Heaviside's Operational Calculus and the Attempts to Rigorise It

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Abstract

At the end of the 19th century OLIVER HEAVISIDE developed a formal calculus of differential operators in order to solve various physical problems. The pure mathematicians of his time would not deal with this unrigorous theory, but in the 20th century several attempts were made to rigorise HEAVISIDE's operational calculus. These attempts can be grouped in two classes. The one leading to an explanation of the operational calculus in terms of integral transformations (BROMWICH, CARSON, VAN DER POL, DOETSCH) and the other leading to an abstract algebraic formulation (LÉVY, MIKUSIŃSKI). Also SCHWARTZ's creation of the theory of distributions was very much inspired by problems in the operational calculus.

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Introduction

HEAVISIDE is said to be the inventor of the operational calculus in most of the literature from the early twentieth century dealing with this calculus. Today we know that this view is wrong, but it reflects the central role that HEAVISIDE has played in the history of this branch of mathematics. His work became the starting point of the development of the operational calculus in this century, his predecessors apparently being for a period totally forgotten. Therefore this study will start with HEAVISIDE after a very brief account of the previous history; for a more detailed discussion see PINCHERLE [1904-16], KOPPELMAN [1971-72], COOPER [1952].

G. W. LEIBNIZ' differential notation made it possible to consider the differential operator as an algebraic quantity independent of the function operated upon. Several mathematicians, among them J. L. LAGRANGE, P. S. LAPLACE, L. ARBOGAST and A. L. CAUCHY, employed this idea, so fundamental for the operational calculus.

An explanation of the success of the algebraic treatment of the differential operators was sought for in other fields of mathematics. LAPLACE [1812] for example explained the operational methods by means of the LAPLACE transformation, whereas CAUCHY [1827] used FOURIER's theorem. This knowledge of the relations between operational calculus and integral transformation was forgotten later in the 19^{th} century but, as we shall see, it was rediscovered in the 20^{th}.

SERVOIS [1814] thought that the reason why algebraic treatment was applicable to differential operators was that the latter obeyed the commutative and distributive laws. Together with the Leibnizian notation this idea was taken over by the English mathematicians and was employed and developed by, among others, D. GREGORY and G. BOOLE. HEAVISIDE knew about the abstract algebraic approach to operational calculus from BOOLE's book *Treatise on Differential Equations* (1859), but he created his own personal version of this discipline.

I. Oliver Heaviside (1850-1925)

1. HEAVISIDE's main contribution to science is his development and reformulation of MAXWELL's electrodynamics, and it was in this context that his mathematical ideas arose. He considered mathematics as an experimental science on a par with physics* and his mathematical reasoning was often guided by physical intuition:

> we shall have, preliminarily, to work by instinct, not by rigorous rules. We have to find out first how things go in the mathematics as well as in the physics. When we have learnt the go of it we may be able to see our way to an understanding of the meaning of the processes... [*EMT* § 239]

This view was incompatible with that of the established mathematicians who, towards the end of the 19^{th} century, spent much effort on rigorising mathematics. The "Cambridge mathematicians" were so indignant at HEAVISIDE's unrigorous use of divergent series that they stopped the publication of a sequence of his papers [see COOPER 1952]. Nevertheless HEAVISIDE continued to use his experimental mathematics:

> Shall I refuse my dinner because I do not fully understand the process of digestion? No, not if I am satisfied with the result. [*EMT* § 225]

* *E.g. EMT* II § 223-226. In the following I shall use the abbreviation *EMT* for *Electromagnetic Theory* and *EP* for *Electrical Papers*.

In his scientific papers one often finds harsh attacks on the "Cambridge mathematicians"; more contemptuous, however, are his remarks about the electrical engineers who found MAXWELL's and his own work too mathematical [*EP* vol. II, preface].

HEAVISIDE's style is very colorful, polemic and capricious:

> How is it possible to be a natural philosopher when a Salvation Army band is performing outside; joyously, it may be, but not most melodiously? [*EMT* I § 4]

This "wandering about guided by circumstances" [*EMT* § 223] makes HEAVISIDE very enjoyable to read, but at the same time it makes it most difficult to get a comprehensive view of his production. HEAVISIDE himself collected his papers in two works: *Electrical Papers* [*EP*] containing the papers from 1873 to 1891 and *Electromagnetic Theory* [*EMT*] (vol. I contains papers from 1891–93; vol. II, 1894–1898; vol. III, 1900–1912).

HEAVISIDE's contributions to mathematics are in two fields, namely vector analysis and operational calculus. The importance which has been attached to his work in the second field can be seen from E. T. WHITTAKER's memorial paper "Oliver Heaviside" (1928):

> We should now place the Operational Calculus with Poincaré's discovery of automorphic functions and Ricci's discovery of the Tensor Calculus as the three most important mathematical advances of the last quarter of the nineteenth century. Applications, extensions and justifications of it constitute a considerable part of the mathematical activity of today.*

In the next sections I shall give an outline of HEAVISIDE's final version of his operational calculus. The account will follow HEAVISIDE's didactic device by introducing the mathematics in a physical context. I shall refer and comment upon three physical examples taken from *EMT* vol. 2; thereafter I shall try to give a systematic view of HEAVISIDE's mathematical method. Section (I.7) will contain a brief sketch of the *progress* of the operational method in HEAVISIDE's writings.

2. Example a. In the treatment of electric networks HEAVISIDE employed his method of "resistance operators". He defined the resistance operator of an electric system to be an operator Z transforming the current C into the voltage e:

$$e = ZC \quad \text{or} \quad C = \frac{1}{Z} e.$$

The resistance operator of a pure resistance R $\left(\text{conductance } \frac{1}{K}\right)$ is the multiplication operator R $\left(\frac{1}{K}\right)$ whereas a coil with self-inductance L has the resistance operator $L\frac{d}{dt}$, and a condenser with permittance S has $Z = \frac{1}{S}\int_0^t \cdot\, dt$. HEAVISIDE

* As pointed out in the introduction, the operational calculus was not discovered in the last quarter of the 19th century, but this was the general opinion in 1928.

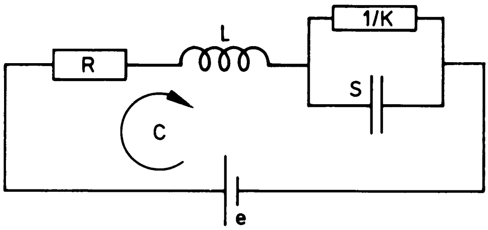


Fig. 1

used the abbreviations $p = \frac{d}{dt}$, $p^{-1} = \int_0^t \cdot \, dt$ which allowed him to denote the two last mentioned operators Lp and $\frac{1}{S} p^{-1}$.

In *EMT* §285 HEAVISIDE considered a coil (L), with an inner resistance (R), and a leaky condenser in sequence influenced by an electro-motive force e (Fig. 1). He wished to determine C in terms of e. He found the resistance operator of the connected system of Fig. 1 by applying the ordinary rules for connecting resistances to the resistance operators. Thus he found the resistance operator Z' of the leaky condenser, represented by a condenser $\frac{1}{Sp}$ and a resistance $\frac{1}{K}$ in parallel, from

$$\frac{1}{Z'} = K + Sp,$$

yielding

$$Z' = \frac{1}{K + Sp}.$$

From this he found the total resistance operator by addition:

$$Z = R + Lp + (K + Sp)^{-1}$$

and hence

$$C = \frac{e}{R + Lp + (K + Sp)^{-1}}. \tag{I.1}$$

HEAVISIDE called this expression for C containing the operator p the operational solution of the problem. His aim was to transform it into a real solution, *i.e.* a function of t. For the process of transformation he invented the new term *algebrize* which in the following will be used in that special sense.

When e is a harmonic oscillation, $e = \sin nt$, HEAVISIDE algebrized (1) as follows (in the special case where $K = S = 0$):

> As regards the simply periodic solutions in these cases, the working is perfectly simple, by means of the property $p^2 = -n^2$ which obtains in simply periodic states, or, which is the same, $p = ni$, applied to reduce the resistance or conductance operator to the standard form $a + bp \ldots$ when e is given …

we have

$$C=\frac{e}{R+Lp}=\frac{(R-Lp)e}{R^2+L^2n^2}$$

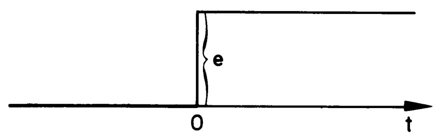
[*EMT* §284]

i.e. for $e=\sin nt$

$$C=\frac{1}{R^2+L^2n^2}(R\sin nt-Ln\cos nt). \tag{I.2}$$

This algebrizing procedure is similar to an AC circuit technique which was relatively often used in HEAVISIDE's time.

To algebrize the expression (1) when e is a constantly impressed force at



$t=0$, HEAVISIDE used another technique (again $K=S=0$):

We have

$$\frac{e}{R+Lp}=\frac{e}{Lp\left(1+\frac{R}{Lp}\right)}=\frac{1}{R}\left\{\frac{R}{Lp}-\left(\frac{R}{Lp}\right)^2+\left(\frac{R}{Lp}\right)^3-\dots\right\}e. \tag{I.3}$$

This is got by expanding the fraction by division. The rest is done by

$$p^{-n}1=\frac{t^n}{\lfloor\underline{n}}$$

which makes, applied to (3),

$$C=\frac{e}{R}\left\{\frac{Rt}{L}-\frac{1}{\lfloor\underline{2}}\left(\frac{Rt}{L}\right)^2+\frac{1}{\lfloor\underline{3}}\left(\frac{Rt}{L}\right)^3-\dots\right\}.$$

[*EMT* §283]

Hence we have

$$C=\frac{e}{R}\left(1-\exp\left(-\frac{Rt}{L}\right)\right). \tag{I.4}$$

It should be remarked that HEAVISIDE used the notation 1 for the HEAVISIDE function $H(t)$ and $\lfloor\underline{n}$ for $n!$

He also algebrized the operational solution by expanding it in ascending powers of p:

$$C=\frac{e}{R+Lp}=\frac{1}{R}\left(1-\frac{Lp}{R}+\left(\frac{Lp}{R}\right)^2-\dots\right)e.$$

As e is constant for $t>0$, HEAVISIDE assumed $p^n e$ to be zero (see (I.§8)); thus

$$C=\frac{e}{R}, \tag{I.5}$$

He noticed that (5) was the limit of (4) as t tended to infinity. The connection between the seemingly contradictory results (4) and (5) will be illustrated more clearly in the next example.

3. Example b. Most of HEAVISIDE's work is devoted to the treatment of continuous systems with which he became acquainted when he worked as an operator for The Great Northern Telegraph Company. For instance in *EMT* §§ 238–42 he considers a semi-infinite cable and a network with resistance operator Z in sequence, operated upon by an e.m.f. $e=H(t)$ (Fig. 2). Here HEAVISIDE neglects self induction in the cable and therefore finds that the potential $V(x,t)$ and the current $C(x,t)$ are connected by the equations

$$-\frac{dC}{dx}=SpV, \qquad -\frac{dV}{dx}=RC, \tag{I.6}$$

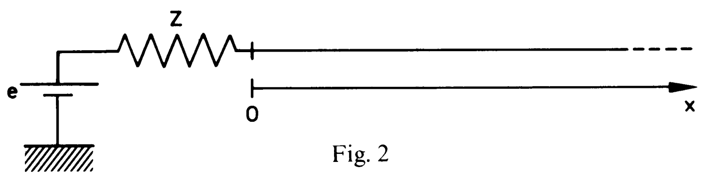


Fig. 2

S being the permittance and R the resistance per unit length. Eliminating C, he gets

$$\frac{d^2V}{dx^2}=RSpV=q^2V \tag{I.7}$$

where q is defined by

$$q^2=RSp. \tag{I.8}$$

Treating q as a constant, he obtains the operational solution of (7)

$$V(x,t)=Ae^{qx}+Be^{-qx}, \tag{I.9}$$

A and B being arbitrary functions of t. They are determined from the boundary conditions at $x=0$ and $x=\infty$ yielding

$$V(x,t)=V_0e^{-qx}, \tag{I.10}$$

where V_0 is the impressed e.m.f. at the end $(x=0)$.

