The Aesthetic Viewpoint in Mathematics*

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Compared with representatives of most other disciplines, mathematicians suffer from a serious handicap. Lawyers, linguists, biologists, chemists, physicians—all these people can discuss their professions with uninitiated laymen. Perhaps they cannot fully explain the deeper problems with which they themselves are wrestling, but they can easily give a comprehensible account of what lies on the surface, and their listeners will be interested and grateful.

Not so in mathematics! It really seems to be true that a special sixth sense is needed to understand mathematics. The few who possess this sense fling themselves passionately into the subject; the rest stay as far away from it as possible or consider it a necessary evil. Of course, this isolation gives mathematicians one advantage: unlike other professionals, they are seldom tempted to burden the uninitiated with shop-talk at social gatherings. But mathematicians do not always enjoy their isolation, and I feel it with particular pain today, because I am so eager to give you an idea of the special charms that captivate me in mathematics. I believe my only hope is to avoid any objective discussion of mathematics and just explain my personal feelings about the subject. I do not think it too presumptuous to offer such a personal confession of faith, because even those closely involved with mathematics can adopt seriously different attitudes toward it. I don't believe that my own point of view has been precisely stated anywhere, though I know that a number of my colleagues share it by and large.

Outsiders usually think of mathematics as an especially dry science. Those who know nothing at all about it picture a mathematician as a kind of calculator. I even remember a famous novel—*Friend Death*, by Emil Strauss—which openly presents the view that the essence of mathematics lies in knowing logarithm tables by heart! Those who know mathematics a little better generally see its main value in the irrefutable certainty of its theorems; they believe that the only thing essential for a mathematician is a keen, unerring intellect. But I want to emphasize as strongly as I can that a true mathematician must above all have imagination. I am quite certain that it is precisely the possession of this imagination that distinguishes the future researcher from the merely talented mathematics student.

Of course the mathematician's imagination is a special, "mathematical" one, but in certain circumstances it is related to that of an artist. I realize that Mathematics and Art sound like polar opposites, and I would feel very heretical in juxtaposing them if I could not appeal to several reliable authorities. Kronecker, for instance, one of the most important mathematicians of the generation before last, said it in a Latin couplet:

> Nos mathematici sumus isti veri poetae,
> Sed quod fingimus nos et probare decet.

That is,

> Poets in truth are we in mathematics,
> But our creations also must be proved.

Still more important for me right now are some passages in the book on group theory by the Zurich mathematician Speiser. In one passage he speaks of the buried mathematical treasures that lie in the music of Bach. I do not want to say much, however, about the celebrated connection between mathematics and music. For one thing, I am not well enough educated in music, though I personally believe that mathematics and music theory are quite comparable. Besides, the whole question has really not been adequately clarified. But Speiser goes on to make a noteworthy remark about mathematics and the fine arts in Egypt (I paraphrase):

> If you really want to judge correctly the level of mathematics among the Egyptians, you should not just look at the computations recorded in their arithmetic books or the

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elementary geometry underlying their system of surveying. Analyze instead all the wonderful ornaments with which they covered their temple walls and statues. Only then will you really be able to appreciate what a lofty mathematical spirit lived in those people.

What is this remark supposed to mean? Please picture, as vividly as you can, a large, ornamented surface—perhaps a temple wall, or a wrought-iron door, or just a rich carpet pattern. What lies behind the special charm that you feel in looking at this decorated surface? The basic form of the ornament may be a very simple figure, perhaps a square or a hexagon or a circle with a triangle in it. This figure recurs again and again, in ever new intricacies, but obeying a certain law. Sometimes the law is immediately apparent, sometimes it only reveals itself to the careful observer; to a large extent the artistic charm of the ornamentation depends on it. And this law can be formulated mathematically.

