matter?! What more do you want? It is quoted by Littlewood [8] in his review of Ramanujan's collected works as of "supreme beauty." I wondered what readers would think of it: but I never supposed that it would rank last, with (19), (20), and (21).

R. P. Lewis illustrated the variety of responses when he suggested that among theorems not included I could have chosen "Most of Ramanujan's work," adding, "(21) is pretty, but easy to prove, and not so deep."

Depth seems not so important to respondents, which makes me feel that my interpretation of depth may be idiosyncratic. I was surprised that theorem (8), which is surely deep, ranks below (5), to which Le Lionnais's argument might apply, but (8) has no simple proof. Is simplicity that important?

(18) also scored poorly. Is it no longer deep or difficult? Alan Laverty and Alfredo Octavio suggested that it would be harder and more beautiful if it answered

the same problem for non-zero squares.

Daniel Shanks once asked whether the quadratic reciprocity law is deep, and concluded that it is not, any longer. Can loss of depth have destroyed the beauty of (24)?

Theme 7: Fields of Interest

Robert Anderssen argued that judgements of mathematical beauty "will not be universal, but will depend on the background of the mathematician (algebraist, geometer, analyst, etc.)"

S. Liu, writing from *Physics Review* (a handful of respondents identified themselves as non-pure-mathematicians), admitted "my answers reflect a preference for the algebraic and number-theoretical over the geometrical, topological, and analytical theorems,' and continued: "I love classical Euclidean geometry—a subject which originally attracted me to mathematics. However, within the context of your questionnaire, the purely geometrical theorems pale by comparison."

Should readers have been asked to respond only to those theorems with which they were extremely familiar? (22) is the only item that should not have been included, because so many left it blank. Was it outside the main field of interest of most respondents, and rated down for that reason?

Theme 8: Differences in Form

Two respondents suggested that $e^{i\pi} + 1 = 0$ was (much) superior, combining "the five most important constants." Can a small and "inessential" change in a theorem change its aesthetic value? How would $i^i = e^{-\pi/2}$ have scored?

Two noted that (19) is equivalent to the irrationality of $\sqrt{3}$ and one suggested that (7) and (19) are equivalent. Equivalent or related?

When inversion is applied to a theorem in Euclidean geometry are the new and original theorems automatically perceived as equally beautiful? I feel not, and naturally not if surprise is an aesthetic variable.

Are a theorem and its dual equally beautiful? Douglas Hofstadter suggested that Desargues's theorem (its own dual) might have been included, and would have given a very high score to Morley's theorem on the trisectors of the angles of a triangle. Now, Morley's theorem follows from the trigonometrical identity,

$$1/4 \sin 3\theta = [\sin \theta]\,[\sin (\pi/3 - \theta)]\,[\sin (\pi/3 + \theta)].$$

How come one particular transformation of this identity into triangle terms is thought so beautiful? Is it partly a surprise factor, which the pedestrian identity lacks?

Theme 9: General versus Specific

Hardly touched on by respondents, the question of general vs. specific seems important to me so I shall quote Paul Halmos [5]: "Stein's (harmonic analysis) and Shelah's (set theory) . . . represent what seem to be two diametrically opposite psychological attitudes to mathematics . . . The contrast between them can be described (inaccurately, but perhaps suggestively) by the words special and general. . . . Stein talked about singular integrals . . . [Shelah] said, early on: 'I love mathematics because I love generality,' and he was off and running, classifying structures whose elements were structures of structures of structures."

Freeman Dyson [4] has discussed what he calls "accidental beauty" and associated it with unfashionable mathematics. Roger Sollie, a physicist, admitted, "I tend to favour 'formulas' involving π," and scored (14) almost as high as (5) and (8). Is π, and anything to do with it, coloured by the feeling that π is unique, that there is no other number like it?

Theme 10: Idiosyncratic Responses

Several readers illustrated the breadth of individual responses. Mood was relevant to Alan Laverty: "The scores I gave to [several] would fluctuate according to mood and circumstance. Extreme example: at one point I was considering giving (13) a 10, but I finally decided it just didn't thrill me very much." He gave it a 2.