From (10) and (6) HEAVISIDE gets

$$C=\frac{q}{R}e^{-qx}V_0=C_0e^{-qx}.$$

C_0 expresses the current at the end of the cable. By applying (8) to $V_0=\frac{R}{q}C_0$, he finds that

$$V_0=\left(\frac{R}{Sp}\right)^{\frac{1}{2}}C_0 \tag{I.11}$$

and concludes that the resistance operator of the cable is $\left(\frac{R}{Sp}\right)^{\frac{1}{2}}$.

So if Z is put between the cable and the earth with the impressed voltage acting, we have

$$C_0 = \frac{e}{Z + \left(\frac{R}{Sp}\right)^{\frac{1}{2}}} \tag{I.12}$$

to express the current through Z and entering the cable. This is because the operators are additive like resistances. Also we have $V_0 = \left(\frac{R}{Sp}\right)^{\frac{1}{2}} C_0$ as before; consequently by (12)

$$V_0 = \frac{e}{1 + Z\left(\frac{Sp}{R}\right)^{\frac{1}{2}}}. \tag{I.13}$$

This finds V_0, the potential at the beginning of the cable, in terms of e [*EMT* § 242].

HEAVISIDE supposes that Z is a pure resistance r and that e is a constant impressed force at $t=0$ (*i.e.*, $e = eH(t)$). He then algebrizes (13) by expanding in ascending powers of p:

$$V_0 = \left\{1 - r\left(\frac{Sp}{R}\right)^{\frac{1}{2}} + r^2 \frac{Sp}{R} - r^3 \left(\frac{Sp}{R}\right)^{\frac{3}{2}} + \dots\right\} e. \tag{I.14}$$

As in the previous examples he puts $p^n e = 0$ when n is a natural number and obtains

$$V_0 = \left\{1 - r\left(1 + \frac{r^2 Sp}{R} + \frac{r^4 S^2 p^2}{R^2} + \dots\right)\left(\frac{Sp}{R}\right)^{\frac{1}{2}}\right\} e. \tag{I.15}$$

HEAVISIDE was then faced with the problem of fractional differentiation. He treated it experimentally, proving that

$$p^{\frac{1}{2}} H(t) = (\pi t)^{-\frac{1}{2}} \qquad (EMT \text{ chap. } 7). \tag{I.16}$$

This formula was known to LACROIX (1819), but HEAVISIDE apparently made its discovery independently.*

From (15) and (16) it follows that

$$V_0 = e - er\left(\frac{S}{R\pi T}\right)^{\frac{1}{2}} \left\{1 - \frac{r^2 S}{2Rt} + 1 \cdot 3\left(\frac{r^2 S}{2Rt}\right)^2 - \dots\right\}. \tag{I.17}$$

HEAVISIDE expressed in his own way the fact that (17) gave an asymptotic series for $V(t)$ for $t \to \infty$, without using the term asymptotic series. He never employed any of the theories about divergent series which were introduced at the end of the 19^{th} century, but he proceeded quite formally. This caused him a great deal of trouble with the "Cambridge mathematicians".

He commented upon (17) in the following way:

* For the history of fractional differentiation see ROSS 1977.

> (17) is unsuitable when t is small enough to make the initial convergency be insufficient. It is said that every bane has its antidote, and some amateur botanists have declared that the antidote is to be found near the bane. We have an example here. The antidote is got by algebrizing (13) in a different way. [*EMT* § 242]

HEAVISIDE then expanded the expression in (13) in descending powers of p

$$V_0 = \begin{cases} \left(\frac{R}{r^2 Sp} + \left(\frac{R}{r^2 Sp}\right)^2 + \dots\right)\left(\frac{r^2 Sp}{R}\right)^{\frac{1}{2}} e \\ -\left(\frac{R}{r^2 Sp} + \left(\frac{R}{r^2 Sp}\right)^2 + \dots\right) e, \end{cases}$$

and applying (16) he obtained

$$V_0 = 2e\left(\frac{Rt}{r^2 S\pi}\right)^{\frac{1}{2}}\left\{1 + \frac{2Rt}{3r^2 S} + \frac{1}{3\cdot 5}\left(\frac{2Rt}{r^2 S}\right)^2 + \dots\right\} - e\left(\exp\frac{Rt}{r^2 S} - 1\right). \quad \text{(I.18)}$$

> We see now that we can calculate V_0 conveniently when t is small. But (18) is bad when t is big. Then we may consider (18) the bane and (17) the antidote. They are complementary, though not mutually destructive.

Returning to Example a, we see that (5) is the asymptotic expansion of (4) for $t \to \infty$ so that the connection between (4) and (5) is also covered by the above remarks.

4. Example c. For a finite cable of length l operated upon by an e.m.f. $e = H(t)$ at one end A and isolated at the other end B HEAVISIDE found the

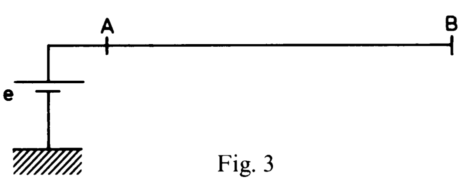


Fig. 3

operational expression for the voltage at any point of the cable to be

$$V(x,t) = \frac{\cosh q(l-x)}{\cosh ql} 1 = \frac{\cos s(l-x)}{\cos sl} H(t) \quad \text{(I.19)}$$

where

$$s^2 = -q^2 = -RSp.$$

He obtained this result from (10) by adding up all the reflected waves from both ends of the cable. (A more direct mathematical way to obtain (19) would be to determine A and B in (9) from the boundary conditions.) In order to algebrize solution (19) HEAVISIDE used his expansion theorem:

> The method may be briefly (though imperfectly) stated as follows: Let $e = ZC$ be the operational solution of an electromagnetic problem; say, for

definiteness, that C is the current at a certain place due to an impressed force, e, at the same or some other place. Let the form of Z be such as to indicate the existence of normal solutions for C. Then, when e is steady, beginning at the moment $t=0$, the C due to e is expressed by

$$C=\frac{e}{Z_0}+e\sum\frac{\varepsilon^{pt}}{p\frac{dZ}{dp}}\,{}^{\star} \qquad \text{(I.20)}$$

to be understood thus: –

In the first place, the Z in the operational solution is an operator, a function of p the time differentiator. But in equation (20) Z is entirely algebraical. Thus, Z_0 is the algebraical function obtained by putting $p=0$ in Z. It is the effective steady resistance to e when, as supposed, e is a voltage, and it is at the place of C. Otherwise it is more general. Then, in the summation, $\frac{dZ}{dp}$ is the ordinary differential coefficient of Z with respect to p as a quantity. Lastly, the summation ranges over all the roots of the algebraical equation $Z=0$, which is, in this respect, the determinantal equation, though Z itself is much more. These special values of p are to be used in $\frac{dZ}{dp}$ as well as explicitly. [*EMT* § 282]

If we apply the operator $Y(p)$ on both sides of (20), we find that the operational expression

$$C=\frac{Y}{Z}1=\frac{Y}{Z}H(t)$$

is algebrized by

$$C=\frac{Y(0)}{Z(0)}+\sum_{Z(\lambda)=0}\frac{Y(\lambda)\,e^{\lambda t}}{\lambda\left.\frac{dZ}{dp}\right|_{p=\lambda}}. \qquad [EMT\ \S 285] \qquad \text{(I.21)}$$

HEAVISIDE employed this extension of the expansion formula to algebrize (19): $V=\frac{\cos s(l-x)}{\cos sl}H(t)=\frac{Y(p)}{Z(p)}H(t)$ [*EMT* § 288]. He found the denominators in the sum (21) to be

$$p\frac{dZ}{dp}=p\frac{d}{dp}\cos sl=p\frac{ds}{dp}\frac{d}{ds}\cos sl=-\frac{1}{2}sl\sin sl$$

so that from (21)

$$V=1-\Sigma\frac{\cos s(l-x)}{\frac{1}{2}sl\sin sl}e^{pt}=1-2\Sigma\frac{\sin sx}{sl}e^{pt} \qquad \text{(I.22)}$$

where p is determined from $RSp=-s^2$ and the summation ranges over the positive roots of $\cos sl=0$, *i.e.* $s=\frac{1}{2}\frac{\pi}{l},\frac{3}{2}\frac{\pi}{l},\dots$ (the negative roots would only lead to repetition).

★ $\varepsilon^{pt}=\exp pt$

The expansion theorem is HEAVISIDE's most important tool in algebrizing procedures. For instance he also used it to give an alternative algebrization of (1).

In *Electromagnetic Theory* HEAVISIDE gave no proof of the expansion theorem, but in *Electrical Papers* he presented two different proofs. The first proof, which led to the discovery of the theorem, is a beautiful example of the experimental mathematics that lay so near to HEAVISIDE's heart. This proof, however, I shall save for an appendix as it does not throw any light on the operational calculus. On the other hand, the second proof is an operational proof, the final version of which can be found in a footnote in *EP* II, p. 226.

Developing $\frac{Y(p)}{Z(p)}$ into partial fractions, HEAVISIDE gets

$$\frac{Y(p)}{Z(p)} H(t) = \sum_{Z(\lambda)=0} \frac{Y(\lambda)}{(p-\lambda)\left.\frac{dZ}{dp}\right|_{p=\lambda}} H(t). \tag{I.23}$$

He then inserts

$$\frac{1}{p-\lambda} H(t) = -\frac{1}{\lambda}(1-e^{\lambda t}),$$

which can be obtained from (3) and (4), in (23), yielding

$$\frac{Y(p)}{Z(p)} H(t) = -\sum_{Z(\lambda)=0} \frac{Y(\lambda)}{\lambda \left.\frac{dZ}{dp}\right|_{p=\lambda}} + \frac{Y(\lambda)}{\lambda \left.\frac{dZ}{dp}\right|_{p=\lambda}} e^{\lambda t}. \tag{I.24}$$

Putting p equal to 0 in (23), he saw that the first term of (24) was equal to $\frac{Y(0)}{Z(0)}$; hence (21) is achieved.

The above deduction of (21) evidently presupposes that $Z(p)$ is a polynomial of higher order than $Y(p)$ and that the former has no multiple or zero roots. HEAVISIDE knew very well that (21) was not universally correct, but he never specified the necessary conditions for its application.

> Now it would be useless to attempt to state a formal enunciation to meet all circumstances. Even supposing that an absolutely perfect knowledge of the subject made it possible to do so, it would be very unpractical. It would be worse – far worse – than that very lengthy enunciation of a theorem in the 5th Book of Euclid, which may be read and re-read fifty times without properly grasping its meaning, which is not much, after all; only something in compound proportion that the modern schoolboy does in a minute or two. It is better to learn the nature and application of the expansion theorem by actual experience and practice. A theorem which has so wide an application is a subject for a treatise rather than a proposition. [*EMT* II § 282]

5. The methods seen in the three examples are typical of HEAVISIDE's way of solving physical problems, which can briefly be described in the following diagram:

Step 1) Formulation of differential equation
Step 2) Operational solution
Step 3) Algebrizing
- 3.1) Harmonic impressed force
- 3.2) Impressed force $= H(t)$
  - 3.2.1) Expansion in descending powers of p
  - 3.2.2) Expansion in ascending powers of p
  - 3.2.3) Expansion formula.