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Perhaps I can make this clear to you with a very simple example. Consider a surface covered with a net of squares; and to keep it from looking too barren, suppose also that a circle is drawn in each square. You then have already a simple but quite usable carpet pattern. We can get this pattern by starting with one circle in one square and repeatedly shifting it horizontally and vertically by a fixed distance (the width of the square). Thus we have a definite system of geometric operations—in our case they are all straight shifts, but in more complicated cases rotations, reflections, and such will also occur. With their help we can produce the whole ornament from a fixed basic figure. This collection of operations represents a mathematical structure, a "group."

Even individual ornaments, not just large patterns, may display mathematical regularities. I am thinking, for instance, of the spiral I saw just this summer on the Viking ship discovered in Oslo. This curve can be obtained by a simple mathematical construction. Fasten one end of a rod where you want the center of the spiral to be, and take a movable point on the rod. Now rotate the rod with fixed speed around the center, and at the same time let the point on the rod move toward the center with a certain decreasing speed. Mathematically, one can roughly call the process the application of a continuous group of transformations. The movable point precisely traces out a spiral; by its construction then the spiral is sent to itself by the group of transformations. This mathematical regularity embodied in the spiral so delighted the Basel mathematician Jakob Bernoulli, who first discovered it, that he had a drawing of the spiral put over his grave and alluded to its regularity in the inscription: EADEM MUTATA RESURGO (Changed, I rise again the same). Personally, I believe that it is precisely the mathematical group behind the spiral that is responsible for its aesthetic value. I may be going too far in this belief, but in any case I have shown you that mathematical regularities can frequently be discovered behind works of art. Thus artists must allow us to say that in some cases they are—unconsciously—mathematicians. This is the meaning of Speiser's remark about the art and mathematics of the Egyptians.

Now, how much do my statements so far support my original assertion? I wanted to show that the imagination and creativity of the mathematician are in some way related to those of the artist. My remarks so far have made it clear that mathematical regularities can lie behind artistic effects, and that in some cases we can identify mathematical accomplishments by artists. But obviously this does not prove the point in question. Someone could well say, "All right, artists sometimes use mathematical ideas. But mathematicians aren't aiming at any kind of artistic effect! When they investigate a group that underlies the construction of some ornament, what do they care about the ornament itself? They just want to explain and elucidate the special properties of the group. The aesthetic effect of the work of art depends on the fact that it does not conceptually analyze the properties of the group; rather it instantaneously presents the viewer with an intuitively comprehensible totality."

But I think anyone who talks like this is basically mistaken about the intentions of the mathematician. Mathematicians are not concerned merely with finding and proving theorems; they also want to arrange and assemble the theorems so that they appear not only correct but evident and compelling. Such a goal, I feel, is aesthetic rather than epistemological. From this point of view the group theorist has a quite different relation to art. A beautiful ornament should indeed set before the viewer an especially striking presentation of the totality of properties of the underlying group. For exactly this reason, a mathematician deeply absorbed in the contemplation of a beautiful ornament may easily be stimulated to an especially elegant investigation of the group involved. Indeed, in some cases a mathematician may find that an appropriate ornament is the most attractive way to present the mathematics with which he is concerned.

To show that these assertions are not empty, we need only look at the work of a man whose name is closely associated with Erlangen: the great mathematician Felix Klein, who died a few years ago. As a new professor here—perhaps standing just where I am now—he presented the "Erlanger Programm" that has since become a classic work in mathematics. In his great investigations of algebra and function theory, he often followed his intuitive genius all too rashly; thus it happened that many of his proofs are flawed and some of his assertions are simply wrong. Yet Klein's presentation always has a particularly captivating charm, and a flawed work by Klein sometimes ranks far above several flawless treatments of the same subject.

What is so alluring in Klein's work? He had a strong power of geometric visualization, and all his investigations were essentially governed by appropriate geometric pictures. For instance, if you look at the drawings in his papers on automorphic functions, you will be astonished by the beauty of these figures, most of which are made up of very simple basic figures like triangles with curved sides. This beauty rests precisely on the fact that these figures illustrate the underlying mathematical relationships in an extremely simple and transparent way. Since Klein builds on these figures, all the results he derives possess that self-evidence which, as we said before, is the goal of mathematical research. Accordingly, Klein's articles will always be worth reading, even for one who knows quite well that they are not always satisfactory in their rigor.