Shirley Ulrich "could not assign comparative scores to the . . . items considered as one group," so split them into geometric items and numeric items, and scored each group separately.

R. S. D. Thomas wrote: "I feel that negativity [(7), (8), (19) and (22)] makes beauty hard to achieve."

Philosophical orientation came out in the response of Jonathan Watson (software designer, philosophy major, reads *Mathematical Intelligencer* for foundational interest): "I am a constructivist . . . and so lowered the score for (3), although you can also express that theorem constructively." He adds, ". . . the questionnaire indirectly raises foundational issues—one theorem is as true as another, but beauty is a human criterion. And beauty is tied to usefulness."

Conclusion

From a small survey, crude in construction, no positive conclusion is safe. However, I will draw the negative conclusion that the idea that mathematicians largely agree in their aesthetic judgements is at best grossly oversimplified. Sylvester described mathematics as the study of difference in similarity and similarity in difference. He was not characterising only mathematics. Aesthetics has the same complexity, and both perspectives require investigation.

I will comment on some possibilities for further research. Hardy asserted that a beautiful piece of mathematics should display generality, unexpectedness, depth, inevitability, and economy. "Inevitability" is perhaps Hardy's own idiosyncracy: it is not in other analyses I have come across. Should it be?

Such lists, not linked to actual examples, perhaps represent the maximum possible level of agreement, precisely because they are so unspecific. At the level of this questionnaire, the variety of responses suggests that individuals' interpretations of those generalities are quite varied. Are they? How? Why?

Halmos's generality-specificity dimension may be compared to this comment by Saunders Mac Lane [9]: "I adopted a standard position—you must specify the subject of interest, set up the needed axioms, and define the terms of reference. Atiyah much preferred the style of the theoretical physicists. For them, when a new idea comes up, one does not pause to define it, because to do so would be a damaging constraint. Instead they talk around about the idea, develop its various connections, and finally come up with a much more supple and richer notion. . . . However I persisted in the position that as mathematicians we must know whereof we speak. . . . This instance may serve to illustrate the point that there is now no agreement as to how to do mathematics. . . ."

Apart from asking—Was there ever?—such differences in approach will almost certainly affect aesthetic judgements; many other broad differences between mathematicians may have the same effect.

Changes over time seem to be central for the individual and explain how one criterion can contradict another. Surprise and mystery will be strongest at the start. An initial solution may introduce a degree of generality, depth, and simplicity, to be followed by further questions and further solutions, since the richest problems do not reach a final state in their first incarnation. A new point of view raises surprise anew, muddies the apparently clear waters, and suggests greater depth or broader generality. How do aesthetic judgements change and develop, in quantity and quality, during this temporal roller coaster?

Poincaré and von Neumann, among others, have emphasised the role of aesthetic judgement as a heuristic aid in the process of mathematics, though liable to mislead on occasion, like all such assistance. How do individuals' judgements aid them in their work, at every level from preference for geometry over analysis, or whatever, to the most microscopic levels of mathematical thinking?

Mathematical aesthetics shares much with the aesthetics of other subjects and not just the hard sciences. There is no space to discuss a variety of examples, though I will mention the related concepts of isomorphism and metaphor. Here is one view of surprise [6]: "Fine writing, according to Addison, consists of sentiments which are natural, without being obvious. . . . On the other hand, productions which are merely surprising, without being natural, can never give any lasting entertainment to the mind."

How might "natural" be interpreted in mathematical terms? Le Lionnais used the same word. Is it truth that is both natural and beautiful? How about Hardy's "inevitable?" Is not group theory an historically inevitable development, and also natural, in the sense that group structures were there to be detected, sooner or later? Is not the naturalness and beauty of such structures related to depth and the role of abstraction, which provides a ground, as it were, against which the individuality of other less general mathematical entities is highlighted?

Mathematics, I am sure, can only be most deeply understood in the context of all human life. In particular, beauty in mathematics must be incorporated into any adequate epistemology of mathematics. Philosophies of mathematics that ignore beauty will be inherently defective and incapable of effectively interpreting the activities of mathematicians [12].

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