Before I comment on the methods contained in the diagram, I would like to emphasize that this schematic arrangement does not claim to cover all of HEAVISIDE's work but only the very essential part related to the operational calculus.

Step 1. The three physical examples we have seen all deal with electrical systems, the most common problem HEAVISIDE's work. However, he also treated other topics in an operational way, for example the problem of the age of the earth [*EMT* § 223–237]. He mostly worked with physical systems influenced by one external force. He supposed that this force described the system completely; this implies that in cases where he did not explicitly state the initial values he implicitly assumed them.

The mathematical models corresponding to his problems could be either ordinary or partial differential equations. Typically, discrete finite networks would lead to ordinary differential equations, whereas continuous systems would give partial differential equations.

Step 2. The substitution $\frac{d}{dt} \curvearrowright p$ transformed the ordinary differential equations into algebraic equations, and the partial differential equations into differential equations in the space variables only. Treating p as an algebraic quantity, HEAVISIDE solved these equations and obtained the operational or symbolic solution to the problem. In this step HEAVISIDE proceeded as if the calculations took place in a field. Still he remarked [*EMT* § 251] that the commutative law was not universally valid for the differential and integral operators, but he made no attempt to find the limits of its validity. As Example a shows, HEAVISIDE often mixed Steps 1 and 2.

Step 3. HEAVISIDE algebrized the operational solutions

$$C = Z(p)\, F(t),$$

i.e. he converted them into ordinary functions of t, for two types of impressed forces $F(t)$:

$$3.1)\ F(t) = \sin nt \quad \text{and} \quad 3.2)\ F(t) = H(t).$$

3.1. In this case the algebraization was obtained by interpreting p^2 to be equal to $-n^2$ (*cf.* Example a). The initial-value conditions were here replaced by the assumption that $C(t)$ varies harmonically with the same frequency as $F(t)$.

This is a reasonable assumption because it is fulfilled for all dissipative systems for large values of t irrespective of prescribed initial values.

3.2. HEAVISIDE only occasionally considered A–C circuits. He was more interested in the transient behavior which could be studied by the aid of the steady impressed force $F(t)=H(t)$. His extensive use of this function justifies the name "HEAVISIDE function" and the symbol $H(t)$ later attached to it. When the impressed force was $H(t)$, HEAVISIDE assumed the system to be at rest for $t<0$. If we confine ourselves to positive times, this physical requirement is equivalent to the following initial-value condition: the unknown function (C in Example a and V in Examples b and c) and its derivatives to a certain order are 0 for $t=0$, the order being defined by the requirement that the further development of the system shall be well determined.

3.2.1. The expansion in descending powers of p and the rule $p^{-n}H(t)=\frac{t^n}{n!}$ would normally give the solution in a power series of t.

3.2.2. By expanding in ascending powers of p and interpreting $p^n H(t)$ to be 0, HEAVISIDE achieved an asymptotic expansion of the solution. Often the algebrizing procedures *3.2.1* and *3.2.2* led to fractional differentiations, which he managed by using the formula $p^{\frac{1}{2}}H(t)=(\pi t)^{-\frac{1}{2}}$. I shall not investigate the questions regarding divergent series or fractional differentiation but only point out that HEAVISIDE's procedures have been partly justified by DOETSCH (1937).

3.2.3. The expansion theorem was, as I have already remarked, the most powerful tool in the algebrizing procedure. Although his operational proof of it applied only to rational functions, HEAVISIDE used it also for transcendental functions (*cf.* Example c).

6. As a final example of HEAVISIDE's methods I am going to show how they apply to a general finite network. HEAVISIDE just sketched this general procedure [*EMT* § 245]. Although general networks were very essential in his discovery of the expansion theorem, he treated only special networks after having made the discovery. Many of HEAVISIDE's successors, however, treated his methods and especially the expansion theorem in connection with general networks, so the following example will have a central position in the next sections.

Example d. We consider a finite network consisting of resistances, coils and condensers. It can be treated by the aid of KIRCHHOFF's circuit law, which states that the sums of the potential drops in each mesh equal the impressed e.m.f. Thus, if we put

$$x_j(t)=\int_0^t C_j(t)\,dt \qquad \text{(I.25)}$$

where C_i is the current in the i^{th} mesh, we get a system of differential equations of the form

$$\begin{aligned}&\left(a_{11}\frac{d^2}{dt^2}+b_{11}\frac{d}{dt}+c_{11}\right)x_1(t)+\cdots+\left(a_{1n}\frac{d^2}{dt^2}+b_{1n}\frac{d}{dt}+c_{1n}\right)x_n(t)=F_1(t),\\&\left(a_{21}\frac{d^2}{dt^2}+b_{21}\frac{d}{dt}+c_{21}\right)x_1(t)+\cdots+\left(a_{2n}\frac{d^2}{dt^2}+b_{2n}\frac{d}{dt}+c_{2n}\right)x_n(t)=F_2(t)=0,\\&\vdots\\&\left(a_{n1}\frac{d^2}{dt^2}+b_{n1}\frac{d}{dt}+c_{n1}\right)x_1(t)+\cdots+\left(a_{nn}\frac{d^2}{dt^2}+b_{nn}\frac{d}{dt}+c_{nn}\right)x_n(t)=F_n(t)=0.\end{aligned} \tag{I.26}$$

where the constants a_{ij}, b_{ij}, c_{ij} represent inductances, Ohmic resistances and capacitances and the F_i's are impressed e.m.f.'s; we have followed HEAVISIDE and put all but F_1 equal to zero. The solution to the problem when there is more than one impressed force can be found as a superposition of such solutions.

The substitution of $p=\frac{d}{dt}$ in (26) yields

$$\begin{aligned}&e_{11}x_1+e_{12}x_2+\cdots+e_{1n}x_n=F_1,\\&e_{21}x_1+e_{22}x_2+\cdots+e_{2n}x_n=0,\\&\vdots\\&e_{n1}x_1+e_{n2}x_2+\cdots+e_{nn}x_n=0\end{aligned} \tag{I.27}$$

where $e_{ij}=a_{ij}p^2+b_{ij}p+c_{ij}$.

This algebraic system of equations is solved by

$$x_j=\frac{D_{1j}(p)}{D(p)}F_1 \tag{I.28}$$

where $D(p)$ is the determinant and D_{ij} is the i,j^{th} subdeterminant of (27). Application of the expansion theorem (21) yields the algebrized version of (28):

$$x_j=\frac{D_{1j}(0)}{D(0)}+\sum_{D(\lambda)=0}\frac{D_{ij}(\lambda)}{\lambda\left.\frac{dD}{dp}\right|_{p=\lambda}}e^{\lambda t}. \tag{I.29}$$

To make a comparison, let us outline how (26) was and still is traditionally solved. First the solution to the corresponding homogeneous equation is found as a linear combination of exponentials of the form $e^{{}_j\lambda t}$ where the ${}_j\lambda$'s are the roots of the determinantal equation $D({}_j\lambda)=0$. This solution presupposes that $D(\lambda)$ has no double roots, a requirement we also meet in the "proof" of the expansion theorem. The general solution of (26) is then obtained by adding one solution to the inhomogeneous equation. At last the coefficients of the $e^{{}_j\lambda t}$'s are determined by the prescribed initial value conditions.

HEAVISIDE's expansion theorem gives immediately the full solution (29) of (26), $\frac{D_{1j}(0)}{D(0)}$ being a special solution of the inhomogenious equation, and, what is more important, it automatically adjusts the arbitrary constants so that $x_j(0)=0$ and $x_j'(0)=0$. HEAVISIDE pointed out that this was a great advantage as it could often be a hard job to adjust the arbitrary constants in the traditional way.

7. In the preceding sections we have seen what HEAVISIDE's operational calculus looked like in 1894–98. In this section I shall make a few remarks on the development of the operational method in HEAVISIDE's writing.

In his early work from the period 1872–1881 he used only traditional mathematical methods. The first trace of an operational way of thinking can be found in the paper "On Induction between parallel Wires" (1881) [*EPT* p. 125] where he used the notation D for $\frac{d}{dt}$ and Δ for $\frac{d}{dx}$. In the same article HEAVISIDE derived from the conjugate property the arbitrary constants in the solution of a homogeneous system of differential equations in terms of the initial values (*cf.* the Appendix). Although in 1884 he used resistance operators in dealing with the differential equations for a general network (a system similar to that of § 6), HEAVISIDE before 1886 only now and then treated his systems operationally.

Before 1886 HEAVISIDE mostly employed the symbol D in connection with the subsidence factor e^{Dt} on which D operates as a multiplication operator. But as he gradually began to name the subsidence factor e^{pt}, it also became natural for him to let p denote the time differentiator, which he did from the end of 1886. At the same time HEAVISIDE's interest in the arbitrary constants in homogenous systems led him to the first version of the expansion theorem, which he proved with the condenser method* in "On the Self Induction of Wires, III" (1886). Only one month later did he prove the expansion theorem operationally by a development in partial fractions.

Thus the expansion theorem was the first algebrizing rule HEAVISIDE discovered. This is understandable because he at first considered p in connection with e^{pt}. Expansions of an operator $Z(p)$ in descending powers of p are found in "On Electromagnetic Waves" (1888) [*EP* II 426–27] whereas I have not found examples of expansions in increasing powers of p before 1892, when it occurs in "On Operators in Physical Mathematics".

The HEAVISIDE function can be traced back to HEAVISIDE's early works. Its importance was still increasing, and in 1892 it obtained its central position in the operational calculus in the above-mentioned series of papers.

8. In Example a (§ 2) we saw that HEAVISIDE put $pH(t)=0$. In other connections, however, he often showed a deeper understanding of $p^n H(t)$ considering it the "function" similar to what we denote by δ^{n-1} [*EMT* § 249, 251, 253, 267, 271]. HEAVISIDE called $pH(t-x)$ "an impulsive function at the moment $t=x$" and the higher derivatives were termed "multiple impulses". Thus HEAVISIDE (1893–95), along with KIRCHHOFF (1891), was the first to introduce the δ-function.

By means of the operational calculus HEAVISIDE was in a position to treat systems in which the impressed force was a δ-function; for if the symbolic solution equalled $Z(p)\,\delta(t)=Z(p)\,p\,H(t)$, the three algebrizing rules could be applied to $Z(p)\,p$. This solution, which HEAVISIDE termed the impulsive solution, is nowadays called the fundamental solution:

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* See the Appendix.

> But knowing the developed impulsive solution ... the solution for a continued force varying anyhow with the time is at once expressible by a definite integral, because the continued force may be regarded as consisting of an infinite series of successive infinitesimal impulses. [*EMT* § 251]

In modern notation HEAVISIDE here states that since

$$\int \delta(y-t)\, f(y)\, dy = f(t) \tag{I.30}$$

for every continuous function f (explicitly stated in *EMT* § 267), we obtain, by a formal application of $Z(p)$,

$$Z(p)\, f(t) = \int Z(p)\, \delta(y-t)\, f(y)\, dy = (E * f)(t) \tag{I.31}$$

where $E(t) = Z(p)\,\delta(t)$ is the fundamental solution. HEAVISIDE, however, did not attach great importance to this equation, which is so central in the modern theory of differential equations.