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This example inevitably brings us to a point that we must discuss more thoroughly. I said that, from an aesthetic viewpoint, a flawed work by Klein sometimes ranks above flawless works by other people. On no account, of course, do I want to assert that mathematicians should aim merely at elegant and attractive presentations without regard to whether the assertions they make are true. Such a claim would plainly be sheer nonsense. The couplet by Kronecker, which I mentioned at the beginning of my talk, quite rightly says in its second line

Sed quod fingimus nos et probare decet.

"But our creations also must be proved." If a mathematician has recognized a relationship intuitively, he must obviously use the strictest self-criticism in checking the proofs by which he shows the correctness of his insight. Yet even here, in the question of rigorous proof, there is a serious difference among mathematicians. There are some for whom a clear and compelling—in other words, beautiful—presentation of the results is not just a secondary aspect but a basic requirement. For others, the value of mathematics seems to lie above all in the irrefutable certainty and logical unassailability of its theorems. You see I am now admitting that this latter view of the significance of mathematics, which in my introduction I called the view of the average educated layman, also has its adherents among mathematicians. For instance David Hilbert of Göttingen, the old master of German mathematics, probably holds this view to some extent. Far be it from me to deny the justifiability of such a viewpoint. I just want to make it clear to you here that I think differently, and to show how the basic viewpoint people adopt toward mathematics can influence the direction of research.

A mathematician who is concerned above all with the irrefutable certainty of his results will try to base his theorems on as few unproved assumptions as possible. Consequently he will only feel secure in geometry, for instance, when he has completely reduced that subject to arithmetic, which mathematicians generally consider the most reliable foundation. He will further subject the foundations of arithmetic to a sharp scrutiny; he will for instance investigate to what extent the set of all infinite decimals can be considered a logically faultless concept; he will try to make do with whole numbers as far as possible; he will even try to reduce the system of whole numbers to something even simpler, perhaps a system of logic. In short, he will devote himself to what people in mathematics nowadays call the study of foundations.

The more aesthetically oriented mathematician will have less interest in the study of foundations, with its painstaking and often necessarily complicated and unattractive investigations. He will of course unfailingly fit his proofs to the rigor of his time, but he will not rack his brains about whether his theorems are proved in a way that will necessarily be considered absolutely flawless under all conditions for all eternity. The student of foundations must be very suspicious when he sees that the proofs of the great mathematicians of the 17th and 18th centuries often no longer seem valid to us today. The aesthetically oriented mathematician takes comfort precisely in the thought that Leibniz and Euler have lost none of their mathematical greatness for us, even though we clearly see that these masters often proved their discoveries in a way that we today can no longer accept as faultless.

However, the mathematical aesthete often meets painful difficulties not only in the discovery of new theorems but also in satisfactory presentation of the results. He cannot allow the reader of his works to be (so to speak) overwhelmed by the weight of the proofs, forced to concede that the stated results are right but unable to escape the feeling that they might equally well have been wrong. He must always avoid what Schopenhauer criticized as "mousetrap-proofs." On the contrary, he must arrange his material so that each theorem necessarily leads to the next, so that the reader, even before he has checked through the proofs in detail, can see at a glance (so to speak) that the results could not possibly have been different. An aesthetically sensitive researcher will therefore in some cases prove a theorem not just once but many times; in fact there are famous mathematical theorems for which more than ten or twenty proofs exist. I believe this state of affairs shows beyond doubt that aesthetic viewpoints play a large role in mathematics.