9. What was the new aspect of HEAVISIDE's treatment of the operational calculus? First of all, as he pointed out clearly, it was highly applicable to electrical engineering. Some historians of mathematics have reduced HEAVISIDE's contributions to the operational calculus to this single point, claiming that he just applied BOOLE's well known theory without adding anything new (see *e.g.* COOPER 1952). To show that HEAVISIDE's investigations contained more than that, let us again consider the three steps in his operational calculus (see the scheme in § 5 above). Step 1 had been a traditional step in mathematical physics since 1700; Step 2 had been characteristic of the operational calculus also before HEAVISIDE, but in Step 3 HEAVISIDE blazed his own trail. The introduction of the standard operand $H(t)$ and the three algebrizing rules were his original contributions to the operational calculus.

His experimental, unrigorous mathematics brought the operational mathematics into disrepute among most contemporary mathematicians, but in the long run it proved to be a challenge to mathematicians to explain how this peculiar method could lead to correct results.*

II. The Method of Function Theory

1. During the sixteen years following the publication of *EMT* II noone, neither mathematician nor engineer, seems to have used HEAVISIDE's ideas, but in the second decade of this century interest in the operational method revived. HEAVISIDE was considered the inventor of the operational calculus, and the studies in the field were at the beginning concentrated on his work. The favorite

* In Kuhnian (KUHN 1962) terminology: HEAVISIDE solved physical puzzles within a mathematical paradigm which was a personal rearrangement of an otherwise abandoned paradigm from about 1800. In the Mathematical paradigm of 1900 his solutions were considered puzzles which were successfully solved in two different ways in the 20th century.

subject was the expansion formula which was mostly known from *EMT* § 282, where it is stated without proof. Until WAGNER in 1925 and VALLASTA in 1926 independently found HEAVISIDE's first proof, it was the general opinion (except for BROMWICH) that HEAVISIDE had no proof of the expansion theorem. For this reason it was regarded as important to find a proof and further to extend the theorem to the case where the denominator of the operator $\frac{Y(p)}{Z(p)}$ had double or zero roots.

The English mathematician T.J.I'A. BROMWICH and the German electrical engineer K.W. WAGNER were the first to publish justifications of HEAVISIDE's work; their works were independent but quite similar, both using function theory. WAGNER [1915–16] concentrated on the expansion formula, whereas BROMWICH [1916] gave a broader explanation of the operational calculus. Moreover, since BROMWICH's paper was the more influential, I shall here outline his ideas.

2. BROMWICH began to think about the operational calculus when he had read *EMT* II, and after sixteen years he gave his interpretation of it in the article "Normal Coordinates in Dynamical Systems" (1916). At first he considers the homogeneous system of differential equations characteristic of a discrete, isolated physical system:

$$\begin{aligned} e_{11}x_1+e_{12}x_2+\cdots+e_{1n}x_n&=0,\\ e_{21}x_1+e_{22}x_2+\cdots+e_{2n}x_n&=0,\\ e_{n1}x_1+e_{n2}x_2+\cdots+e_{nn}x_n&=0, \end{aligned} \tag{II.1}$$

where

$$e_{ij}=a_{ij}\frac{d^2}{dt^2}+b_{ij}\frac{d}{dt}+c_{ij}, \qquad i=1,2,\ldots,n,\ j=1,2,\ldots,n$$

with prescribed, inhomogeneous initial values,

$$x_j(0)=u_j, \qquad \frac{d}{dt}x_j(0)=v_j. \tag{II.2}$$

BROMWICH then assumes the x_j's to be given by the complex integral:

$$x_j=\frac{1}{2\pi i}\int_K e^{\lambda t}\xi_j\,d\lambda, \tag{II.3}$$

K being a closed curve enclosing all the poles of $\xi_j(\lambda)$. By inserting (3) into (1) he finds the equations for the ξ_j's:

$$\int_K e^{\lambda t}p_i\,d\lambda=0, \qquad i=1,2,\ldots,n, \tag{II.4}$$

where

$$\begin{aligned} \lambda_{11}\xi_1+\lambda_{12}\xi_2+\cdots+\lambda_{1n}\xi_n&=p_1,\\ \lambda_{21}\xi_1+\lambda_{22}\xi_2+\cdots+\lambda_{2n}\xi_n&=p_2,\\ &\vdots\\ \lambda_{n1}\xi_1+\lambda_{n2}\xi_2+\cdots+\lambda_{nn}\xi_n&=p_n, \end{aligned} \tag{II.5}$$

and

$$\lambda_{ij} = a_{ij}\lambda^2 + b_{ij}\lambda + c_{ij}.$$

To find the p_i's in terms of the initial values, BROMWICH expands $\xi_j(\lambda)$ in a LAURENT series for large values of $|\lambda|$:

$$\xi_j = \frac{X_j}{\lambda} + \frac{Y_j}{\lambda^2} + \frac{Z_j}{\lambda^3} + \dots, \tag{II.6}$$

positive powers being omitted because they do not contribute to (3). From (3) and (2) we get

$$u_j = \frac{1}{2\pi i} \int_K \xi_j \, d\lambda, \qquad v_j = \frac{1}{2\pi i} \int_K \lambda \xi_j \, d\lambda; \tag{II.7}$$

hence

$$X_j = u_j, \qquad Y_j = v_j.$$

To make p_i satisfy the requirement (4) BROMWICH assumes the p_i's to be polynomials. They can then be found by inserting

$$\xi_j = \frac{u_j}{\lambda} + \frac{v_j}{\lambda^2} + O\left(\frac{1}{\lambda^3}\right)$$

in the Equations 5 and by omitting negative powers of λ:

$$p_i = (a_{i1}\lambda + b_{i1}) u_i + a_{i1} v_i + \dots + (a_{in}\lambda + b_{in}) u_n + a_{in} v_n. \tag{II.8}$$

The solution of (5) considered as equations in the ξ_j's yields

$$\xi_j = \frac{1}{D(\lambda)} \{D_{1j}(\lambda) p_1 + D_{2j}(\lambda) p_2 + \dots + D_{nj} p_n\} \tag{II.9}$$

where the p_i's are found from (8) and the D_{ij}'s are the subdeterminants of the determinant D corresponding to the system (5). To check that the boundary conditions are not violated by assuming p_i to be a polynomial BROMWICH proves (7) from (9).

When the ξ_j's have been found, a calculation of residues determines x_j from (3) and the homogenous equation (1) is solved.

3. BROMWICH next treats an inhomogenous system with homogenous initial-value conditions, a problem that is much more closely related to the HEAVISIDE calculus. The impressed forces are assumed to have the form $P_i e^{\mu t}$:

$$\begin{aligned} e_{11}x_1 + e_{12}x_2 + \dots + e_{1n}x_n &= P_1 e^{\mu t}, \\ e_{21}x_1 + e_{22}x_2 + \dots + e_{2n}x_n &= P_2 e^{\mu t}, \\ &\vdots \\ e_{n1}x_1 + e_{n2}x_2 + \dots + e_{nn}x_n &= P_n e^{\mu t}. \end{aligned} \tag{II,1'}$$

Again (3) is substituted in (1′) yielding

$$\int_K e^{\lambda t} p_i \, d\lambda = P_i e^{\mu t}, \qquad i = 1, 2, \dots, n, \tag{II.10}$$

the p_i's being defined from (5) as before. Equation (10) is satisfied by

$$p_i = \frac{P_i}{(\lambda - \mu)}. \tag{II.11}$$

Of course any analytic function could be added to $\frac{P_i}{(\lambda-\mu)}$, but a test shows that (11) leads to the required homogeneous initial values. By applying (11) to (5) the equations for the ξ_j's are obtained:

$$\begin{aligned} \lambda_{11}\xi_1 + \lambda_{12}\xi_2 + \cdots + \lambda_{1n}\xi_n &= \frac{P_1}{\lambda-\mu}, \\ &\vdots \\ \lambda_{n1}\xi_1 + \lambda_{n2}\xi_2 + \cdots + \lambda_{nn}\xi_n &= \frac{P_n}{\lambda-\mu}, \end{aligned}$$

and they have the solution

$$\xi_j = \frac{1}{D(\lambda)}\left(D_{1j}(\lambda)P_1 + \cdots + D_{nj}P_n\right)\frac{1}{\lambda-\mu} \tag{II.9'}$$

from which x_j can be found from (3).

BROMWICH then considers the special case treated by HEAVISIDE, where one constant force is impressed at $t=0$. For $t>0$ this is equivalent to putting $P_2 = P_3 = \cdots = P_n = 0$ and $\mu = 0$ in (1'). In this case (9') inserted into (3) gives

$$x_j = \frac{1}{2\pi i}\int_K \frac{D_{1j}P_1}{\lambda D(\lambda)} e^{\lambda t}\, d\lambda. \tag{II.12}$$

If $D(\lambda)$ has no multiple or zero roots a calculation of residues leads to HEAVISIDE's expansion theorem (I.29). BROMWICH's formula (12) however, is more comprehensive as it allows him to treat multiple and zero roots as well. From (12) it is also possible to derive HEAVISIDE's other algebrizing rules. This was done by JEFFREYS [1927].

4. Letting r tend to infinity in the path K shown in Fig. 4, BROMWICH transforms the integral (3) into

$$x_j(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda t}\,\xi_j(\lambda)\, d\lambda \tag{II.3''}$$

for $t>0$ when all the poles of ξ_j lie to the left of the line $\mathrm{Re}\,\lambda = c$. This transformation is correct only if $\xi_j(\lambda)$ tends quickly to zero for $|\lambda| \to \infty$, for instance as $O\left(\frac{1}{\lambda^3}\right)$, a requirement that is fulfilled when the initial values are homogeneous.

The formula (3'') proves valuable when BROMWICH applies his method to the solution of partial differential equations. For it often happens that the ξ_j's have infinitely many poles which cannot be enclosed in a closed path (*e.g.*, § I

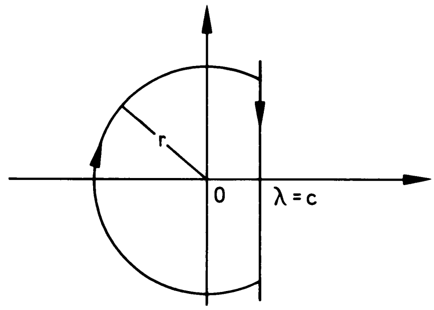


Fig. 4

Example c). In normal physical systems, however, the real part of the poles will be bounded from above – in dissipative systems bounded by zero – so that a path as in (3″) can still be used. BROMWICH remarks about the treatment of partial differential equations:

> It will be observed that the foregoing does not profess to give a complete proof of (3″): all that we have done is to establish an analogy between (3″) and the formulae which we have proved for discrete systems.

The main idea in the method of function theory is that the integral representation of x_j, (3) or (3″), transforms the given normal system of differential equations into a system of algebraic equations – or a partial differential equation into a differential equation one degree lower – the differential operator $\frac{d}{dt}$ being transformed into multiplication by the independent variable λ. Moreover the initial values will enter the new equations (5) and (8). However, the application of the method is difficult because the inverse of (3) or (3″), which would transform the given equation into the easier one, is not given explicitly.

5. In his article of 1916 BROMWICH presented the method of function theory as a substitute for HEAVISIDE's calculus. In a long series of applications published in the Proceedings of the London Mathematical Society and the Proceedings of the Cambridge Philosophical Society (references in JEFFREYS 1927), however, he realized that HEAVISIDE's operational method was the easiest one to use; thus he recommended its application. But for two reasons the paper of 1916 was still central in BROMWICH's work. First, because it contained "the final rigorous proof" of HEAVISIDE's operational calculus; second, because BROMWICH could derive a new algebrizing rule from (12) as follows.

From (I.28) we know that the operational solution of (1′) when $P_2 = P_3 = \cdots = 0$ and $\mu = 0$ is

$$x_j = \frac{D_{1j}(p) P_1}{D(p)}, \tag{II.13}$$

and from (12) that the ordinary solution is

$$x_j = \frac{1}{2\pi i} \int_K \frac{D_{1j}(\lambda) P_1}{\lambda D(\lambda)} e^{\lambda t} d\lambda.$$

Therefore, if the operational solution of a problem is $f(p)$, BROMWICH could find the ordinary solution $g(t)$ from the integral:

$$g(t)=\frac{1}{2\pi i}\int_K \frac{f(p)}{p} e^{pt}\,dp. \tag{II.14}$$

When the operand for $f(p)$ is not stated explicitly, it is assumed to be $H(t)$, i.e. $g(t)=f(p)H(t)$. In 1927 BROMWICH proved that the HEAVISIDE function was given by the integral

$$H(t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{pt}}{p}\,dp, \qquad \sigma, t>0, \tag{II.15}$$

and by applying $f(p)$ formally inside the integral sign in (15) he had a short proof of (14). A similar idea was the basis of WAGNER's proof of HEAVISIDE's expansion theorem.

Thus in his later works BROMWICH proceeded as follows: he calculated with $p=\frac{d}{dt}$ as an algebraic quantity (*cf.* HEAVISIDE's Steps 1 and 2) and finally interpreted the symbolic solution by the aid of (14) or by HEAVISIDE's algebrizing rules. This second version of the method of function theory was applied by a lot of engineers and was published in a book by one of BROMWICH's fellow mathematicians at St. John's College, Cambridge: HAROLD JEFFREYS (1927). Function theory seems to have troubled some engineers, for instance HEAVISIDE to whom BROMWICH sent his method. [See H. J. JOSEPH's notes in *EMT* III, 2nd ed.] Other scientists gravitating more toward mathematics advocated the original version of the method of function theory as presented in BROMWICH's paper of 1916. Among these was CARSLAW who discussed this with BROMWICH and JEFFREYS in a very interesting article in the Mathematical Gazette 1928–29. JEFFREYS concluded:

> Whether it is better to introduce them [the complex integrals] for this purpose when the operational solution has been found, as Bromwich and I would do, or to bring them in at the very beginning, as Prof. Carslaw does, is a matter of taste and printing expenses. No logical question is involved.

III. Carson's Integral Equation

1. JOHN R. CARSON, who worked for the American Telephone and Telegraph Company, developed another significant reformulation of HEAVISIDE's operational calculus in a series of papers. He collected his ideas in the book *Electric Circuit Theory* (1926). We shall follow the development of his version of the operational calculus, beginning with his first paper on the subject: "On a General Expansion Theorem for Transient Oscillations of a Connected System" (1917) in which the following traditional proof was given, to all appearances independently of BROMWICH's and WAGNER's proofs. To a network with the equation (I.26) CARSON applied the impressed force $F_1(t)=E_1 e^{pt}$, where p is a complex constant. He then assumed the solutions to be of the form $x_j(t)=\xi_j e^{pt}$, and by cancelling e^{pt} in the system (I.26) he obtained a system of algebraic equations in the ξ_j's. CARSON solved that by traditional means and found the following particular solution to (I.26),

$$x_j(t)=E_1\,\frac{D_{1j}(p)}{D(p)}\,e^{pt}, \tag{III.1}$$

and the general solution by adding the solution of the homogeneous equation,

$$x_j(t)=\sum_{D({}_kp)=0} {}_kA_j\,e^{{}_kpt}. \tag{III.2}$$

By developing $\frac{D_{1j}(p)}{D(p)}$ in partial fractions and adjusting ${}_kA_j$ so that $x_j(0)=\frac{d}{dt}x_j(0)=0$ he obtained the following expansion theorem:

$$x_j=E_1\left\{\frac{D_{1j}(p)}{D(p)}\,e^{pt}-\sum_{D({}_kp)=0}\frac{D_{1j}({}_kp)}{(p-{}_kp)\,D'({}_kp)}\,e^{{}_kpt}\right\}. \tag{III.3}$$

For $p=0$ this is reduced to HEAVISIDE's expansion formula (I.29) about which CARSON wrote in the main text: "HEAVISIDE states this theorem without proof." However, in a footnote he corrects this statement: "Since the above was written Mr. H. W. Nichols has called my attention to the fact that HEAVISIDE derives his Expansion theorem in his Electrical Papers vol. II, p. 373". The current produced by a constant impressed e.m.f. at $t=0$ was called the indicial admittance by CARSON. It is the derivative of the above mentioned solution x_j (*cf.* I.25). The importance of the indicial admittance $A(t)$ was stressed in CARSON's next paper [1919] where he showed that the current $I(t)$ corresponding to an arbitrary force $F_1(t)=E(t)$ could be determined by the aid of the formula

$$I(t)=\frac{d}{dt}\int_0^t A(t-\tau)\,E(\tau)\,d\tau=\frac{d}{dt}\int_0^t E(t-\tau)\,A(\tau)\,d\tau \tag{III.4}$$

(*cf.* HEAVISIDE's formula (I.31)).

In a footnote in his book of 1926 he stated

> This important theorem, independently derived and published by the author, is actually the equivalent of a much older theorem in dynamics due to Duhamel.

2. But not until his paper in the Bell System's Technical Journal of 1922 can we find the most essential element in CARSON's reinterpretation of the operational calculus, namely an integral equation for the indicial admittance. He deduced this by applying DUHAMEL's principle to a network with the impressed e.m.f. $E(t)=e^{pt}$. From (III.1) and (III.2) CARSON knew that this e.m.f. corresponded to a current

$$I(t)=\frac{e^{pt}}{Z(p)}+\sum_{Z({}_kp)=0} {}_kA'_j\,e^{{}_kpt} \tag{III.5}$$

where $Z(p)$ is the impedance of the system $\left(=\frac{D(p)}{p D_{11}(p)}\right)$. From (III.4) CARSON got

$$\begin{aligned}I(t)&=\frac{d}{dt}e^{pt}\int_0^t A(\tau)e^{-p\tau}d\tau\\&=\frac{d}{dt}\left(e^{pt}\int_0^\infty A(\tau)e^{-p\tau}d\tau-e^{pt}\int_t^\infty A(\tau)e^{-p\tau}d\tau\right)\\&=pe^{pt}\int_0^\infty A(\tau)e^{-p\tau}d\tau-pe^{pt}\int_t^\infty A(\tau)e^{-p\tau}d\tau+A(t).\end{aligned}\tag{III.6}$$

By combining (III.5) and (III.6) for $I(t)$ and dividing through by $e^{p\tau}$ he found

$$\begin{aligned}&\frac{1}{Z(p)}+\Big(\sum_{Z({}_kp)=0}{}_kA_j'e^{{}_kpt}\Big)e^{-pt}\\&\quad=p\int_0^\infty A(\tau)e^{-p\tau}d\tau-p\int_t^\infty A(\tau)e^{-p\tau}d\tau+A(t)e^{-pt}.\end{aligned}$$

> This equation is valid for all values of t. Consequently if we set $t=\infty$ and if the real part of p is positive, only the first term on the right and the left hand side of the equation remain [as $\mathrm{Re}({}_kp)>0$ in all dissipative systems] and we get:
>
> $$\frac{1}{pZ(p)}=\int_0^\infty A(t)e^{-pt}dt. \tag{III.7}$$
>
> This is an integral equation valid for all positive real values of p which completely determines the indicial admittance $A(t)$.
>
> [CARSON 1926, p. 19]

Having proved (III.7), CARSON proceeded as follows toward the solution of circuit problems: He first found the operational expression of the indicial admittance

$$A=\frac{1}{Z(p)}H(t)$$

in the same way as HEAVISIDE did in Steps 1 and 2. Then he determined the indicial admittance from the integral equation (III.7), and at last he found the current resulting from the given e.m.f. from DUHAMEL's integral (III.4).* CARSON's method was applicable only to homogeneous initial values in contrast to the original method of BROMWICH.

3. To facilitate the operational solution CARSON in his book of 1926 tabulated solutions to (III.7) for different $Z(p)$. More solutions could be obtained from the tabulated ones by means of the following theorem due to BOREL:

If the functions $f(t)$, $f_1(t)$, $f_2(t)$ are defined by the integral equations

* (III.7) can be found in *EMT* § 526, but HEAVISIDE used it to calculate integrals by the aid of operational methods.

$$F(p) = \int_0^\infty f(t)\, e^{-pt}\, dt$$

$$F_1(p) = \int_0^\infty f_1(t)\, e^{-pt}\, dt$$

$$F_2(p) = \int_0^\infty f_2(t)\, e^{-pt}\, dt$$

and if the functions F, F_1 and F_2 satisfy the relation

$$F(p) = F_1(p)\, F_2(p)$$

then

$$f(t) = \int_0^t f_1(\tau)\, f_2(t-\tau)\, d\tau.$$

The BOREL theorem enabled CARSON to calculate the response to an arbitrary e.m.f., without taking the detour around the indicial admittance:

If the operational equation

$$h = \frac{1}{H(p)}$$

expresses the response of a network to a "unit e.m.f." and if an arbitrary e.m.f. E impressed at time $t=0$ is expressible by the operational equation

$$E = V(p)$$

or the infinite integral

$$\int_0^\infty E(t)\, e^{-pt}\, dt = \frac{V(p)}{p}$$

then the response x of the network to the arbitrary force is given by the operational equation

$$x = \frac{V(p)}{H(p)}$$

and $x(t)$ is determined by the integral equation

$$\frac{1}{p}\,\frac{V(p)}{H(p)} = \int_0^\infty x(t)\, e^{-pt}\, dt.$$

CARSON also proved that HEAVISIDE's three algebrizing rules could be derived from (III.7) under certain conditions.

From a modern point of view CARSON's procedure can be interpreted in terms of integral transformations; however it should be pointed out that CARSON did not consider (III.7) to be a transformation of a function of t into a function of p.

4. BROMWICH, WAGNER, and CARSON were not the only ones who worked with HEAVISIDE's ideas, but from a mathematical point of view their results were by far the most interesting. It is difficult to explain why so many people suddenly took an interest in HEAVISIDE's methods, which had remained unnoticed for a long period after their publication. On the other hand there can be no doubt that the long period of neglect was due to HEAVISIDE's obscure way of presenting them, for soon after the methods had been presented in a clearer way, they were taken up by the engineers. From 1920 onwards periodicals of electrical engineering and physics were full of operational calculations; applications to electrical engineering were in the majority, but there were others as well.

The growth in the applications of the operational calculus was intensified after the publication of textbooks at the end of the twenties [CARSON 1926, JEFFREYS 1927, BERG 1929, BUSH 1929].

IV. Integral Transformations

1. BROMWICH's second method and CARSON's method treat differential equations with $H(t)$ as the right-hand side in a similar way. First the operational solution $f(p)$ is obtained by using HEAVISIDE's Steps 1 and 2, and thereafter the ordinary solution $h(t)$ is found. At this last step the methods differ. If the method of function theory is used, $h(t)$ is found directly from the complex integral

$$h(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{f(p)}{p}e^{pt}\,dp \qquad \text{(IV.1)}$$

whereas CARSON found $h(t)$ indirectly as the solution of the integral equation

$$f(p)=p\int_{0}^{\infty}h(t)\,e^{-pt}\,dt. \qquad \text{(IV.2)}$$

We have seen that the connection between these two methods and HEAVISIDE's method had been clear from the very beginning; but the relations between them were not derived explicitly until 1927, by H. W. MARCH. He remarked that the FOURIER-LAPLACE theorem,

$$v(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{xy}\int_{0}^{\infty}e^{-yz}\,v(z)\,dz\,dy, \qquad \text{(IV.3)}$$

implied that $h(t)$ found from BROMWICH's formula (1) was a solution to CARSON's integral equation (2), as should be if both methods were correct. In other words, (1) and (2) are inverse integral transformations.* We shall call (2) the modified LAPLACE transformation and denote it by L'.