I have not yet discussed whether these aesthetic aspects can seriously influence the further development of mathematics, the derivation of new propositions and the creation of new theories. I want to give you an example to show that such possibilities do indeed occur. At the same time I can show you the mathematician's special *mathematical* sense of beauty—a sense that is probably the special sixth sense, which I said at the beginning was necessary for understanding mathematics. I started my observations with geometry, but in this example we turn instead to elementary number theory.

You doubtless know that every natural number can be factored into its simplest components, the "prime" numbers, which themselves cannot be split further into factors. (For instance, $21 = 3 \cdot 7$ and $60 = 2 \cdot 2 \cdot 3 \cdot 5$.) You probably also know that this factorization can be done in only one way, so that for instance there does not exist any other factorization of 21 in which (say) the factor 5 would appear. Most of you are probably quite unmoved by this fact. But a mathematician thinks it is beautiful that the natural numbers can be factored so simply, and he is happy to find similar regularities in other domains. This is the case, for instance, in the domain of polynomials—the expressions formed from letters that plagued you all in junior or senior high school. For instance, the factorization $a^2 - b^2 = (a + b)(a - b)$ is a unique splitting of the expression $a^2 - b^2$ into the irreducible factors $(a + b)$ and $(a - b)$.

In other areas where it would be very nice, a mathematician no longer finds such factorizations. In the domain of all infinite decimals, for instance, he sees at once that it would be senseless to expect such a factorization. Hence he will search still more eagerly for subdomains of that large set of decimals in which he can again hope for such simple and beautiful factorization laws. The most promising such subdomains are systems formed from what are called "algebraic" numbers. I cannot explain here in any detail what algebraic numbers are; let me just tell you that for instance the square roots and cube roots whose "extraction" you probably toiled over in school are among them. With such algebraic numbers one can in fact find similar factorizations, except that (to begin with) they don't come out so nicely unique as with the ordinary natural numbers. The difficulties that occur here were first noticed in the first half of the last century.

Now comes the decisive point. Kummer, the man who discovered these difficulties, was not satisfied just to note with resignation that they existed. He held to the conviction that the algebraic numbers had to have a structure just as simple and beautiful as that of the ordinary whole numbers; they only needed to be approached in the right way. It was for him an aesthetic ideal, so to speak, that there had to be some sort of simplest components into which algebraic numbers could be uniquely decomposed. He did not rest until he was indeed able to get such a factorization, at least in the simplest cases, by introducing ideal factors alongside the numbers actually given. His work was then extended and completed by the great mathematician Dedekind, who lived until 1914 and was the creator of general ideal theory.

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The appearance of the word "ideal" in mathematics has amused many of my nonmathematical friends when I've mentioned it to them. How could "ideals" be the subject of mathematical analysis? But you see it is not an accident that mathematicians speak of ideals in number theory. These "ideals" and "ideal numbers" of course are not ethical ideals, but they owe their introduction to an aesthetic ideal. Naturally, I do not know whether Kummer and Dedekind had exactly this idea in mind when they coined the name "ideal." But in any case the word looks particularly appropriate from our aesthetic point of view, whereas otherwise it would stand out strangely from other mathematical terms.

There is another fact that shows how important the aesthetic point of view was for Dedekind. For years after he had discovered the main theorem of general ideal theory, he did not publish it, even though he had a logically faultless proof. This proof did not seem to him transparent enough; it did not satisfy his aesthetic demands, and so he held back from publication. Thus he was concerned not only with finding theorems but equally with finding the right presentation of them, a presentation satisfying the mathematical sense of beauty. Here I should emphasize again, as strongly as I can, that mathematics is indebted to Kummer and especially to Dedekind for enormous advances in content as well as form. Precisely because he placed such a high value on elegant presentation in perfect form, Dedekind created methods of such flexibility that they can be applied in many different areas of mathematics. Even today they provide the basis for continuing progress. I myself must name Dedekind as the teacher who had by far the greatest influence on me, even though I knew him only through his works.