In Sections II and III we saw that BROMWICH used the inverse of L', and CARSON implicitly used L', but without considering it to be a transformation.

* In fact (3) is valid only if $x>0$ while the integrals yield 0 if $x<0$. This is so because the correct integral formula is

$$v(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{xy}\int_{-\infty}^{\infty}e^{-yz}\,v(z)\,dz\,dy. \qquad \text{(IV.3')}$$

The first to use this transform consistently as a substitution for and an explanation of HEAVISIDE's operational calculus was VAN DER POL, who was employed in the "Philips Gloeilampenfabriken" in Holland. In a sequence of papers starting in 1929 he solved differential equations in t by transforming them by means of L', *i.e.* into equations in p, the solution of which he transformed back into the t domain by the transformation L'^{-1}, *i.e.* (1).

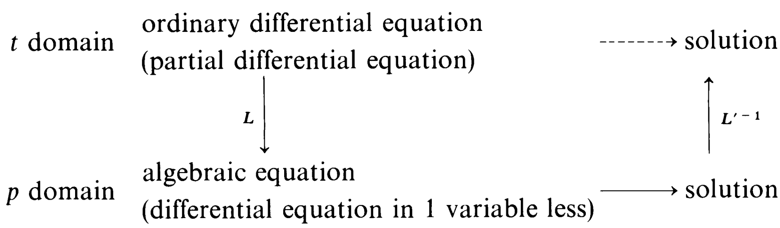


Fig. 5

VAN DER POL introduced the notation

$$f(p) \doteqdot h(t)$$

for the connection (2) or (1). For a differential equation with constant coefficients, homogeneous initial-value conditions and $H(t)$ as the right-hand side it is evident that VAN DER POL by transforming the equation into the p domain obtained HEAVISIDE's operational equation. Hence the two methods were equivalent when they were both applicable. However the transformation method had several advantages:

1) HEAVISIDE's formal rules for the formation of the operational equation, that is $\frac{\partial}{\partial t} \curvearrowright p$ and the dropping of $H(t)$, were replaced by the rigorous application of (2).

2) It was easier to delimit the method; for instance (2) should be defined for all the functions entering into the equation.

3) VAN DER POL's method was applicable to equations with non-constant coefficients and non-constant right-hand sides.

4) Also non-homogeneous initial values could be taken into account. That had also been the case with the method of function theory (Section II), but there the initial values had entered into the equations in a rather strange way. It became much more straightforward when the transformation (2) was directly applied; for if $f(p) \doteqdot h(t)$ a partial integration shows that

$$p f(p) - p h(0) \doteqdot \frac{d}{dt} h(t). \tag{IV.4}$$

Therefore by repeated use of partial integration the initial values would enter directly into the transformed equation.

VAN DER POL made long tables of functions and their modified LAPLACE transforms and published a textbook on the transformation method (1950).* His work made it much easier for users to learn the operational calculus.

2. There was, however, another person who contributed even more to the spreading of integral transformation theory in connection with the operational calculus, namely the German mathematician GUSTAV DOETSCH. The mathematicians or electricians we have considered until now were all led from HEAVISIDE's operational calculus towards integral transformations. DOETSCH went the opposite way. He had already applied the LAPLACE transformation

$$f(p)=L(h)(p)=\int_0^\infty h(t)\,e^{-pt}\,dt$$

to the solution of integral and differential equations before connecting it with the operational calculus. This he first did in a review (1930) of CARSON's book *Electric Circuit Theory*.** It is therefore only natural that he did not accept the unrigorous operational method, and he pointed out that CARSON ought to have transformed the equations from the start instead of postponing the integral equation to the last step in the argument.

In his book *Theorie und Anwendung der Laplace-Transformation* (1937) his remarks about the HEAVISIDE calculus were very deprecatory. He considered his predecessors' works from a point of view of LAPLACE transformations, and although he proved several of HEAVISIDE's procedures, he thought that it was just an accident that they worked [p. 338]. From the preceding sections and from the following comments on MIKUSIŃSKI's book it should be clear that this judgment was much too hard.

DOETSCH's book is a very rigorous and clear compilation of what was known about the LAPLACE transformation in 1937. To make the applications as broad as possible he sought to minimise the assumptions of the theorems, for which reason the proofs are often very complicated. For engineers and others who were not so interested in a high standard of rigor, he published several practically orientated books on the subject and some tables of functions and their LAPLACE transformations. The engineers quickly learned the method, and from about the middle of the thirties most works on the operational calculus dealt with the LAPLACE transformation or with the modified LAPLACE transformation. This change in the meaning of the expression "operational calculus" must be due to VAN DER POL, who used this term for his method, and to the engineers, who regarded the transformation method as a reformulation of

* In the textbook "Operational Calculus Based on the Two-sided Laplace-Integral" VAN DER POL applied the transformation (2) with the integration interval $]-\infty, \infty[$. This had the advantage that the transformation (1) became the proper inverse, also for functions which were not 0 for $t<0$. On the other hand the treatment of initial value problems was complicated because all entering functions had to be multiplied by $H(t)$.

** It would be very interesting to know how the theory of the LAPLACE transformation developed from the time of LAPLACE till that of DOETSCH, a development which was apparently without connection with the operational calculus.

HEAVISIDE's ideas. DOETSCH made a sharp distinction between the unrigorous operational calculus and his own method.

3. Even before the LAPLACE transformation had been considered in connection with the operational calculus, NORBERT WIENER had made the FOURIER transformation – or an extension of it – the basis of a treatment of the operational calculus (1926). His paper was praised by many other writers for its rigor, but in practice the method was too difficult. Further, the method was not fit for a treatment of the initial-value problem, which was of such great practical importance, because it used the two-sided FOURIER transformation (with the range of integration $(-\infty, \infty)$); this can easily be seen from (4).

4. Common to all the integral transformations employed in connection with the operational calculus is the transformation of the time differentiator $\frac{d}{dt}$ into the multiplication operator αp (plus possible initial value terms) where α is a constant. In his doctoral dissertation from 1934 the Dutch mathematician H. B. J. FLORIN showed that an integral transformation of the form

$$f(p) = \int K(p, t)\, h(t)\, dt \tag{IV.5}$$

had this property if its kernel is of the form

$$K(p, t) = e^{-\alpha p t} B(p). \tag{IV.6}$$

If the transformed formulae are to be in accordance with HEAVISIDE's symbolic formulae we must require also that the transform of $H(t)$ be 1. FLORIN showed that this is the case when $\alpha = 1$ and $B(p) = p$, *i.e.* if (5) is the modified LAPLACE transformation (2).

If $\alpha = 1$ and $B(p) = 1$ in (6), then (5) is the LAPLACE transformation. It is therefore clear that HEAVISIDE's and DOETSCH's formulas differ by a factor p.

If $\alpha = i$ and $B(p) = 1$ we get the FOURIER integral.

5. BROMWICH's and CARSON's work led to a reinterpretation and an explanation of the operational calculus in terms of integral transformations. This development rested heavily on HEAVISIDE's introduction of a standard operand $H(t)$. For when the operator $f(p)$ always operated on the same operand, this operand could be left out, as HEAVISIDE himself did, and $f(p)$ could then be interpreted as a function of the variable p, namely the modified LAPLACE transform of the function

$$h(t) = f(p)\, H(t).$$

As a curiosity it can be mentioned that if HEAVISIDE had chosen the impulsive function $\delta(t)$ as his standard operand instead of $H(t)$, the corresponding integral transformation would have been the LAPLACE transformation.

V. Algebraic Explanations, Mikusinski's Operational Calculus, and Schwartz's Theory of Distributions

1. In the preceding three sections we have followed the development of HEAVISIDE's methods that led to the integral transformations. Another "proof" of HEAVISIDE's procedures consisted in showing that more traditional methods lead to the same results as those obtained by HEAVISIDE. CARSON's first proof of HEAVISIDE's expansion theorem (III.1) provides such an example. A similar one was given by CASPER [1925]. Most engineers, however, sought the explanation of HEAVISIDE's calculus in algebra. HEAVISIDE's results were often explained by formal algebraic manipulations with the operator symbols [see *e.g.* COHEN 1922], but these investigations cannot be considered justifications of the operational methods, but are only applications of them.

Other authors went into the problem more deeply. They examined which algebraic laws were used in the operational calculus and tried to prove that differential and integral operators obeyed these laws. The commutative law made a smooth interpretation along these lines impossible, because the differential and integral operators do not commute, as had been stated by BOOLE in his book on differential equations [BOOLE 1859]. Thus

$$p^{-1}pf=\int_0^t \frac{d}{ds}f(s)\,ds=f(t)-f(0) \tag{V.1}$$

whereas

$$p^{-1}pf=\frac{d}{dt}\int_0^t f(s)\,ds=f(t). \tag{V.2}$$

These difficulties were overcome in different ways.

Some authors accepted this non-commutative algebra and calculated within it (for instance GAUSTER 1930, LOCHER 1934). One of the rules for calculations with differential and integral operators, which was emphasized by GAUSTER, was that one was allowed only to reduce a fraction "von rechts oben nach links unten".

Other authors tried to avoid the difficulties by changing the operators or the operands so that the commutative law was fulfilled.

Thus the American E. J. BERG, in his book *Heaviside's Operational Calculus* (1929), pointed out that the commutative law can be obtained if all operands are multiplied by $H(t)$ and $H(0)$ is defined to be 0. Namely, we have

$$\begin{aligned} p^{-1}pf(t)H(t)&=\int_0^t \frac{d}{ds}\big(f(s)H(s)\big)\,ds\\ &=\int_0^t f'(s)H(s)\,ds+\int_0^t f(s)H'(s)\,ds\\ &=[f(s)H(s)]_0^t-\int_0^t f(s)H'(s)\,ds+\int_0^t f(s)H'(s)\,ds\\ &=f(t)H(t)-f(0)H(0)=f(t)H(t)\end{aligned} \tag{V.3}$$

and also

$$pp^{-1}f(t)H(t)=\frac{d}{dt}\int_0^t f(s)H(s)\,ds=f(t)H(t). \tag{V.4}$$

Therefore all the normal algebraic rules could be applied; only one problem remained which was the meaning of H'.

FLORIN, who also gave an account of the algebraic approach to the operational calculus, obtained commutativity by redefining the operator p. In some connections he used p for $\frac{d}{dt}$ just as HEAVISIDE had done, but in the treatment of differential equations he translated $\frac{d}{dt}f$ into $pf-pf(0)$. He then achieved

$$\begin{aligned} p^{-1}pf(t)&=\int_0^t pf(s)\,ds=\int_0^t \left(\frac{d}{ds}f(s)+pf(0)\right)ds \\ &=f(t)-f(0)+p^{-1}pf(0)=f(t) \end{aligned} \tag{V.5}$$

if we define $p^{-1}pf(0)$ to be $f(0)$ and

$$pp^{-1}f(t)=\frac{d}{dt}\int_0^t f(s)\,ds+p\int_0^0 f(s)\,ds=f(t). \tag{V.6}$$

From the algebraic investigations it became evident, just as in transformation theory, that the initial values played a special part. HEAVISIDE's method had yielded solutions with homogeneous initial values, because in his calculations he had made operations which assumed f and some of the derivatives to vanish in 0.

2. In 1926 PAUL LÉVY opened a different approach to the algebraic treatment of the operational calculus by connecting it with the convolution integral

$$(f*g)(t)=\int_0^t f(t-u)\,g(u)\,du. \tag{V.7}$$

Corresponding to a function f LÉVY introduced the convolution operator F, which acts on a function g as follows:

$$F(g)=f*g.$$

To the constant function 1 corresponds the integral operator

$$I\,g(t)=\int_0^t g(u)\,du, \tag{V.8}$$

and $1*1*\cdots*1$ (α times) corresponds to

$$I^\alpha g(t)=\int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)}\,g(u)\,du. \tag{V.9}$$

Then LÉVY introduced the operators I^{-n} such that

$$I^{-n} I^{\alpha} = I^{\alpha-n} \quad \text{for } \alpha > 0 \tag{V.10}$$

and he thought that "il faut convenir que I^{-n} représente une dérivation d'ordre n". In fact this definition of I^{-n} does not follow from (10) and it gave the usual non-commutativity

$$I^n I^{-n} = f(t) - \left(f(0) + f'(0)\,t + \cdots + \frac{f'(0)}{n!}\,t^n \right). \tag{V.11}$$

In LÉVY's paper an operator F was thus identified with an element in the convolution ring with the compositions

$$(f+g)(t) = f(t) + g(t), \tag{V.12}$$

$$(f * g)(t) = \int_0^t f(u)\, g(t-u)\, du. \tag{V.13}$$

This idea, however, did not become very fertile for the understanding of the operational calculus until it was extended to a direct treatment of operators of the form

$$\frac{F}{G}.$$

LÉVY was not able to do this. He extended the ring by the differential operators I^{-n} but he could only interpret other expressions of the form $\frac{I}{G}$ by making a doubtful expansion rather like what HEAVISIDE had done. There was another defect in LÉVY's work – as well as in most of his predecessors' – namely that the underlying class of functions was not specified.

3. From the middle of the thirties it seems that the only accepted approach to the operational calculus was that using the LAPLACE-transformation, and the algebraic one was apparently abandoned until 1950. In that year the Polish mathematician MIKUSIŃSKI published his paper "Sur les fondements du calcul opératoire". His ideas, which were developed in a textbook in 1953 (translated into English in 1959), were a rediscovery, a perfection, and an extension of LÉVY's ideas. However, MIKUSIŃSKI does not refer to LÉVY. He considers the ring of continuous complex-valued functions defined for $t \geqq 0$ with the compositions (12) and (13). According to TITCHMARSH (1926) this ring has no zero divisors, so that it can be extended to a field. MIKUSIŃSKI calls the elements in the smallest field-extension operators and shows that the inverse of the unit function 1 with respect to $*$ is an operator s whose product with an absolutely continuous function is given by

$$s * f = s * f(0) + f' \tag{V.14}$$

where $f(0)$ is a function with the constant value $f(0)$. Thus the inverse of the integral operator $1 * \cdot$ is not just the differential operator but includes the addition of a term depending on f's initial value. The formula (14) is the same as FLORIN's translation rule for $\frac{d}{dt}$, but in MIKUSIŃSKI's treatment (14) is a consequence of a fundamental idea whereas for FLORIN it was only an *ad hoc* hypothesis.

In the field of operators, which encloses both functions and differential and integral operators in one algebraic structure, MIKUSIŃSKI can now rigorously use the methods which had until then only had a formal character. Formula (14) even enables him to extend the differential operator to continuous functions that are not differentiable in the classical sense. For example the HEAVISIDE function $H(t)$ ($=1$ in the considered domain $t \geqq 0$) has the "derivative" $s * 1$, which according to the definition of s is the unit element in the field of operators. Hence this unit element is the rigorization of the mysterious δ "function" which HEAVISIDE's unit operand had introduced in to the operational calculus (for a further account of MIKUSIŃSKI's method see FREUDENTHAL 1969).

4. However, MIKUSIŃSKI was not the first to rigorize the δ-function. His work had been preceded by SCHWARTZ's theory of distributions.

As early as 1925 the American J. J. SMITH made an attempt to solve the problem of the δ-function, which he saw as the main challenge of the operational calculus. In a series of papers in *Journal of the Franklin Institute* he developed a new kind of analysis, the "Theory of H-Functions", the main object of which was the treatment of pointwise multiple-valued or infinite functions. But his theory of H-functions was both unrigorous and unfitted for practical computation.

At the end of the twenties the problem of the δ-function became more important because DIRAC had made it a central tool in the quantum mechanics [DIRAC 1926]. It was he who gave it the name δ-function. He used it as a counterpart to KRONECKER's δ in the description of continuous systems of orthogonal eigenvalues. J. VON NEUMANN avoided the mathematical difficulties with "DIRAC's" δ-function by formulating the quantum mechanics in terms of operators in abstract spaces [1927], but nevertheless DIRAC's formulation remained in use.

A desire to solve the increasingly urgent problem of the δ-function was an important motive for SCHWARTZ in his development of the distribution theory [1945]. He also wanted to put HEAVISIDE's calculus on a firm basis, but his approach was not so directly guided by the operational ideas as was MIKUSIŃSKI's. First of all SCHWARTZ wanted to enlarge the concept of function, and for this purpose he considered the space $\mathscr{D}'$ of all continuous linear functionals on $\mathscr{C}_c^\infty(\mathbb{R})$, the infinitely often differentiable real functions with compact support. He called the elements of $\mathscr{D}'$ distributions. L^1_{loc} (the space of locally integrable functions) was considered a subspace of $\mathscr{D}'$ by identifying a L^1_{loc} function f with the functional

$$\varphi \curvearrowright \int_{-\infty}^{\infty} f(x)\,\varphi(x)\,dx.$$

The δ-distribution was defined as

$$\delta(\varphi)=\varphi(0) \qquad \text{for } \varphi\in\mathscr{C}_c^\infty(R).$$

SCHWARTZ pointed out that δ does not correspond to any L^1_{loc} function, but still, if symbolically we write

$$\delta(\varphi)=\int_{-\infty}^{\infty}\delta(x)\,\varphi(x)\,dx,$$

then $\delta(x)$ must have the properties characteristic of the δ-"function". SCHWARTZ extended differentiation to all of $\mathscr{D}'$, making it the transpose of differentiation in $\mathscr{C}_c^\infty$:

$$T'(\varphi)=-T(\varphi') \qquad \text{for } T\in\mathscr{D}',\ \varphi\in\mathscr{C}_c^\infty.$$

He also introduced convolution for distributions whereby the operator D^n was represented by $\delta^n *$. This enabled him to treat distributions in a way similar to the one used later by MIKUSIŃSKI in dealing with operators. Therefore distribution theory gave a rigorization of the algebraic approach to the operational calculus.

Distribution theory was also useful for the approach to operational calculus employing integral transformations, as it proved to be a problem that the domain of definition for the transformations was not so large as could be wished. Beginning with his first paper, from 1945 onward SCHWARTZ tried to extend the LAPLACE and FOURIER transformations, but not until two years later did he find the space of "tempered" or "spherical" distributions $\mathscr{S}'$ [SCHWARTZ 1947], which is the space best suited to describing these transformations. He extended the FOURIER transformation to an isomorphism of $\mathscr{S}'$ onto itself and the LAPLACE transformation to all distributions T for which there exists a $\xi_0\in\mathbb{R}$ such that $e^{\xi t}T\in\mathscr{S}'$ for all $\xi>\xi_0$.

In modern applications of the operational calculus the most used approach is the method of LAPLACE transformations with the extension made by SCHWARTZ.

Final Remarks

I have followed the development of operational calculus without mentioning the theory of linear operators in HILBERT and BANACH spaces and the theory of VON NEUMANN algebras which developed in the period 1900–1940. The reason is that apparently the development of the two types of operator theory did not influence each other noticeably. This may seem strange, but several reasons can be mentioned for this lack of interaction.

First, there is the historical reason that the two theories developed from different problems. The operational calculus had its source in the practical manipulation of differential equations, whereas operators in BANACH spaces developed from a theoretical interest in the solution of integral equations.

Secondly, it can be noticed that the problem in the two theories were of a different nature. The operational calculus was a method successful in practice but lacking a natural rigorous interpretation, whereas the theory of integral equations had clear concepts but no effective methods.

Closely connected with this there was a social reason for the separation of the two theories. As we have seen, the operational calculus developed at the fringe of the main mathematical stream and was partly used by practitioners, whereas the theory of operators in BANACH spaces occupied a central position in the mathematics of the 20th century and was created by "real" mathematicians.

The development of the operational calculus gives an illustrative example of how a practical problem – long distance telegraphy – influenced a mathematical theory. It shows that one must be very cautious not to simplify the relations between technology and mathematics. I have already pointed out in the introduction that it was not the technological problems which gave rise to the invention of the operational calculus, and it has just been noted that this practically inspired discipline was never integrated in the theory of operators developing from internal mathematical problems. However, the engineering method inspired one essential field of mathematics, namely the theory of distributions, but one should keep in mind that there were several other sources for this theory. These I shall discuss in another paper.

Appendix

Heaviside's First Proof of His Expansion Theorem

This proof is a most illustrative example of how physically guided experimental mathematics may lead to the discovery of new mathematical theorems. The steps leading HEAVISIDE to the theorem are scattered over several papers from the period 1881–1886. The succeeding account partly follows VALLARTA [1926].

1. *The problem* is to express how a physical system reacts to an impressed force at $t=0$. But in his approach to the theorem HEAVISIDE considered nothing but isolated systems and introduced the impressed force only in the final argument from 1886. This shows that he was not consciously hunting for the expansion theorem until this year. Let the isolated system be described by the differential equations (I.26) and (I.27) with $F_1=0$. HEAVISIDE considers these equations to be representatives of both an electrical system as in (I § 6) and a mechanical system with the generalized coordinates x_i. If the determinant $D(p)$ of the system has n different roots ${}_1p, {}_2p, {}_3p, \ldots, {}_np$ then the homogeneous equation (I.26) has the "normal" solutions

$$ {}_jx_i(t) = {}_jq_i \exp({}_jpt) \qquad i, j = 1, 2, \ldots, n, \tag{A.1} $$

and thus the general solution is

$$ x_i(t) = \sum_j {}_jA\, {}_jq_i \exp({}_jpt). \tag{A.2} $$

(In the following the indices behind the letters enumerate the generalized coordinates, and indices in front of the letters enumerate different roots of $D(p)$ or different normal systems.) HEAVISIDE wants to find an expression for the ${}_jA$'s corresponding to a given initial state.

2. For this purpose he uses *the conjugate property*, which was discovered independently by ROUTH and HEAVISIDE. HEAVISIDE first mentions the property in 1881 (*EP* I, p. 127–29) in connection with an electrical system and later on uses it in 1882 in different connections [*EP* I, p. 141–48], and in 1885 [*EP* I, p. 520–25]. In 1886 [*EP* II, p. 201–26] he derived it from mechanics in the following way:

The equation of total activity in the system (I.26) is, with all $F_i=0$,

$$0=\sum_i F_i \dot{x}_i = Q+\dot{U}+\dot{T} \tag{A.3}$$

where

$$\begin{aligned} &\text{the dissipativity} && Q=b_{11}\dot{x}_1^2+2b_{12}\dot{x}_1\dot{x}_2+b_{22}\dot{x}_2^2+\dots,\\ &\text{the potential energy} && U=\tfrac{1}{2}(c_{11}x_1^2+2c_{12}x_1x_2+c_{22}x_2^2+\dots),\\ &\text{the kinetic energy} && T=\tfrac{1}{2}(a_{11}\dot{x}_1^2+2a_{12}\dot{x}_1\dot{x}_2+a_{22}\dot{x}_2^2+\dots) \end{aligned} \tag{A.4}$$

if $a_{ij}=a_{ji}$, $b_{ij}=b_{ji}$ and $c_{ij}=c_{ji}$ which will be supposed in the following.