Thus the aesthetic attitude in no way condemns mathematicians who adopt it to unproductiveness. On the contrary, it spurs them on to grasp new and unexplored areas, to create out of seeming chaos a harmonious order satisfying the sense of mathematical beauty. I can certainly tell you that for me personally it is mainly this urge that draws me on to further mathematical work. I see a seemingly confused picture, and I sense behind it a hidden order that will suddenly reveal itself if only I can find the magic word that will compel the individual pieces to fit themselves together into a harmonious whole. Then I simply cannot stop playing around with the material, hoping that one day enlightenment will come to me.

Finally, I think it is important in general to emphasize that aesthetic points of view play a large role for mathematicians. You may be surprised to learn that there are movements in mathematics just as in art, movements that attack each other and denounce the works of the opposing party, pronouncing them not false, to be sure, but uninteresting and worthless. At the moment, for instance, mathematicians are split into two great camps, the troops of the "concrete" and the troops of the "abstract," and the "abstract" often come under sharp attack from the "concrete."

What is this great opposition between "concrete" and "abstract"? Thinking of the words in their ordinary senses, you might suppose that "concrete" mathematics is concerned with investigations that have immediate practical application, while "abstract" mathematics is more devoted to intellectual games. The people who use "concrete" as praise and "abstract" as blame probably want you to understand the words in this way. But if you look more carefully, you will often see that purely abstract investigations sooner or later yield results useful in practice, and even more often you will observe that the problems studied in concrete mathematics are significant only from a mathematical viewpoint, not in practice. You might then think that this much-emphasized antithesis between concrete and abstract was altogether meaningless. This would not be right either; there definitely are profound differences—but they are differences in taste. And the best proof that we are dealing with an aesthetic question is that I can most easily explain these different tastes by using an analogy from architecture.

Some people like heavily ornamented buildings; they hate bare surfaces and place great importance on having the largest possible number of beautiful details to admire. Others, the adherents of modern objectivity, are concerned above all with the great "line," and do their best to renounce ornamental accessories. If we transferred the mathematical terms to art, we would call the first taste concrete and the second abstract. Incidentally, my previous remarks have shown you that I am among the abstract mathematicians. All along I have emphasized simplicity, clarity, and great "line" as characteristics of mathematical beauty. If I had a more "concrete" taste, I would instead have spoken of diversity, variegation, and the like.

It seems to me very important to realize that the opposition between abstract and concrete mathematics simply comes down to a difference of taste. As the proverb says, there's no disputing over tastes. Instead of pronouncing the work of the other side worthless, a mathematician should content himself with stating that the problems of the other party seem unattractive to him, not denying that these same problems may perhaps look very beautiful when seen from a different aesthetic point of view.

But I must not stray from my topic, and I must remember that I am not addressing a peace conference for mathematicians. My last remarks, which seem so comforting to me, may have terrified you. Some of you may be saying to yourselves, "Until now we always thought that the ultimate goal of mathematics was its application to practical problems. Now we see that some, perhaps very many mathematicians have a quite different idea. For them the major role is played by so-called aesthetic considerations. These may indeed be extremely attractive for a narrow circle of initiates, but they will never mean anything at all to us laymen. Such mathematics right from the start counts only on the understanding of a small circle. Is it worth anything at all?"

And so you see we have come back to where we started: how painfully we mathematicians sense our isolation. The more we ourselves are enraptured by the beauties of mathematics, the more we regret that we can bring so few people to share our pleasure. But at least those of us in the school of abstract mathematics have one consolation: as we make our presentations clearer and more transparent, they automatically become easier to understand. Bear in mind that 400 years ago, arithmetic was a difficult art! So great an educator as Melanchthon did not trust the average student to penetrate into the secrets of fractions. Yet now every child in elementary school must master them. Perhaps eventually the beauties of higher arithmetic, of which I have tried to give you some indication in this talk, will be accessible to every educated person. Doubtless this idea seems Utopian to you, and perhaps it is. But it consoles me for the painful recognition that many of my best friends now have no real idea of the charm that has drawn me wholeheartedly to mathematics.