We now suppose that the system oscillates as a superposition of two normal solutions ${}_1x_i+{}_2x_i$ where ${}_1x_i={}_1q_i\exp({}_1pt)$ and ${}_2x_i={}_2q_i\exp({}_2pt)$. Then the dissipativity and the potential and kinetic energy can be decomposed

$$Q={}_1Q+{}_2Q+{}_{12}Q, \qquad U={}_1U+{}_2U+{}_{12}U, \qquad T={}_1T+{}_2T+{}_{12}T$$

where ${}_jQ$, ${}_jU$, ${}_jT$ are the energies of the systems j ($j=1,2$) existing separately and

$$\begin{aligned} {}_{12}Q&=2(b_{11}\,{}_1\dot{x}_1\,{}_2\dot{x}_1+b_{22}\,{}_1\dot{x}_2\,{}_2\dot{x}_2+b_{12}({}_1\dot{x}_1\,{}_2\dot{x}_2+{}_1\dot{x}_2\,{}_2\dot{x}_1)+\dots),\\ {}_{12}U&=c_{11}\,{}_1x_1\,{}_2x_1+c_{22}\,{}_1x_2\,{}_2x_2+c_{12}({}_1x_1\,{}_2x_2+{}_1x_2\,{}_2x_1)+\dots,\\ {}_{12}T&=a_{11}\,{}_1\dot{x}_1\,{}_2\dot{x}_1+a_{22}\,{}_1\dot{x}_2\,{}_2\dot{x}_2+a_{12}({}_1\dot{x}_1\,{}_2\dot{x}_2+{}_1\dot{x}_2\,{}_2\dot{x}_1)+\dots \end{aligned} \tag{A.5}$$

are called the mutual energies.

If, instead of (3), we calculate the mutual activities $\sum_i {}_1F_i\,{}_2\dot{x}_i$ and $\sum_i {}_2F_i\,{}_1\dot{x}_i$ and insert ${}_jF_i$ from equation (I.26), we get for ${}_jF_i=0$

$$0=\sum_i {}_1F_i\,{}_2\dot{x}_i=\tfrac{1}{2}\,{}_{12}Q+{}_2p\,{}_{12}U+{}_1p\,{}_{12}T, \tag{A.6a}$$

$$0=\sum_i {}_2F_i\,{}_1\dot{x}_i=\tfrac{1}{2}\,{}_{12}Q+{}_1p\,{}_{12}U+{}_2p\,{}_{12}T \tag{A.6b}$$

because differentiation with respect to t in system j consists of a multiplication with ${}_jp$. By subtracting (6a) from (6b) we have

$$0=({}_1p-{}_2p)({}_{12}U-{}_{12}T).$$

Thus for different normal states $({}_1p \neq {}_2p)$

$$_{12}U = {}_{12}T.$$

This equality of the mutual potential and kinetic energy is called the conjugate property.

3. The problem of *determining the* ${}_jA$'s in (2) from the conjugate property was taken up by HEAVISIDE in 1881 in the article where the conjugate property was presented for the first time. Since this first note is difficult to follow, I shall outline his derivation from 1885 (*EP* I, pp. 520–25).

The mutual potential and kinetic energies of the r^{th} normal system in (2) and the given initial state, that is (5) with ${}_rx_i$ substituted for ${}_1x_i$ and $x_i(0)$ substituted for ${}_2x_i$, have the form

$$_{0r}U = {}_1A\,{}_{1r}U + {}_2A\,{}_{2r}U + {}_3A\,{}_{3r}U + \ldots, \tag{A.7a}$$

$$_{0r}T = {}_1A\,{}_{1r}T + {}_2A\,{}_{2r}T + {}_3A\,{}_{3r}U + \ldots. \tag{A.7b}$$

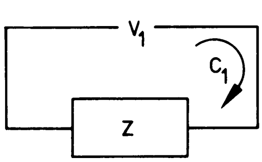
By subtraction we get

$$_{0r}U - {}_{0r}T = {}_rA({}_{rr}U - {}_{rr}T) \tag{A.8}$$

because all the terms $({}_{jr}U - {}_{jr}T)$ for $j \neq r$ are zero according to the conjugate property. Whence

$$_rA = \frac{{}_{0r}U - {}_{0r}T}{{}_{rr}U - {}_{rr}T}. \tag{A.9}$$

4. In 1886 (*EP* II, pp. 215–218) HEAVISIDE used what he called the condensor method to derive the effects of an impressed force from (9). I shall go through his clarified and generalized version from the following year (*EP* II, pp. 371–74). Suppose that (I.26) is the system of equations corresponding to an electric network in which one e.m.f. $F_1 = V_1 = e(t)$ is impressed in the first mesh where it produces the current C_1 (System 1).



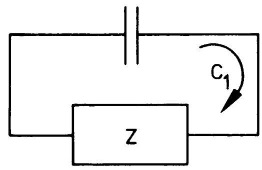
System 1

V_1 and C_1 are then connected through $V_1 = Z(p)\,C_1$, the resistance operator $Z(p)$ being given by (I.28) and (I.25)

$$Z(p) = \frac{D(p)}{p\,D_{11}(p)}.$$

If $D(p)$ has no zero root, the normal systems $x_i \exp({}_jpt)$ of the corresponding homogeneous equation are determined by $Z({}_jp) = 0$.

In order to apply the conjugate property HEAVISIDE replaces the external e.m.f. V_1 by a condensor with permittance S and initial voltage e, thus creating the following isolated auxiliary System 2:



System 2

System Z is supposed to be free from current and charges at $t=0$. Differentiating (2) and putting ${}_jw_1 = {}_jp\,{}_jq_1$, we get the current C_1 in System 2

$$C_1 = px_1 = \sum_{Z'({}_jp)=0} {}_jA\,{}_jw_1 \exp({}_jpt) \tag{A.10}$$

where the resistance operator $Z'(p)$ of the total System 2 is

$$Z'(p) = (Sp)^{-1} + Z(p). \tag{A.11}$$

A current in mesh one of the form ${}_jw_1 \exp({}_jpt)$ gives rise to a voltage over the condensor of magnitude

$${}_ju_1 \exp({}_jpt) = -\frac{{}_jw_1 \exp({}_jpt)}{S\,{}_jp} \tag{A.12}$$

so that the total voltage over the condensor is

$$V = \sum_{Z'({}_jp)=0} {}_jA\,{}_ju_1 \exp({}_jpt) = \sum_{Z'({}_jp)=0} -\frac{{}_jw_1 \exp({}_jpt)}{S\,{}_jp}. \tag{A.13}$$

Next, HEAVISIDE determines the A_j's from (9) and the following physical arguments

a. The Numerator. The mutual kinetic energy ${}_{0r}T$ is zero since the initial current is supposed to be zero. At $t=0$ there is only accumulated charge on the condensor so that

$${}_{0r}U - {}_{0r}T = {}_{0r}U = Se\,{}_ru_1. \tag{A.14}$$

b. The Denumerator. From Maxwell's equations HEAVISIDE derives

$${}_{rs}U - {}_{rs}T = \frac{{}_ru_1\,{}_sw_1 - {}_su_1\,{}_rw_1}{{}_sp - {}_rp} \quad \text{for } r \neq s.$$

(*EP* II, p. 203–06). Whence

$${}_{rs}U - {}_{rs}T = {}_sw_1\,{}_rw_1 \frac{\left(\frac{{}_ru_1}{{}_rw_1} - \frac{{}_su_1}{{}_sw_1}\right)}{{}_sp - {}_rp} = -{}_sw_1\,{}_rw_1 \frac{(Z'({}_rp) - Z'({}_sp))}{{}_rp - {}_sp}.$$

HEAVISIDE lets ${}_s p$ tend to ${}_r p$ and obtains

$$ {}_{rr}U - {}_{rr}T = -({}_r w_1)^2 \left. \frac{dZ'(p)}{dp} \right|_{p={}_r p}. \tag{A.15} $$

Combining (14) and (15) with (9), he gets

$$ {}_r A = -\frac{S\, {}_r u_1\, e}{({}_r w_1)^2 \left. \dfrac{dZ'(p)}{dp} \right|_{p={}_r p}}. $$

According to (12) ${}_r w_1 = -S\, {}_r p\, {}_r u_1$, implying that

$$ {}_r A = \frac{e}{{}_r w_1\, {}_r p \left. \dfrac{dZ'(p)}{dp} \right|_{p={}_r p}} $$

which inserted into (10) gives

$$ C_1 = \sum_{Z'({}_j p)=0} \frac{e}{p \left. \dfrac{dZ'(p)}{dp} \right|_{p={}_j p}} \exp({}_j p t). \tag{A.16} $$

"Now increase S infinitely, keeping e constant. Z' ultimately becomes Z; but in doing so one root of $Z'=0$ becomes zero". For if $Z'(p)=0$, (11) is reduced to

$$ pZ(p) = -\frac{1}{S}; $$

hence

$$ pZ(p)=0 \qquad \text{for } S=\infty; $$

from this we see that $Z'(p)$ has the same roots as $Z(p)$ plus the root $p=0$. "We have, by (11), and remembering that $Z'=0$

$$ p\frac{d}{dp}Z'(p) = -(Sp)^{-1} + p\frac{d}{dp}Z(p) = Z(p) + p\frac{d}{dp}Z(p) \tag{A.17} $$

so, when $S=\infty$ and $Z=0$, we have $p\dfrac{d}{dp}Z'(p) = p\dfrac{d}{dp}Z(p)$ for all roots except the one just mentioned [*i.e.* 0], in which case p tends to zero and $\dfrac{d}{dp}Z(p)$ is finite, making in the limit $\dfrac{d}{dp}Z'(p) = Z(0)$ by (17).... Therefore finally"

$$ C_1 = \frac{e}{Z(0)} + \sum_{Z({}_j p)=0} \left. \frac{e}{p\dfrac{dZ}{dp}} \right|_{p={}_j p} \exp({}_j p t). \tag{A.18} $$

In the limit $S=\infty$ the condensor is converted into a constant e.m.f., so that (18) expresses the current in System 1. Thus HEAVISIDE has derived the expansion theorem (I.20) in the case where $Y=1$.

Note added in proof. Professor TRUESDELL has drawn my attention to the work of H. BATEMAN on the LAPLACE transformation. BATEMAN was among the pioneers of the application of LAPLACE transformation to differential equations. He published on it as early as 1910 in Proc. Cambridge Phil. Soc. vol. 15, pp. 423–27. According to his own statement he sent a short list of LAPLACE integrals to G. R. CARSON in which he recommended the use of the LAPLACE transformation after the appearance of CARSON [1919] (*cf.* ERDÉLYI's Obituary Notice "Harry Bateman 1882–1946" in Obituary Notices of Fellows of the Royal Society vol. **5**, 1948, p. 598 and Footnote 111 in BATEMAN's paper "The Control of an Elastic Fluid" Bull. Amer. Math. Soc. vol. **51**, 1945, pp. 601–646). Thus it seems likely that CARSON's use of (III 7) was inspired by BATEMAN. However, CARSON did not refer to BATEMAN.

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